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BOUNDEDNESS FOR MULTILINEAR INTEGRAL OPERATORS ON TRIEBEL-LIZORKIN AND LEBESGUE SPACES

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Abstract

The boundedness for the multilinear operators associated to some integral operators on Triebel-Lizorkin and Lebesgue spaces are obtained. The operators include Littlewood-Paley operators, Marcinkiewicz operators, and Bochner-Riesz operator.

1. Introduction

As the development of singular integral operators, their commutators and multilinear operators have been well studied (see [1-7]). From [2, 13], we know that the commutators and multilinear operators generated by the singular integral operators and the Lipschitz functions are bounded on the Triebel-Lizorkin and Lebesgue spaces. The purpose of this paper is

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to introduce some multilinear operators associated to certain non-convolution type integral operators and prove the boundedness properties for the multilinear operators on the Triebel-Lizorkin and Lebesgue spaces. The operators include Littlewood-Paley operators, Marcinkiewicz operators, and Bochner-Riesz operator.

2. Notations and Theorem

In this paper, we will study a class of multilinear operators associated to some nonconvolution type integral operators as following.

Let m_j be the positive integers $(j = 1, \cdots, l)$, $m_1 + \cdots + m_l = m$ and A_j be the functions on $R^n (j = 1, \cdots, l)$. Set

$$R_{m_j+1}(A_j;\ x,\ y) = A_j(x) - \sum_{|\alpha| \le m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x-y)^\alpha.$$

Let $F_t(x, y)$ define on $R^n \times R^n \times [0, +\infty)$. Set

$$F_t(f)(x) = \int_{R^n} F_t(x,\ y) f(y) dy,$$

and

$$F_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j;\ x,\ y)}{|x-y|^m} F_t(x,\ y) f(y) dy,$$

for every bounded and compactly supported function f. Let H be the Banach space $H = \{h : \|h\| < \infty\}$ such that, for each fixed $x \in R^n$, $F_t(f)(x)$ and $F_t^A(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H. Then, the multilinear operator associated to F_t is defined by

$$T^A(f)(x) = \|F_t^A(f)(x)\|,$$

where F_t satisfies: for fixed $\varepsilon > 0$ and $0 \leq \delta < n$,

$$\|F_t(x, y)\| \leq C|x - y|^{-n+\delta},$$

and

$$\|F_t(y, x) - F_t(z, x)\| \leq C|y - z|^{\varepsilon}|x - z|^{-n-\varepsilon+\delta},$$

if $2|y - z| \leq |x - z|$. We define that $T(f)(x) = \|F_t(f)(x)\|$.

Note that when $m = 0$, T^A is just the multilinear commutator of T and A (see [8-11, 16]). While when $m > 0$, it is non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2-6]). The purpose of this paper is to study the boundedness properties for the multilinear operator T^A on Triebel-Lizorkin and Lebesgue spaces. In Section 4, some applications of Theorem in this paper are given.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well-known that (see [14, 15])

$$f^{\#}(x) = \sup_{x \in Q} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

For $1 \leq p < \infty$ and $0 \leq \eta < n$, let

$$M_{\eta, p}(f)(x) = \sup_{x \in Q} \left(\frac{1}{|Q|^{1-p\eta/n}} \int_Q |f(y)|^p dy \right)^{1/p}.$$

For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}$ be the homogeneous Triebel-Lizorkin space (see [13]). The Lipschitz space $\dot{\Lambda}_\beta$ is the space of functions f such that

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_{\substack{x,\, h \in R^n \\ h \neq 0}} \left|\Delta_h^{[\beta]+1} f(x)\right| / |h|^\beta < \infty,$$

where Δ_h^k denotes the k-th difference operator (see [13]).

Now we can state our theorem as following:

Theorem. *Let* $0 < \beta < \min(1/l, \varepsilon/l)$ *and* $D^\alpha A_j \in \dot{\Lambda}_\beta$ *for all* α *with* $|\alpha| = m_j$ *and* $j = 1, \cdots, l$. *Suppose that* T *is bounded from* $L^r(R^n)$ *to* $L^s(R^n)$ *for any* $0 \leq \delta < n$, $1 < r < n/\delta$ *and* $1/r - 1/s = \delta/n$. *Then*

(a) T^A *is bounded from* $L^p(R^n)$ *to* $\dot{F}_q^{l\beta,\infty}(R^n)$ *for any* $0 \leq \delta < n$, $1 < p < n/\delta$, $1/p - 1/q = \delta/n$;

(b) T^A *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$ *for any* $0 \leq \delta < n - l\beta$, $1 < p < n/(\delta + l\beta)$ *and* $1/p - 1/q = (\delta + l\beta)/n$.

