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SOME WEAKER FORMS OF CONTINUITY IN BITOPOLOGICAL ORDERED SPACES

A. F. SAYED

Department of Mathematics
Al-Lith University College
Umm Al-Qura University
P. O. Box 112, Al-Lith 21961
Makkah Al Mukarramah
Kingdom of Saudi Arabia
e-mail: dr.afsayed@hotmail.com

Abstract

The main purpose of the present paper is to introduce and study some weaker forms of continuity in bitopological ordered spaces. Such as pairwise *I*-continuous maps, pairwise *D*-continuous maps, pairwise *B*-continuous maps, pairwise *I*-open maps, pairwise *D*-open maps, pairwise *B*-open maps, pairwise *I*-closed maps, pairwise *D*-closed maps, and pairwise *B*-closed maps.

1. Introduction

Singal and Singal [4] initiated the study of bitopological ordered spaces. Raghavan ([2], [3]) and other authors have contributed to development and construct some properties of such spaces. In 2002, Veera Kumar [5] introduced *I*-continuous maps, *D*-continuous maps,

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B-continuous maps, I-open maps, D-open maps, B-open maps, I-closed maps, D-closed maps, and B-closed maps for topological ordered spaces together with their characterizations. Nachbin [1] initiated the study of topological ordered spaces in 1965. A topological ordered space is a triple $(X, \tau, \leq)$, where τ is a topology on X and $\leq$ is a partial order on X. In this paper, we introduce pairwise I-continuous maps, pairwise D-continuous maps, pairwise B-continuous maps, pairwise I-open maps, pairwise D-open maps, pairwise B-open maps, pairwise I-closed maps, pairwise D-closed maps, and pairwise B-closed maps for bitopological ordered spaces together with their characterizations as a generalization of that were studied for topological ordered spaces by Veera Kumar [5].

2. Preliminaries

Let $(X, \leq)$ be a partially ordered set (i.e., a set X together with a reflexive, antisymmetric, and transitive relation). For a subset $A \subseteq X$, we write

$$L(A) = \{y \in X : y \leq x \text{ for some } x \in A\},$$

$$M(A) = \{y \in X : x \leq y \text{ for some } x \in A\}.$$

In particular, if A is a singleton set, say $\{x\}$, then we write $L(x)$ and $M(x)$, respectively. A subset A of X is said to be decreasing (resp., increasing) if $A = L(A)$ (resp., $A = M(A)$). The complement of a decreasing (resp., an increasing) set is an increasing (resp., a decreasing) set. A mapping $f : (X, \leq) \to (X^*, \leq^*)$ from a partially ordered set $(X, \leq)$ to a partially ordered set $(X^*, \leq^*)$ is increasing (resp., a decreasing) if $x \leq y$ in X implies $f(x) \leq^* f(y)$ (resp., $f(y) \leq^* f(x)$), where f is called an order isomorphism if it is an increasing bijection such that f^{-1} is also increasing.

A bitopological ordered space [4] is a quadruple consisting of a bitopological space (X, τ_1, τ_2), and a partial order $\leq$ on X; it is denoted as $(X, \tau_1, \tau_2, \leq)$. The partial order $\leq$ said to be closed (resp., weakly closed) [2] if its graph $G(\leq) = \{(x, y) : x \leq y\}$ is closed in the product topology $\tau_i \times \tau_j$ (resp., $\tau_1 \times \tau_2$), where $i, j = 1, 2;\ i \neq j$, or equivalently, if $L(x)$ and $M(x)$ are τ_1-closed, where $i = 1, 2$ (resp., $L(x)$ is τ_1-closed and $M(x)$ is τ_2-closed), for each $x \in X$.

For a subset A of a bitopological ordered space $(X, \tau_1, \tau_2, \leq)$,

$H_i^l(A) = \bigcap\{F \mid F$ is τ_i-decreasing closed subset of X containing $A\}$,

$H_i^m(A) = \bigcap\{F \mid F$ is τ_i-increasing closed subset of X containing $A\}$,

$H_i^b(A) = \bigcap\{F \mid F$ is a closed subset of X containing A with $F = L(F) = M(F)\}$,

$O_i^l(A) = \bigcup\{G \mid G$ is τ_i-decreasing open subset of X contained in $A\}$,

$O_i^m(A) = \bigcup\{G \mid G$ is τ_i-increasing open subset of X contained in $A\}$,

$O_i^b(A) = \bigcup\{G \mid G$ is both τ_i-increasing and τ_i-decreasing open subset of X contained in $A\}$.

