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OPERATORS *P* AND *T* IN *PT*-SYMMETRIC QUANTUM THEORY

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Abstract

In this paper, we discuss the notions of operators P and T in PT symmetric quantum theory. Using the definition of operator T, we classify it into two types and study their influence on PT symmetry. Based on the operator T, we discuss the characters and forms of operator P in 2×2 and 3×3 quantum systems.

1. Introduction

The PT symmetric quantum theory was put forward by professor Bender et al. in 1998 [1], they pointed out that non-Hermitian Hamiltonian posses real eigenvalues provided they respect unbroken PT symmetry. PT symmetry refers to the parity - time symmetry, where P and T stand for parity and time reversal, respectively. Recently, there has been a great deal of interest in studying PT-symmetric quantum

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theory [2-11]. In standard quantum mechanics the observable is represented by Hermitian operators, but in PT-symmetric quantum system there mainly discuss non-Hermitian Hamiltonian.

In this paper, from the angle of mathematics, we analyze operators P and T of the PT-symmetric quantum theory and discuss their forms and characters. We then discuss the forms of operator P when the non-Hermitian Hamiltonian satisfies PT symmetry.

2. Basic Concepts

In this section, we discuss the basic concepts of operator P, operator T and PT symmetry. In quantum mechanics, $\hat{x}$ stands for coordinate operator and $\hat{p}$ stands for momentum operator, their algorithm are as follows:

$$(\hat{x}f)(x, t) = xf(x, t),$$

$$(\hat{p}f)(x, t) = -i\frac{\partial}{\partial x}f(x, t).$$

We call operate P parity operator [3], if it satisfies $P\hat{x}P = -\hat{x}$ and $P\hat{p}P = -\hat{p}$. We call operate T time inversion operator [3], if it satisfies $T\hat{x}T = -\hat{x}$, $T\hat{p}T = -\hat{p}$, and $TiT = -i(i = \sqrt{-1})$, which was conjugate linear operator. It is easy to know that operators P and T are all projection operators, namely, $P^2 = T^2 = I$ (identify operator). Meanwhile, operator P commutates to operator T : $[P, T] = PT - TP = 0$.

If H is an $n \times n$ square matrix, and

$$H = H^{PT} = PTHPT, \tag{1}$$

then we say that H is PT-symmetric operator [1].

Using commutator, formula (1) can be rewritten as

$$[H, PT] = HPT - PTH = 0 \quad \text{or} \quad HPT = PTH. \tag{2}$$

3. Operators *T* and *P*

This section presents the forms of operators P and T using their definitions.

3.1. Classification of operator *T* and its influence on *PT* symmetry

From the definition of operator T, we can easily know that it is anti-linear, namely, conjugate linear. Therefore, operator T can be classified into the following two types.

Definition 1. For any vector $(x_1, x_2, \cdots, x_n)^t \in C^n$ (t means the transpose), we call operator T is the first type operator T, if it satisfies the following condition:

$$T\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_n \end{pmatrix},$$

where $\overline{x}$ represents the complex conjugate of x, denoted by T_1. If the operator T satisfies the following condition:

$$T\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -\overline{x}_1 \\ -\overline{x}_2 \\ \vdots \\ -\overline{x}_n \end{pmatrix},$$

we say that T is the second type operator T, denoted by T_2. Obviously, $T_1^2 = T_2^2 = I$.

Next, we discuss the influence of the classification of operator T on PT symmetry.

Theorem 1. *Assuming that H is a Hamiltonian of 2×2 quantum system, if H meets PT symmetry, then $P\overline{H} = HP$ no matter $T = T_1$ or $T = T_2$.*

Proof. Suppose that

$$H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}.$$

Since H meets PT symmetry, if $T = T_1$, then $PT_1H = HPT_1$, hence

$$PT_1HT_1 = HPT_1^2 = HP. \tag{3}$$

For any $(x, y)^t \in \mathbb{C}^2$, we have

$$T_1HT_1 = \begin{pmatrix} x \\ y \end{pmatrix} = T_1\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = T_1\begin{pmatrix} a\overline{x} + b\overline{y} \\ c\overline{x} + d\overline{y} \end{pmatrix} = \begin{pmatrix} \overline{a}x + \overline{b}y \\ \overline{c}x + \overline{d}y \end{pmatrix}$$

$$= \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \overline{H}\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix},$$

then $T_1H\overline{T_1} = \overline{H}$. From (3), we have

$$P\overline{H} = HP. \tag{4}$$

Similarly, if $T = T_2$, we have

$$PT_2HT_2 = HPT_2^2 = HP. \tag{5}$$

For any $(x, y)^t \in \mathbb{C}^2$, we have

$$T_2HT_2 = \begin{pmatrix} x \\ y \end{pmatrix} = T_2\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -\overline{x} \\ -\overline{y} \end{pmatrix} = T_2\begin{pmatrix} -(a\overline{x} + b\overline{y}) \\ -(c\overline{x} + d\overline{y}) \end{pmatrix} = \begin{pmatrix} \overline{a}x + \overline{b}y \\ \overline{c}x + \overline{d}y \end{pmatrix}$$

$$= \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \overline{H}\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}.$$

Hence, $T_2HT_2 = \overline{H}$, then from (5), we have $P\overline{H} = HP$.