3. Proof of Theorem

To prove the theorem, we need the following lemmas:

Lemma 1 (see [13]). *For* $0 < \beta < 1$, $1 < p < \infty$, *we have*

$$\|f\|_{\dot{F}_p^{\beta,\infty}} \approx \left\|\sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q|\,dx\right\|_{L^p}$$

$$\approx \left\|\sup_{\cdot \in Q} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c|\,dx\right\|_{L^p}.$$

Lemma 2 (see [13]). *For* $0 < \beta < 1$, $1 \le p \le \infty$, *we have*

$$\|b\|_{\dot{\Lambda}_\beta} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| dx \approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - b_Q|^p dx \right)^{1/p}.$$

Lemma 3 (see [13]). *For* $b \in \dot{\Lambda}_\beta$, $0 < \beta < 1$, $0 \le \eta < n$ *and* $1 < r < \infty$, *we have*

$$\|(b - b_Q) f \chi_Q\|_{L^r} \le C \|b\|_{\dot{\Lambda}_\beta} |Q|^{1/r + \beta/n - \eta/n} M_{\eta, r}(f).$$

Lemma 4 (see [1]). *Suppose that* $1 \le r < p < n/\delta$ *and* $1/q = 1/p - \eta/n$. *Then*

$$\|M_{\eta, r}(f)\|_{L^q} \le C \|f\|_{L^p}.$$

Lemma 5 (see [5]). *Let* A *be a function on* R^n *and* $D^\alpha A \in L^q(R^n)$ *for* $|\alpha| = m$ *and some* $q > n$. *Then*

$$|R_m(A; x, y) \le C|x - y|^m \sum_{|\alpha| = m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x, y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ *is the cube centered at x and having side length* $5\sqrt{n}|x - y|$.

Proof of Theorem (a). We first prove the sharp estimate for T^A as following:

$$\frac{1}{|Q|^{1 + l\beta/n}} \int_Q |T^A(f)(x) - C_0| dx \le CM_{\delta, r}(f)(\tilde{x}),$$

for $1 < r < p < n/\delta$ and some constant C_0. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha| = m} \frac{1}{\alpha!} (D^\alpha A_j)_{\tilde{Q}} x^\alpha$, then $R_m(A_j; x, y) = R_m(\tilde{A}_j; x, y)$ and $D^\alpha \tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned}
F_t^A(f)(x) &= \int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j+1}(\widetilde{A}_j;\ x,\ y)}{|x-y|^m} F_t(x,\ y) f(y) dy \\
&= \int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j+1}(\widetilde{A}_j;\ x,\ y)}{|x-y|^m} F_t(x,\ y) f_2(y) dy \\
&\quad + \int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j}(\widetilde{A}_j;\ x,\ y)}{|x-y|^m} F_t(x,\ y) f_1(y) dy \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\widetilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \widetilde{A}_1(y) F_t(x, y) f_1(y) dy \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\widetilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \widetilde{A}_2(y) F_t(x, y) f_1(y) dy \\
&\quad + \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\,\alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\widetilde{A}_1(y) D^{\alpha_2}\widetilde{A}_2(y)}{|x-y|^m} \\
&\qquad \times F_t(x,\ y) f_1(y) dy,
\end{aligned}$$

then

$$\left| T^A(f)(x) - T^{\widetilde{A}}(f_2)(x_0) \right| = \left| \|F_t^A(f)(x)\| - \|F_t^{\widetilde{A}}(f_2)(x_0)\| \right|$$

$$\begin{aligned}
&\le \|F_t^A(f)(x) - F_t^{\widetilde{A}}(f_2)(x_0)\| \le \left\| \int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j}(\widetilde{A}_j;\ x,\ y)}{|x-y|^m} F_t(x,\ y) f_1(y) dy \right\| \\
&+ \left\| \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\widetilde{A}_2;\ x,\ y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \widetilde{A}_1(y) F_t(x,\ y) f_1(y) dy \right\| \\
&+ \left\| \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\widetilde{A}_1;\ x,\ y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \widetilde{A}_2(y) F_t(x,\ y) f_1(y) dy \right\|
\end{aligned}$$