Clearly, $H_i^m(A)$ (resp., $H_i^l(A)$, $H_i^b(A)$) is the smallest τ_i-increasing (resp., τ_i-decreasing, both τ_i-increasing and τ_i-decreasing) closed set containing A. Moreover $\overline{A}_i \subseteq H_i^m(A) \subseteq H_i^b(A)$, where $\overline{A}_i$ stands for the τ_i-closure of A in $(X, \tau_1, \tau_2, \leq)$, $i = 1, 2$. Further A is τ_i-decreasing (resp., τ_i-increasing) closed if and only if $A = H_i^m(A) = H_i^l(A)$.

Clearly, $O_i^m(A)$ (resp., $O_i^l(A)$, $O_i^b(A)$) is the largest τ_i-increasing (resp., τ_i-decreasing, both τ_i-increasing and τ_i-decreasing) open set contained in A. Moreover $O_i^b(A) \subseteq O_i^m(A) \subseteq A_i^o$ and $O_i^b(A) \subseteq O_i^l(A)$, where A_i^o denotes the τ_i-interior of A in $(X, \tau_1, \tau_2, \leq)$, $i \neq j$. If A and B are two τ_1 subsets of a bitopological ordered space $(X, \tau_1, \tau_2, \leq)$, $i \neq j$ such that $A \subseteq B$, then $O_i^m(A) \subseteq O_i^m(B) \subseteq B_i^o$. $\Omega(O_i^m(X))$ (resp., $\Omega(O_i^l(X))$, $\Omega(O_i^b(X))$) denotes the collection of all τ_i-increasing (resp., τ_i-decreasing, both τ_i-increasing and τ_i-decreasing) open subset of a bitopological ordered space $(X, \tau_1, \tau_2, \leq)$.

3. Pairwise *I*-continuous, Pairwise *D*-continuous and Pairwise *B*-continuous Maps

Definition 3.1. A function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq)$ is called a pairwise *I*-continuous (resp., a pairwise *D*-continuous, a pairwise *B*-continuous) map if $f^{-1}(G) \in \Omega(O_i^m(X))$ (resp., $f^{-1}(G) \in \Omega(O_i^l(X))$, $f^{-1}(G) \in \Omega(O_i^b(X))$), whenever G is an i-open subset of $(X^*, \tau_1^*, \tau_2^*, \leq)$, $i = 1, 2$.

It is evident that every pairwise x-continuous map is pairwise continuous for $x = I, D, B$ and that every pairwise *B*-continuous map is both pairwise *I*-continuous and pairwise *D*-continuous.

Example 3.2. Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $\tau_2 = \{\emptyset, X, \{c\}\}$ and $\leq = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$. Clearly, $(X, \tau_1, \tau_2, \leq)$ is a bitopological ordered space. Let f be the identity map from $(X, \tau_1, \tau_2, \leq)$ onto itself. $\{b\}$ is τ_1-open and $\{c\}$ is τ_2-open, but $f^{-1}(\{b\}) = \{b\}$ is neither a τ_1-increasing nor a τ_1-decreasing open set and also $f^{-1}(\{c\}) = \{c\}$ is neither a τ_2-increasing nor a τ_2-decreasing open set. Thus f is not pairwise x-continuous for $x = I, D, B$. However f is continuous.

The following example supports that a pairwise D-continuous map need not be a pairwise B-continuous map.

Example 3.3. Let $X = \{a, b, c\} = X^*$, $\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} = \tau_1^*$, $\tau_2 = \{\emptyset, X, \{c\}\} = \tau_2^*$, $\leq = \{(a, a), (b, b), (c, c), (a, c)\}$ and $\leq^* = \{(a, a), (b, b), (c, c), (a, b), (a, c), (b, c)\}$. Let g be the identity map from $(X, \tau_1, \tau_2, \leq)$ onto $(X^*, \tau_1^*, \tau_2^*, \leq)$, g is not pairwise B-continuous. However g is a pairwise D-continuous map.

The following example supports that a pairwise I-continuous map need not be a pairwise B-continuous map.

Example 3.4. Let $X = \{a, b, c\} = X^*$, $\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $\tau_1^* = \{\emptyset, X^*, \{a\}\}$, $\tau_2 = \{\emptyset, X, \{c\}\}$, $\tau_2^* = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$ and $\leq = \{(a, a), (b, b), (c, c), (a, b), (a, c), (c, b)\} = \leq^*$. Define $h : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq)$ by $h(a) = b$, $h(b) = a$, and $h(c) = c$, where h is pairwise I-continuous but not a pairwise B-continuous map.