Theorem 2. *In finite dimensional space, any operator P, which is commutate to operator T, is a real matrix.*

Proof. Since any linear transformation can be presented by square matrix, we assume that

$$P = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad a_{ij} \in \mathbb{C}. \tag{6}$$

For operator T_1, we have

$$PT_1 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = P \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{pmatrix} = \begin{pmatrix} a_{11}\bar{x}_1 + a_{12}\bar{x}_2 + \cdots + a_{1n}\bar{x}_n \\ a_{21}\bar{x}_1 + a_{22}\bar{x}_2 + \cdots + a_{2n}\bar{x}_n \\ \vdots \\ a_{n1}\bar{x}_1 + a_{n2}\bar{x}_2 + \cdots + a_{nn}\bar{x}_n \end{pmatrix}, \tag{7}$$

and

$$T_1 P \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \overline{a_{11}x_1} + \overline{a_{12}x_2} + \cdots + \overline{a_{1n}x_n} \\ \overline{a_{21}x_1} + \overline{a_{22}x_2} + \cdots + \overline{a_{2n}x_n} \\ \vdots \\ \overline{a_{n1}x_1} + \overline{a_{n2}x_2} + \cdots + \overline{a_{nn}x_n} \end{pmatrix}. \tag{8}$$

Note that $PT_1 - T_1P = 0$, from (7) and (8), we have

$$a_{ij} = \bar{a}_{ij}, \quad i, j = 1, 2, \cdots, n.$$

Hence, operator P is a real matrix.

For operator T_2, we have

$$PT_2\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = P\begin{pmatrix} -\bar{x}_1 \\ -\bar{x}_2 \\ \vdots \\ -\bar{x}_n \end{pmatrix} = \begin{pmatrix} -a_{11}\bar{x}_1 - a_{12}\bar{x}_2 - \cdots - a_{1n}\bar{x}_n \\ -a_{21}\bar{x}_1 - a_{22}\bar{x}_2 - \cdots - a_{2n}\bar{x}_n \\ \vdots \\ -a_{n1}\bar{x}_1 - a_{n2}\bar{x}_2 - \cdots - a_{nn}\bar{x}_n \end{pmatrix}, \tag{9}$$

and

$$T_2P\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -\overline{a_{11}x_1} - \overline{a_{12}x_2} - \cdots - \overline{a_{1n}x_n} \\ -\overline{a_{21}x_1} - \overline{a_{22}x_2} - \cdots - \overline{a_{2n}x_n} \\ \vdots \\ -\overline{a_{n1}x_1} - \overline{a_{n2}x_2} - \cdots + \overline{a_{nn}x_n} \end{pmatrix}. \tag{10}$$

Note that $PT_2 - T_2P = 0$, so by (9) and (10), we have

$$a_{ij} = \bar{a}_{ij}, \quad i, j = 1, 2, \cdots, n.$$

So the operator P is a real matrix.

3.2. Operator P in 2×2 quantum system

From Theorem 2, we know that any operator P in 2×2 and 3×3 system is a real matrix, we may assume that the general form of operator P in 2×2 system is as follows:

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}. \tag{11}$$

Since operator P satisfies $P^2 = I$, we have

$$P^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{12}$$

so

$$\begin{cases} a^2 + bc = 1, \\ b(a+d) = 0, \\ c(a+d) = 0, \\ d^2 + bc = 1. \end{cases} \tag{13}$$

If $b = 0$, then $a^2 = d^2 = 1$, we have

$$P = \begin{pmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{pmatrix} \quad \text{or} \quad P = \begin{pmatrix} \pm 1 & 0 \\ c & \pm 1 \end{pmatrix}, \quad c \in \mathbb{R}. \tag{14}$$

If $b \neq 0$, then $a = -d$, we have

$$P = \begin{pmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{pmatrix}, \quad a, b \in \mathbb{R}. \tag{15}$$

So the concrete forms of operator P are (14) and (15) in 2×2 system. In particular, if operator P is a real symmetric matrix, then it has the following expressions:

$$P = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}, \quad \alpha \in \mathbb{R}. \tag{16}$$

3.3. Operator *P* in 3×3 quantum system

We can also discuss the operator P in 3×3 system using a similar way of 3.2. If the operator P in 3×3 system is the following real symmetric matrix:

$$P = \begin{pmatrix} a & b & c \\ b & e & f \\ c & f & g \end{pmatrix}, \quad a, b, c, e, f, g \in \mathbb{R}. \tag{17}$$

For $P^2 = I,$ we have the following equations:

$$\begin{cases} a^2 + b^2 + c^2 = 1, \\ b^2 + e^2 + f^2 = 0, \\ c^2 + f^2 + g^2 = 0, \\ ab + be + cf = 0, \\ ac + bf + cg = 0, \\ bc + fe + gf = 0. \end{cases} \tag{18}$$

The above Equations (18) is a six-member quadratic equation group, which calculation is complex. With the aid of software program, the results of Equations (18) can be calculated. The following is the firstly three solutions we select as an example:

$$P_1 = \begin{pmatrix} -z & \sqrt{1-z^2} & 0 \\ \sqrt{1-z^2} & z & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{19}$$

$$P_2 = \begin{pmatrix} -z & -\sqrt{1-z^2} & 0 \\ -\sqrt{1-z^2} & z & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{20}$$

$$P_3 = \begin{pmatrix} -z & \sqrt{1-z^2} & 0 \\ \sqrt{1-z^2} & z & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{21}$$

where $|z| \leq 1$ and $z \in \mathbb{R}$.

4. Conclusion

We first discuss the general forms of operators P and T according to their definitions, we then classify the operator T into two types. Next we prove that $P\overline{H} = HP$ no matter the types of operator T in finite dimensional space with the Hamiltonian H meets PT symmetry for the same operator P. Finally, we conclude that the operator P commutate to operator T must be a real matrix, and we present the concrete forms of operator P in 2×2 and 3×3 quantum systems.

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