$$+ C\left\|\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\widetilde{A}_1(y)D^{\alpha_2}\widetilde{A}_2(y)}{|x-y|^m}F_t(x,\ y)f_1(y)dy\right\|$$

$$+ \mid \left|T^{\widetilde{A}}(f_2)(x) - T^{\widetilde{A}}(f_2)(x_0)\right|$$

$$:= I_1(x) + I_2(x) + I_3(x) + I_4(x) + I_5(x),$$

thus

$$\frac{1}{|Q|^{1+2\beta/n}}\int_Q\left|T^A(f)(x) - T^{\widetilde{A}}(f_2)(x_0)\right|dx$$

$$\leq \frac{1}{|Q|^{1+2\beta/n}}\int_Q\left\|\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\widetilde{A}_j;\ x,\ y)}{|x-y|^m}F_t(x,\ y)f_1(y)dy\right\|dx$$

$$+ \frac{C}{|Q|^{1+2\beta/n}}\int_Q\left\|\sum_{|\alpha_1|=m_1}\int_{R^n}\frac{R_{m_2}(\widetilde{A}_2;\ x,\ y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\widetilde{A}_1(y)F_t(x,\ y)f_1(y)dy\right\|dx$$

$$+ \frac{C}{|Q|^{1+2\beta/n}}\int_Q\left\|\sum_{|\alpha_2|=m_2}\int_{R^n}\frac{R_{m_1}(\widetilde{A}_1;\ x,\ y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\widetilde{A}_2(y)F_t(x,\ y)f_1(y)dy\right\|dx$$

$$+ \frac{C}{|Q|^{1+2\beta/n}}$$

$$\int_Q\left\|\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\widetilde{A}_1(y)D^{\alpha_2}\widetilde{A}_2(y)}{|x-y|^m}F_t(x,\ y)f_1(y)dy\right\|dx$$

$$+ \frac{1}{|Q|^{1+2\beta/n}}\int_Q\left|T^{\widetilde{A}}(f_2)(x) - T^{\widetilde{A}}(f_2)(x_0)\right|dx$$

$$:= I_1 + I_2 + I_3 + I_4 + I_5.$$

Now, let us estimate I_1, I_2, I_3, I_4, and I_5, respectively. First, by Lemma 5 and Lemma 2, we get, for $x \in Q$ and $y \in \widetilde{Q}$,

$$|R_m(\widetilde{A}_j;\ x,\ y)| \le C|x-y|^m \sum_{|\alpha|=m} \sup_{x\in\widetilde{Q}} |D^\alpha A_j(x) - (D^\alpha A_j)_{\widetilde{Q}}|$$

$$\le C|x-y|^m |Q|^{\beta/n} \sum_{|\alpha|=m} \|D^\alpha A_j\|_{\dot{\Lambda}_\beta},$$

thus, by the $(L^r,\ L^s)$-boundedness of T with $1 < r < n/\delta$ and $1/s = 1/r - \delta/n$, we obtain, using Hölder's inequality,

$$I_1 \le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\dot{\Lambda}_\beta}\right) \frac{1}{|Q|}\int_Q |T(f_1)(x)|\,dx$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\dot{\Lambda}_\beta}\right) \left(\frac{1}{|Q|}\int_Q |T(f_1)(x)|^s\,dx\right)^{1/s}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\dot{\Lambda}_\beta}\right) |Q|^{-1/s}\left(\int_{\widetilde{Q}} |f(x)|^r\,dx\right)^{1/r}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\dot{\Lambda}_\beta}\right) \left(\frac{1}{|\widetilde{Q}|^{1-r\delta/n}}\int_{\widetilde{Q}} |f(y)|^r\,dy\right)^{1/r}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\dot{\Lambda}_\beta}\right) M_{\delta,r}(f)(\widetilde{x}).$$