Thus we have the following diagram:

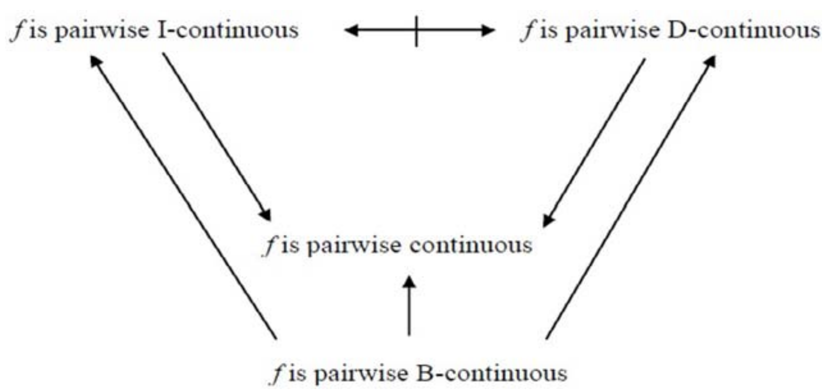

Figure 1.

For a function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq)$, where $P \to Q$ (resp., $P \not\leftrightarrow Q$) represents P implies Q but Q need not imply P (resp., P and Q are independent of each other).

The following theorem characterizes pairwise I-continuous maps.

Theorem 3.5. *For a function* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq)$, *the following statements are equivalent*:

(1) *f is pairwise I-continuous.*

(2) $f(H_i^l(A)) \subseteq \overline{(f(A))_i}$ *for any* $A \subseteq X$, $\quad i = 1, 2$.

(3) $H_i^l(f^{-1}(B)) \subseteq f^{-1}(\overline{B})_i$ *for any* $B \subseteq X^*$, $\quad i = 1, 2$.

(4) *For every* τ_i^**-closed subset* K *of* $(X^*, \tau_1^*, \tau_2^*, \leq)$, $f^{-1}(K)$ *is a* τ_i*-decreasing closed subset of* $(X, \tau_1, \tau_2, \leq)$, $i = 1, 2$.

Proof. (1) $\Rightarrow$ (2): Since $X^* \setminus \overline{(f(A))_i}$ is τ_i-open in X^* and f is pairwise I-continuous, then $f^{-1}(X \setminus \overline{(f(A))_i})$ is a τ_i-increasing open set in X. Then $X \setminus f^{-1}(X \setminus \overline{(f(A))_i})$ is a τ_i-decreasing closed subset of X. Since $X \setminus f^{-1}(X \setminus \overline{(f(A))_i}) = f^{-1}(\overline{(f(A))_i})$, then $f^{-1}(\overline{(f(A))_i})$ is a τ_i-decreasing closed subset of X. Since $A \subseteq f^{-1}(\overline{(f(A))_i})$ and is the smallest τ_i-decreasing closed set containing A, then $H_i^l(A) \subseteq f^{-1}(\overline{(f(A))_i})$. $f(f^{-1}(\overline{(f(A))_i}) \subseteq \overline{(f(A))_i}$. Thus $H_i^l(A) \subseteq \overline{(f(A))_i}$.

(2) $\Rightarrow$ (3): Let $A = f^{-1}(B)$. Then $f(A) = f(f^{-1}(B)) \subseteq B$. This implies $(\overline{f(A)})_i \overline{B}_i$. Now $H_i^l(f^{-1}(B)) \subseteq H_i^l(A) \subseteq f^{-1}(f(H_i^l(A))) \subseteq f^{-1}\,(\overline{f(A)})_i$ [by (2) in this Theorem 3.5]. But $f^{-1}(\overline{f(A)})_i \subseteq f^{-1}(\overline{B}_i)$. Thus $H_i^l(f^{-1}(B)) \subseteq f^{-1}(\overline{B}_i)$.

(3) $\Rightarrow$ (4): $H_i^l(f^{-1}(K)) \subseteq f^{-1}(\overline{K}_i)$ for any τ_i^*-closed set K of $(X^*, \tau_1^*, \tau_2^*, \leq)$. Thus $f^{-1}(K)$ is a τ_i-decreasing closed in $(X, \tau_1, \tau_2, \leq)$, whenever K is a τ_i^*-closed set in $(X^*, \tau_1^*, \tau_2^*, \leq)$.

(4) $\Rightarrow$ (1): Let G be a τ_i^*-open set in $(X^*, \tau_1^*, \tau_2^*, \leq)$. Then $f^{-1}(X \setminus (G))$ is a τ_i-decreasing closed set in $(X, \tau_1, \tau_2, \leq)$, since $X^* \setminus (G)$ is a closed set in $(X^*, \tau_1^*, \tau_2^*, \leq)$. But $X \setminus (f^{-1}(G)) = f^{-1}(X \setminus G)$. Thus $X \setminus (f^{-1}(G))$ is a τ_i-decreasing closed set in $(X, \tau_1, \tau_2, \leq)$. So $f^{-1}(G)$ is a τ_i-increasing open set in $(X, \tau_1, \tau_2, \leq)$. Thus f is pairwise I-continuous. □

The following two theorems characterize pairwise D-continuous maps and pairwise B-continuous maps, whose proofs are similar to as that of the above Theorem 3.5.

Theorem 3.6. *For a function* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq)$, *the following statements are equivalent*:

(1) *f is pairwise D-continuous.*

(2) $f(H_i^m(A)) \subseteq \overline{(f(A))}_i$ *for any* $A \subseteq X$, $i = 1, 2$.

(3) $H_i^m(f^{-1}(B)) \subseteq f^{-1}(\overline{B})_i$ *for any* $B \subseteq X^*$, $i = 1, 2$.

(4) *For every* τ_i^**-closed subset* K *of* $(X^*, \tau_1^*, \tau_2^*, \leq)$, $f^{-1}(K)$ *is a* τ_i*-increasing closed subset of* $(X, \tau_1, \tau_2, \leq)$, $i = 1, 2$.

Theorem 3.7. *For a function* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq)$, *the following statements are equivalent*:

(1) *f is pairwise B-continuous.*

(2) $f(H_i^b(A)) \subseteq \overline{(f(A))}_i$ *for any* $A \subseteq X$, $i = 1, 2$.

(3) $H_i^b(f^{-1}(B)) \subseteq f^{-1}(\overline{B})_i$ *for any* $B \subseteq X^*$, $i = 1, 2$.

(4) *For every* τ_i^**-closed subset* K *of* $(X^*, \tau_1^*, \tau_2^*, \leq)$, $f^{-1}(K)$ *is both* τ_i*-increasing and* τ_i*-decreasing closed subset of* $(X, \tau_1, \tau_2, \leq)$, $i = 1, 2$.

Theorem 3.8. *Let* $f : (X, \tau_1, \tau_2, \leq_1) \to (y, \nu_1, \nu_2, \leq_2)$ *and* $g : (y, \nu_1, \nu_2, \leq_2) \to (Z, \eta_1, \eta_2, \leq_3)$ *be any two mappings. Then*

(1) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ *is pairwise x-continuous for* $x = I, D, B$.

(2) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ *is pairwise x-continuous and g is pairwise continuous for* $x = I, D, B$.

(3) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ *is pairwise x-continuous and g is pairwise y-continuous for* $x, y \in \{I, D, B\}$.

4. Pairwise *I*-open, Pairwise *D*-open and Pairwise *B*-open Maps

Definition 4.1. A function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ is called a pairwise *I*-open (resp., a pairwise *D*-open, a pairwise *B*-open) map if $f(G) \in \Omega(O_i^m(X^*))$ (resp., $f(G) \in \Omega(O_i^l(X^*))$, $f(G) \in \Omega(O_i^b(X^*))$), whenever G is a τ_i-open subset of (X, τ_1, τ_2), $i = 1, 2$.

It is evident that every pairwise x-open map is a pairwise open map for $x = I, D, B$ and that every pairwise B-open map is both pairwise I-open and pairwise D-open.

The following example shows that a pairwise open map need not be pairwise x-open for $x = I, D, B$.

Example 4.2. Let $(X, \tau_1, \tau_2, \leq)$ and f be as in the Example 3.2, f is a pairwise open map but f is not pairwise x-open for $x = I, D, B$.

The following example shows that a pairwise D-open map need not be a pairwise B-open map.

Example 4.3. Let $X, X^*, \tau_1, \tau_2, \tau_1^*, \tau_2^* \leq$ and $\leq^*$ be as in the Example 3.3. Let θ be the identity map from $(X, \tau_1, \tau_2, \leq)$ onto $(X^*, \tau_1^*, \tau_2^*, \leq^*)$, θ is pairwise D-open but not a pairwise B-open map.

The following example shows that a pairwise I-open map need not be a pairwise B-open map.