For I_2, using Lemma 3 and Hölder's inequality, we get, for $1 < r < n/\delta$ and $1/s = 1/r - \delta/n$,

$$I_2 \le C\sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{\dot{\Lambda}_\beta} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|^{1+\beta/n}}\int_Q |T(D^{\alpha_1}\widetilde{A}_1 f_1)(x)|\,dx$$

$$\le C\sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{\dot{\Lambda}_\beta} \sum_{|\alpha_1|=m_1} \|T((D^{\alpha_1} A - (D^{\alpha_1} A)_{\widetilde{Q}})f_1)\|_{L^s} |Q|^{-\beta/n-1/s}$$

$$\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{\dot{\Lambda}_\beta} |Q|^{-\beta/n-1/s} \sum_{|\alpha_1|=m} \|(D^{\alpha_1} A - (D^{\alpha_1} A)_{\widetilde{Q}}) f_1\|_{L^r}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\dot{\Lambda}_\beta} \right) M_{\delta,r}(f)(\widetilde{x}).$$

For I_3, similar to the proof of I_2, we get

$$I_3 \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\dot{\Lambda}_\beta} \right) M_{\delta,r}(f)(\widetilde{x}).$$

Similarly, for I_4, set $r = s\mu_3$ for $1 < s < n/\delta$, $\mu_1, \mu_2, \mu_3 > 1$, $1/\mu_1 + 1/\mu_2 + 1/\mu_3 = 1$ and $1/t = 1/s - \delta/n$, we obtain

$$I_4 \leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|^{1+2\beta/n}} \int_Q |T(D^{\alpha_1} \widetilde{A}_1 D^{\alpha_2} \widetilde{A}_2 f_1)(x)| dx$$

$$\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-2\beta/n-1/t} \left(\int_{R^n} |T(D^{\alpha_1} \widetilde{A}_1 D^{\alpha_2} \widetilde{A}_2 f_1)(x)|^t dx \right)^{1/t}$$

$$\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-2\beta/n-1/t} \left(\int_{R^n} |D^{\alpha_1} \widetilde{A}_1(x) D^{\alpha_2} \widetilde{A}_2(x) f_1(x)|^s dx \right)^{1/s}$$

$$\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-2\beta/n} \left(\frac{1}{|\widetilde{Q}|} \int_{\widetilde{Q}} |D^{\alpha_1} \widetilde{A}_1(x)|^{s\mu_1} dx \right)^{1/s\mu_1}$$

$$\times \left(\frac{1}{|\widetilde{Q}|} \int_{\widetilde{Q}} |D^{\alpha_2} \widetilde{A}_2(x)|^{s\mu_2} dx \right)^{1/s\mu_2} \left(\frac{1}{|\widetilde{Q}|^{1-r\delta/n}} \int_{\widetilde{Q}} |f(x)|^{s\mu_3} dx \right)^{1/s\mu_3}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\dot{\Lambda}_\beta} \right) M_{\delta,r}(f)(\widetilde{x}).$$

For I_5, we write

$$F_t^{\widetilde{A}}(f_2)(x) - F_t^{\widetilde{A}}(f_2)(x_0)$$

$$= \int_{R^n} \left(\frac{F_t(x, y)}{|x-y|^m} - \frac{F_t(x_0, y)}{|x_0 - y|^m} \right) \prod_{j=1}^{2} R_{m_j}(\widetilde{A}_j; x, y) f_2(y) dy$$

$$+ \int_{R^n} \left(R_{m_1}(\widetilde{A}_1; x, y) - R_{m_1}(\widetilde{A}_1; x_0, y) \right) \frac{R_{m_2}(\widetilde{A}_2; x, y)}{|x_0 - y|^m} F_t(x_0, y) f_2(y) dy$$

$$+ \int_{R^n} \left(R_{m_2}(\widetilde{A}_2; x, y) - R_{m_2}(\widetilde{A}_2; x_0, y) \right) \frac{R_{m_1}(\widetilde{A}_1; x_0, y)}{|x_0 - y|^m} F_t(x_0, y) f_2(y) dy$$

$$- \sum_{|\alpha_1| = m_1} \frac{1}{\alpha_1 !} \int_{R^n} D^{\alpha_1} \widetilde{A}_1(y) f_2(y)$$

$$\times \left[\frac{R_{m_2}(\widetilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} F_t(x, y) - \frac{R_{m_2}(\widetilde{A}_2; x_0, y)(x_0 - y)^{\alpha_1}}{|x_0 - y|^m} F_t(x_0, y) \right] dy$$