Example 4.4. Let $X, X^*, \tau_1, \tau_2, \tau_1^*, \tau_2^*, \leq$ and $\leq^*$ be as in the Example 3.4. Define $\varphi : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ by $\varphi(a) = b$, $\varphi(b) = a$, and $\varphi(c) = c$, φ is a pairwise I-open map but not a pairwise B-open map.

Thus we have the following diagram:

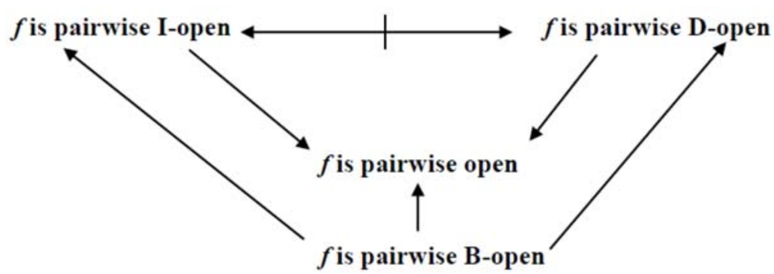

Figure 2.

For a function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$, where $P \to Q$ (resp., $P \not\leftrightarrow Q$) represents P implies Q but Q need not imply P (resp., P and Q are independent of each other).

Before characterizing pairwise I-open (resp., pairwise D-open, pairwise B-open) maps, we establish the following useful lemma:

Lemma 4.5. *Let A be any subset of a bitopological ordered space $(X, \tau_1, \tau_2, \leq)$. Then*

(1) $X \setminus H_i^l(A) = O_i^m(X \setminus A), \quad i = 1, 2.$

(2) $X \setminus H_i^m(A) = O_i^l(X \setminus A), \quad i = 1, 2.$

(3) $X \setminus H_i^b(A) = O_i^b(X \setminus A), \quad i = 1, 2.$

Proof. (1) $X \setminus H_i^l(A) = X \setminus \bigcap\{F | F$ is a τ_i-decreasing closed subset of X containing $A\} = \bigcup\{X \setminus F | F$ is a τ_i-decreasing closed subset of X containing $A\} = \bigcup\{G | G$ is a τ_i-increasing open subset of X contained in $X \setminus A\} = O_i^m(X \setminus A)$.

The proofs for (2) and (3) are analogous to that of (1) and so omitted.

□

The following theorem characterizes pairwise I-open functions.

Theorem 4.6. *For any function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq)$, the following statements are equivalent*:

(1) *f is a pairwise I-open map.*

(2) $f(A_i^o) \subseteq O_i^m(f(A))$ *for any* $A \subseteq X$, $i = 1, 2$.

(3) $(f^{-1}(B))_i^o \subseteq f^{-1}(O_i^m(B))$ *for any* $B \subseteq X^*$, $i = 1, 2$.

(4) $f^{-1}(H_i^l(B)) \subseteq H_i^l(f^{-1}(B))$ *for any* $B \subseteq X^*$, $i = 1, 2$.

Proof. (1) $\Rightarrow$ (3): Since $(f^{-1}(B))_i^o$ is τ_i-open in X and f is pairwise I-open, then $f((f^{-1}(B))_i^o)$ is a τ_i-increasing open set in X^*. Also $f(f^{-1}(B))_i^o \subseteq f(f^{-1}(B)) \subseteq B$. Then $f(f^{-1}(B))_i^o \subseteq O_i^m(B)$ since $O_i^m(B)$ is the largest τ_i-increasing open set contained in B. Therefore $(f^{-1}(B))_i^o \subseteq f^{-1}(O_i^m(B))$.

(3) $\Rightarrow$ (4): Replacing B by $X \setminus B$ in (3), we get $(f^{-1}(X \setminus B))_i^o \subseteq f^{-1}(O_i^m(X \setminus B))$. Since $f^{-1}(X \setminus B) = X \setminus (f^{-1}(B))$, then $(X \setminus (f^{-1}(B)))_i^o \subseteq f^{-1}(O_i^m(X \setminus B))$. Now $X \setminus (H_i^l(f^{-1}(B))) = O_i^m(X \setminus (f^{-1}(B))) \subseteq (X \setminus (f^{-1}(B)))_i^o \subseteq f^{-1}(O_i^m(X \setminus (B))) = f^{-1}(X \setminus (H_i^l(B))) = X \setminus (f^{-1}(H_i^l(B)))$ using the above Lemma 4.5. Therefore $f^{-1}(H_i^l(B)) \subseteq H_i^l(f^{-1}(B))$.