$$- \sum_{|\alpha_2| = m_2} \frac{1}{\alpha_2 !} \int_{R^n} D^{\alpha_2} \widetilde{A}_2(y) f_2(y)$$

$$\times \left[\frac{R_{m_1}(\widetilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} F_t(x, y) - \frac{R_{m_1}(\widetilde{A}_1; x_0, y)(x_0 - y)^{\alpha_2}}{|x_0 - y|^m} F_t(x_0, y) \right] dy$$

$$+ \sum_{|\alpha_1| = m_1, |\alpha_2| = m_2} \frac{1}{\alpha_1 ! \alpha_2 !} \int_{R^n} \left[\frac{(x-y)^{\alpha_1 + \alpha_2}}{|x-y|^m} F_t(x, y) - \frac{(x_0 - y)^{\alpha_1 + \alpha_2}}{|x_0 - y|^m} F_t(x_0, y) \right]$$

$$\times D^{\alpha_1} \widetilde{A}_1(y) D^{\alpha_2} \widetilde{A}_2(y) f_2(y) dy$$

$$= I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)}.$$

By Lemma 5 and the following inequality, for $b \in \dot{\Lambda}_\beta$,

$$|b(x) - b_Q| \le \frac{1}{|Q|} \int_Q \|b\|_{\dot{\Lambda}_\beta} |x-y|^\beta dy \le \|b\|_{\dot{\Lambda}_\beta} (|x - x_0| + d)^\beta,$$

we get

$$|R_{m_j}(\widetilde{A}_j;\, x,\, y)| \le \sum_{|\alpha|=m_j} \|D^{\alpha} A_j\|_{\dot{\Lambda}_\beta} (|x-y|+d)^{m_j+\beta}.$$

Note that $|x-y| \sim |x_0 - y|$ for $x \in Q$ and $y \in R^n \setminus \widetilde{Q}$, we obtain, by the condition of F_t,

$$\|I_5^{(1)}\|$$

$$\le C\int_{R^n\setminus\widetilde{Q}} \left(\frac{|x-x_0|}{|x_0-y|^{m+n+1-\delta}} + \frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{m+n+\varepsilon-\delta}} \right) \prod_{j=1}^{2} |R_{m_j}(\widetilde{A}_j;\, x,\, y)|\,|f(y)|\,dy$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\dot{\Lambda}_\beta} \right)$$

$$\times \sum_{k=0}^{\infty} \int_{2^{k+1}\widetilde{Q}\setminus 2^k\widetilde{Q}} \left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta-2\beta}} + \frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{n+\varepsilon-\delta-2\beta}} \right) |f(y)|\,dy$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\dot{\Lambda}_\beta} \right) |Q|^{2\beta/n}$$

$$\times \sum_{k=0}^{\infty} (2^{k(2\beta-1)} + 2^{k(2\beta-\varepsilon)}) \frac{1}{|2^k\widetilde{Q}|^{1-\delta/n}} \int_{2^k\widetilde{Q}} |f(y)|\,dy$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\dot{\Lambda}_\beta} \right) |Q|^{2\beta/n}$$

$$\times \sum_{k=0}^{\infty} (2^{k(2\beta-1)} + 2^{k(2\beta-\varepsilon)}) \left(\frac{1}{|2^k\widetilde{Q}|^{1-r\delta/n}} \int_{2^k\widetilde{Q}} |f(y)|^r\,dy \right)^{1/r}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\dot{\Lambda}_\beta} \right) |Q|^{2\beta/n} M_{\delta,r}(f)(x).$$

For $I_5^{(2)}$, by the formula (see [5]):

$$R_{m_j}(\widetilde{A}_j;\, x,\, y) - R_{m_j}(\widetilde{A}_j;\, x_0,\, y) = \sum_{|\eta|<m_j} \frac{1}{\eta!} R_{m_j-|\eta|}(D^\eta \widetilde{A}_j;\, x,\, x_0)(x-y)^\eta,$$

and by Lemma 5, we get

$$\|I_5^{(2)}\| \le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{\dot{\Lambda}_\beta} \right) \sum_{k=0}^{\infty} \int_{2^{k+1}\widetilde{Q} \setminus 2^k \widetilde{Q}} \frac{|x-x_0|}{|x_0-y|^{n+1-\delta-2\beta}} |f(y)| dy$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{\dot{\Lambda}_\beta} \right) |Q|^{2\beta/n} M_{\delta,r}(f)(\widetilde{x}).$$

Similarly,

$$\|I_5^{(3)}\| \le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{\dot{\Lambda}_\beta} \right) |Q|^{2\beta/n} M_{\delta,r}(f)(\widetilde{x}).$$

For $I_5^{(4)}$, similar to the estimates of $I_5^{(1)}$ and $I_5^{(2)}$, we obtain

$$\|I_5^{(4)}\| \le C \sum_{|\alpha_1|=m_1} \int_{R^n \setminus \widetilde{Q}} \left\| \frac{(x-y)^{\alpha_1} F_t(x,\, y)}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1} F_t(x_0,\, y)}{|x_0-y|^m} \right\|$$

$$\times |R_{m_2}(\widetilde{A}_2;\, x,\, y)| |D^{\alpha_1} \widetilde{A}_1(y)| |f(y)| dy$$

$$+ C \sum_{|\alpha_1|=m_1} \int_{R^n \setminus \widetilde{Q}} |R_{m_2}(\widetilde{A}_2;\, x,\, y) - R_{m_2}(\widetilde{A}_2;\, x_0,\, y)|$$

$$\times \frac{\|(x_0-y)^{\alpha_1} F_t(x_0,\, y)\|}{|x_0-y|^m} |D^{\alpha_1} \widetilde{A}_1(y)| |f(y)| dy$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}A_j\|_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}\sum_{k=0}^{\infty}(2^{k(2\beta-1)}+2^{k(2\beta-\varepsilon)})$$

$$\times\left(\frac{1}{|2^k\tilde{Q}|^{1-r\delta/n}}\int_{2^k\tilde{Q}}|f(y)|^r dy\right)^{1/r}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}A_j\|_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}M_{\delta,r}(f)(\tilde{x}).$$

Similarly,

$$\|I_5^{(5)}\|\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}A_j\|_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}M_{\delta,r}(f)(\tilde{x}).$$

For $I_5^{(6)}$, we get

$$\|I_5^{(6)}\|\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n\setminus\tilde{Q}}\left\|\frac{(x-y)^{\alpha_1+\alpha_2}F_t(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1+\alpha_2}F_t(x_0,y)}{|x_0-y|^m}\right\|$$

$$\times|D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||f(y)|dy$$

$$\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}+\frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{n+\varepsilon-\delta}}\right)$$

$$\times|D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||f(y)|dy$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\dot{\Lambda}_\beta}\right)$$

$$\times\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta-2\beta}}+\frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{n+\varepsilon-\delta-2\beta}}\right)|f(y)|dy$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\dot{\Lambda}_\beta}\right)|Q|^{2\beta/n}M_{\delta,r}(f)(x).$$

Thus

$$|T^{\tilde{A}}(f_2)(x) - T^{\tilde{A}}(f_2)(x_0)| \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} |Q|^{2\beta/n} M_{\delta,r}(f)(x),$$

and

$$I_5 \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} M_{\delta,r}(f)(x).$$

We now put these estimates together, and taking the supremum over all Q such that $\tilde{x} \in Q$, and using Lemmas 1 and 4, we obtain

$$\|T^A(f)\|_{\dot{F}_q^{2\beta,\infty}} \le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\dot{\Lambda}_\beta} \right) \|f\|_{L^p}.$$

This completes the proof of **(a)**.

For **(b)**, by using the same argument as in proof of **(a)**, we obtain

$$\frac{1}{|Q|} \int_Q |T^A(f)(x) - T^{\tilde{A}}(f_2)(x_0)| dx \le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\dot{\Lambda}_\beta} \right) M_{\delta+2\beta,r}(f),$$

thus, we get the sharp estimate of T_A as following:

$$(T^A(f))^\# \le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\dot{\Lambda}_\beta} \right) M_{\delta+2\beta,r}(f).$$

Now, using Lemma 4, we get

$$\|T^A(f)\|_{L^q} \le C \|(T^A(f))^\#\|_{L^q} \le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\dot{\Lambda}_\beta} \right) \|M_{\delta+2\beta,r}(f)\|_{L^q}$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\dot{\Lambda}_\beta} \right) \|f\|_{L^p}.$$

This completes the proof of **(b)** and the theorem.