(4) $\Rightarrow$ (3): All the steps in (3) $\Rightarrow$ (4) are reversible.

(3) $\Rightarrow$ (2): Replacing B by $f(A)$ in (3), we get $(f^{-1}(f(A)))_i^o \subseteq f^{-1}(O_i^m(f(A)))$. Since $A_i^o \subseteq (f^{-1}(f(A)))_i^o$, then we have $A_i^o \subseteq f^{-1}(O_i^m(f(A)))$. This implies that $f(A_i^o) \subseteq f(f^{-1}(O_i^m(f(A)))) \subseteq O_i^m(f(A))$. Hence $f(A_i^o) \subseteq O_i^m(f(A))$.

(2) $\Rightarrow$ (1): Let G be any τ_i-open subset of X. Then $f(G) = f(G_i^o) \subseteq O_i^m(f(G))$. So $f(G)$ is a τ_i^*-increasing open set in X^*. Therefore f is a pairwise I-open map. □

The following two theorems give characterizations for D-open maps and B-open maps, whose proofs are similar to as that of the above Theorem 4.6.

Theorem 4.7. *For any function* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq)$, *the following statements are equivalent*:

(1) *f is a pairwise D-open map*.

(2) $f(A_i^o) \subseteq O_i^l(f(A))$ *for any* $A \subseteq X$, $i = 1, 2$.

(3) $(f^{-1}(B))_i^o \subseteq f^{-1}(O_i^l(B))$ *for any* $B \subseteq X^*$, $i = 1, 2$.

(4) $f^{-1}(H_i^m(B)) \subseteq H_i^m(f^{-1}(B))$ *for any* $B \subseteq X^*$, $i = 1, 2$.

Theorem 4.8. *For any function* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq)$, *the following statements are equivalent*:

(1) *f is a pairwise B-open map*.

(2) $f(A_i^o) \subseteq O_i^b(f(A))$ *for any* $A \subseteq X$, $i = 1, 2$.

(3) $(f^{-1}(B))_i^o \subseteq f^{-1}(O_i^b(B))$ *for any* $B \subseteq X^*$, $i = 1, 2$.

(4) $f^{-1}(H_i^b(B)) \subseteq H_i^b(f^{-1}(B))$ *for any* $B \subseteq X^*$, $i = 1, 2$.

Theorem 4.9. *Let* $f : (X, \tau_1, \tau_2, \leq_1) \to (y, \nu_1, \nu_2, \leq_2)$ *and* $g : (y, \nu_1, \nu_2, \leq_2) \to (Z, \eta_1, \eta_2, \leq_3)$ *be any two mappings. Then*

(1) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ *is pairwise x-open if f is pairwise open and g is pairwise x-open for* $x = I, D, B$.

(2) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ *is pairwise x-open if both f and g are pairwise x-open for* $x = I, D, B$.

(3) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ *is pairwise x-open if f is pairwise y-open and g is pairwise x-open for* $x, y \in \{I, D, B\}$.

Proof. Omitted. □

5. Pairwise *I*-closed, Pairwise *D*-closed and Pairwise *B*-closed Maps

Definition 5.1. A function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ is called a pairwise *I*-closed (resp., a pairwise *D*-closed, a pairwise *B*-closed) map if $f(G) \in \Omega(H_i^m(X^*))$(resp., $f(G) \in \Omega(H_i^l(X^*))$, $f(G) \in \Omega(H_i^b(X^*))$), whenever G is a τ_i-open subset of (X, τ_1, τ_2), where $\Omega(H_i^m(X^*))$ (resp., COmega$(H_i^l(X^*))$, $\Omega(H_i^b(X^*))$ is the collection of all τ_i-increasing (resp., τ_i-decreasing, both τ_i-increasing and τ_i-decreasing) closed subsets of $(X^*, \tau_1^*, \tau_2^*, \leq^*)$, $i = 1, 2$.

Clearly, every pairwise x-closed map is a pairwise closed map for $x = I, D, B$ and every pairwise *B*-closed map is both pairwise *I*-closed and pairwise *D*-closed. The following example shows that a pairwise closed map need not be pairwise x-closed for $x = I, D, B$.

Example 5.2. Let $(X, \tau_1, \tau_2, \leq)$ and *f* be as in the Example 3.2, *f* is a pairwise closed map but *f* is not pairwise x-closed for $x = I, D, B$.