4. Applications

Now we give some applications of results in this paper.

Application 1. Littlewood-Paley operator.

Fixed $0 \leq \delta < n$, $\varepsilon > 0$ and $\mu > (3n + 2 - 2\delta)/n$. Let ψ be a fixed function which satisfies:

(1) $\int_{R^n} \psi(x) dx = 0$;

(2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1-\delta)}$;

(3) $|\psi(x + y) - \psi(x)| \leq C|y|^{\varepsilon}(1 + |x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$.

We denote that $\Gamma(x) = \{(y, t) \in R_+^{n+1} : |x - y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley multilinear operators are defined by

$$g_\psi^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi^A(f)(x) = \left[\iint_{\Gamma(x)} |F_t^A(f)(x, y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

and

$$g_\mu^A(f)(x) = \left[\iint_{R_+^{n+1}} \left(\frac{t}{t + |x - y|}\right)^{n\mu} |F_t^A(f)(x, y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j; x, y)}{|x - y|^m} \psi_t(x - y) f(y) dy,$$

$$F_t^A(f)(x, y) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j; x, z)}{|x - z|^m} f(z) \psi_t(y - z) dz,$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi(f)(x) = \left(\iint_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

and

$$g_\mu(f)(x) = \left(\iint_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

which are the Littlewood-Paley operators (see [15]). Let H be the space

$$H = \left\{h : \|h\| = \left(\int_0^\infty |h(t)|^2 dt/t\right)^{1/2} < \infty\right\},$$

or

$$H = \left\{h : \|h\| = \left(\iint_{R_+^{n+1}} |h(y, t)|^2 dydt/t^{n+1}\right)^{1/2} < \infty\right\},$$

then for each fixed $x \in R^n$, $F_t^A(f)(x)$ and $F_t^A(f)(x, y)$ may be viewed as the mapping from $[0, +\infty)$ to H, and it is clear that

$$g_\psi^A(f)(x) = \|F_t^A(f)(x)\|, \quad g_\psi(f)(x) = \|F_t(f)(x)\|,$$

$$S_\psi^A(f)(x) = \left\|\chi_{\Gamma(x)} F_t^A(f)(x, y)\right\|, \quad S_\psi(f)(x) = \left\|\chi_{\Gamma(x)} F_t(f)(y)\right\|,$$

and

$$g_\mu^A(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t^A(f)(x, y)\right\|, \; g_\mu(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t(f)(y)\right\|.$$

It is easily to see that g_ψ^A, S_ψ^A, and g_μ^A satisfy the conditions of Theorem (see [8, 10, 11]), thus Theorem holds for g_ψ^A, S_ψ^A, and g_μ^A.

Application 2. Marcinkiewicz operator.

Fixed $0 \le \delta < n$, $\lambda > \max(1, 2n/(n+2-2\delta))$ and $0 < \gamma \le 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$. The Marcinkiewicz multilinear operators are defined by

$$\mu_\Omega^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

$$\mu_S^A(f)(x) = \left[\iint_{\Gamma(x)} |F_t^A(f)(x, y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2},$$

and

$$\mu_\lambda^A(f)(x) = \left[\iint_{R_+^{n+1}} \left(\frac{t}{t + |x-y|} \right)^{n\lambda} |F_t^A(f)(x, y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{|x-y| \le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y) dy,$$

and

$$F_t^A(f)(x, y) = \int_{|y-z| \le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; y, z)}{|y-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-\delta-1}} f(z) dz.$$

Set

$$F_t(f)(x) = \int_{|x-y| \le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y) dy.$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

$$\mu_S(f)(x) = \left(\iint_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2},$$

and

$$\mu_\lambda(f)(x) = \left(\iint_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2},$$

which are the Marcinkiewicz operators (see [16]). Let H be the space

$$H = \left\{ h : \|h\| = \left(\int_0^\infty |h(t)|^2 dt / t^3 \right)^{1/2} < \infty \right\},$$

or

$$H = \left\{ h : \|h\| = \left(\iint_{R_+^{n+1}} |h(y, t)|^2 dydt / t^{n+3} \right)^{1/2} < \infty \right\}.$$