The following example shows that a pairwise *I*-closed map need not be a pairwise *B*-closed map.

Example 5.3. Let $X, X^*, \tau_1, \tau_2, \tau_1^*, \tau_2^*, \leq$ and $\leq^*$ be as in the Example 4.3, θ is pairwise *I*-closed but not a pairwise *B*-closed map.

The following example shows that a pairwise *I*-closed map need not be a pairwise *B*-closed map.

Example 5.4. Let $X, X^*, \tau_1, \tau_2, \tau_1^*, \tau_2^*, \leq, \leq^*$ and φ be as in the Example 4.4, φ is a pairwise D-closed map but not a pairwise B-closed map.

Thus we have the following diagram:

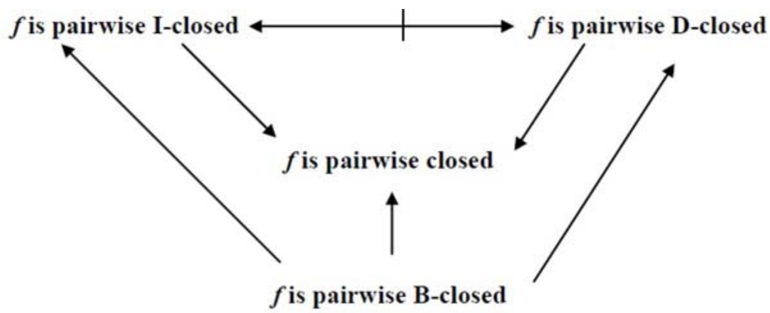

Figure 3.

For a function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$, where $P \to Q$ (resp., $P \nleftrightarrow Q$) represents P implies Q but Q need not imply P (resp., P and Q are independent of each other).

The following theorem characterizes I-closed maps.

Theorem 5.5. *Let $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ be any map. Then f is pairwise I-closed if and only if $H_i^m(f(A)) \subseteq f(\overline{A}_i)$ for every $A \subseteq X$, $i = 1, 2$.*

Proof. Necessity: Since f is pairwise I-closed, then $f(\overline{A}_i)$ is a τ_i-increasing closed subset of X and $f(A) \subseteq f(\overline{A}_i)$. Therefore $H_i^m(f(A)) \subseteq f(\overline{A}_i)$ since $H_i^m(f(A))$ is the smallest τ_i-increasing closed set in X^* containing $f(A)$.

Sufficiency: Let F be any τ_i-closed subset of X. Then $f(F) \subseteq H_i^m(f(F)) \subseteq f(\overline{F}_i) = f(F)$. Thus $f(F) = H_i^m(f(F))$. So $f(F)$ is a τ_i-increasing closed subset of X^*. Therefore f is a pairwise I-closed map.

□

The following two theorems characterize pairwise D-closed maps and pairwise B-closed maps.

Theorem 5.6. *Let* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ *be any map. Then* f *is pairwise D-closed if and only if* $H_i^l(f(A)) \subseteq f(\overline{A}_i)$ *for every* $A \subseteq X$, $i = 1, 2$.

Proof. Omitted. □

Theorem 5.7. *Let* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ *be any map. Then* f *is pairwise B-closed if and only if* $H_i^b(f(A)) \subseteq f(\overline{A}_i)$ *for every* $A \subseteq X$, $i = 1, 2$.

Proof. Omitted. □

Theorem 5.8. *Let* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ *be a pairwise bijection map. Then*

(1) *f is pairwise I-open if and only if f is pairwise D-closed.*

(2) *f is pairwise I-closed if and only if f is pairwise D-open.*

(3) *f is pairwise B-open if and only if f is pairwise B-closed.*

Proof. (1) Necessity: Let F be any τ_i-closed subset of X. Then $f(X \setminus F)$ is a τ_i^*-increasing open subset of X^* since f is a pairwise I-open map and $(X \setminus F)$ is a τ_i-open subset of X. Since f is a pairwise bijection, then we have $f(X \setminus F) = X \setminus (f(F))$. So $f(F)$ is a τ_i^*-decreasing closed subset of X^*. Therefore f is a pairwise D-closed.

Sufficiency: Let G be any τ_i-open subset of X. Then $f(X \setminus G)$ is a τ_i-decreasing closed subset of X^* since f is a pairwise D-closed map and $(X \setminus G)$ is a τ_i-closed subset of X. Since f is a pairwise bijection, then we have that $f(X \setminus G) = X \setminus f(G)$. So $f(G)$ is a τ_i-increasing open subset of X^*. Therefore f is a pairwise I-open map.

The proofs for (2) and (3) are similar to that of (1). □

Theorem 5.9. *Let* $f : (X, \tau_1, \tau_2, \leq_1) \to (y, \nu_1, \nu_2, \leq_2)$ *and* $g : (y, \nu_1, \nu_2, \leq_2) \to (Z, \eta_1, \eta_2, \leq_3)$ *be any two mappings. Then*

(1) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ *is pairwise x-closed if f is pairwise closed and g is pairwise x-closed for* $x = I, D, B$.

(2) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ *is pairwise x-closed if both f and g are pairwise x-closed for* $x = I, D, B$.

(3) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ *is pairwise x-closed if f is pairwise y-closed and g is pairwise x-closed for* $x, y \in \{I, D, B\}$.

Theorem 5.10. *Let* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ *be a pairwise bijection map. Then the following statements are equivalent*:

(1) *f is a pairwise I-open map.*

(2) *f is a pairwise D-closed map.*

(3) f^{-1} *is a pairwise I-continuous.*

Theorem 5.11. *Let* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ *be a pairwise bijection map. Then the following statements are equivalent*:

(1) *f is a pairwise D-open map.*

(2) *f is a pairwise I-closed map.*

(3) f^{-1} *is a pairwise D-continuous.*

Theorem 5.12. *Let* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ *be a pairwise bijection map. Then the following statements are equivalent*:

(1) *f is a pairwise B-open map.*

(2) *f is a pairwise B-closed map.*

(3) f^{-1} *is a pairwise B-continuous.*

Theorem 5.13. *Let* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ *be a pairwise I-closed map and* $B, C \subseteq X^*$. *Then*

(1) *If U is a* τ_i*-open neighbourhood of* $f^{-1}(B)$, *then there exists a* τ_i*-decreasing open neighbourhood V of B such that* $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U$, $i = 1, 2$.

(2) *If* $f^{-1}(B)$ *and* $f^{-1}(C)$ *have disjoint* τ_i*-neighbourhoods, then* $f^{-1}(B)$ *and* $f^{-1}(C)$ *have disjoint* τ_i*-decreasing open neighbourhoods,* $i = 1, 2$.

Proof. (1) Let U be a τ_i-open neighbourhood of $f^{-1}(B)$. Take $X^* \setminus V = f(X \setminus U)$. Since *f* is a pairwise *I*-closed map and $(X \setminus U)$ is a τ_i-closed set, then $X^* \setminus V = f(X \setminus U)$ is a τ_i-increasing closed subset of X^*. Thus *V* is an τ_i-decreasing open subset of X^*. Since $f^{-1}(B)U$, then $X^* \setminus V = f(X \setminus U) \subseteq f(f^{-1}(X^* \setminus B)) \subseteq X^* \setminus B$. So $B \subseteq V$. Thus V is a τ_i^*-decreasing open neighbourhood of B. Further $X \setminus U \subseteq f^{-1}(f(X \setminus U)) = f^{-1}(X^* \setminus V) = X^* \setminus (f^{-1}(V))$. Thus $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U$.

□

Theorem 5.14. *Let* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ *be a pairwise D-closed map and* $B, C \subseteq X^*$. *Then*

(1) *If* U *is a* τ_i*-open neighbourhood of* $f^{-1}(B)$, *then there exists a* τ_i*-decreasing open neighbourhood V of B such that* $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U$, $i = 1, 2$.

(2) *If* $f^{-1}(B)$ *and* $f^{-1}(C)$ *have disjoint* τ_i*-neighbourhoods, then* $f^{-1}(B)$ *and* $f^{-1}(C)$ *have disjoint* τ_i*-increasing open neighbourhoods,* $i = 1, 2$.

Theorem 5.15. *Let* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ *be a pairwise B-closed map and* $B, C \subseteq X^*$. *Then*

(1) *If* U *is a* τ_i*-open neighbourhood of* $f^{-1}(B)$, *then there exists a* τ_i*-open neighbourhood* V *of* B, *which are both* τ_i*-increasing and* τ_i*-decreasing, such that* $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U$, $i = 1, 2$.

(2) *If* $f^{-1}(B)$ *and* $f^{-1}(C)$ *have disjoint* τ_i*-neighbourhoods, then* $f^{-1}(B)$ *and* $f^{-1}(C)$ *have disjoint* τ_i*-open neighbourhoods, which are both* τ_i*-increasing and* τ_i*-decreasing,* $i = 1, 2$.

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