Then, it is clear that

$$\mu_\Omega^A(f)(x) = \|F_t^A(f)(x)\|, \quad \mu_\Omega(f)(x) = \|F_t(f)(x)\|,$$

$$\mu_S^A(f)(x) = \left\| \chi_{\Gamma(x)} F_t^A(f)(x, y) \right\|, \quad \mu_S(f)(x) = \left\| \chi_{\Gamma(x)} F_t(f)(y) \right\|,$$

and

$$\mu_\lambda^A(f)(x) = \left\| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} F_t^A(f)(x, y) \right\|, \mu_\lambda(f)(x) = \left\| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} F_t(f)(y) \right\|.$$

It is easily to see that μ_Ω^A, μ_S^A, and μ_λ^A satisfy the conditions of Theorem (see [8, 10, 11]), thus Theorem holds for μ_Ω^A, μ_S^A, and μ_λ^A.

Application 3. Bochner-Riesz operator.

Let $\delta > (n-1)/2$, $B_t^\delta(f)\hat{}(\xi) = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n} B^\delta (z/t)$ for $t > 0$. Set

$$F_{\delta,t}^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, y)}{|x-y|^m} B_t^\delta(x-y) f(y) dy.$$

The maximal Bochner-Riesz multilinear operator are defined by

$$B_{\delta,*}^{A}(f)(x) = \sup_{t>0} |B_{\delta,t}^{A}(f)(x)|.$$

We also define that

$$B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^{\delta}(f)(x)|,$$

which is the maximal Bochner-Riesz operator (see [12]). Let H be the space $H = \{h : \|h\| = \sup_{t>0} |h(t)| < \infty\}$, then

$$B_{\delta,*}^{A}(f)(x) = \|B_{\delta,t}^{A}(f)(x)\|, \quad B_*^{\delta}(f)(x) = \|B_t^{\delta}(f)(x)\|.$$

It is easily to see that $B_{\delta,*}^{A}$ satisfies the conditions of Theorem, thus Theorem holds for $B_{\delta,*}^{A}$.

References

[1] S. Chanillo, A note on commutators, Indiana Univ. Math. J. 31 (1982), 7-16.

[2] W. G. Chen, Besov estimates for a class of multilinear singular integrals, Acta Math. Sinica 16 (2000), 613-626.

[3] J. Cohen, A sharp estimate for a multilinear singular integral on R^n, Indiana Univ. Math. J. 30 (1981), 693-702.

[4] J. Cohen and J. Gosselin, On multilinear singular integral operators on R^n, Studia Math. 72 (1982), 199-223.

[5] J. Cohen and J. Gosselin, A BMO estimate for multilinear singular integral operators, Illinois J. Math. 30 (1986), 445-465.

[6] R. Coifman and Y. Meyer, Wavelets, Calderón-Zygmund and Multilinear Operators, Cambridge Studies in Advanced Math., 48, Cambridge University Press, Cambridge, 1997.

[7] S. Janson, Mean oscillation and commutators of singular integral operators, Ark. Math. 16 (1978), 263-270.

[8] L. Z. Liu, Triebel-Lizorkin space estimates for multilinear operators of sublinear operators, Proc. Indian Acad. Sci. (Math. Sci.) 113 (2003), 379-393.

[9] L. Z. Liu, The continuity of commutators on Triebel-Lizorkin spaces, Integral Equations and Operator Theory 49 (2004), 65-76.

[10] L. Z. Liu, Boundedness of multilinear operator on Triebel-Lizorkin spaces, Inter J. of Math. and Math. Sci. 5 (2004), 259-272.

[11] L. Z. Liu, Boundedness for multilinear Littlewood-Paley operators on Triebel-Lizorkin spaces, Methods and Applications of Analysis 10(4) (2004), 603-614.

[12] S. Z. Lu, Four Lectures on Real H^p Spaces, World Scientific, River Edge, NI, 1995.

[13] M. Paluszynski, Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss, Indiana Univ. Math. J. 44 (1995), 1-17.

[14] E. M. Stein, Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals, Princeton Univ. Press, Princeton NJ, 1993.

[15] A. Torchinsky, The Real Variable Methods in Harmonic Analysis, Pure and Applied Math., 123, Academic Press, New York, 1986.

[16] A. Torchinsky and S. Wang, A note on the Marcinkiewicz integral, Colloq. Math. 60(61) (1990), 235-243.

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