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SHARP MAXIMAL FUNCTION INEQUALITY FOR ITERATED COMMUTATOR RELATED TO SINGULAR INTEGRAL OPERATORS SATISFYING A VARIANT OF HÖRMANDER'S CONDITION

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Abstract

In this paper, we establish the sharp maximal function estimate for the iterated commutator related to some singular integral operators satisfying a variant of Hörmander's condition. As the application, we obtain the boundedness of the iterated commutator on Lebesgue and Morrey spaces.

1. Introduction

As the development of singular integral operators, their commutators have been well studied (see [4], [7], [14]). Let T be the Calderón-Zygmund singular integral operator, a classical result of Coifman et al. (see [4]) states that the commutator $[b, T](f) = T(bf) - bT(f)$ (where $b \in BMO(R^n)$)

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is bounded on $L^p(R^n)$ for $1 < p < \infty$. In [8], some singular integral operators satisfying a variant of Hörmander's condition are introduced, and the boundedness for operators are obtained (see [8], [16]). The main purpose of this paper is to establish the sharp maximal function estimate for the iterated commutator related to the singular integral operators satisfying a variant of Hörmander's condition. As the application, we obtain the boundedness of the iterated commutator on Lebesgue and Morrey spaces.

2. Notations and Results

First let us introduce some notations (see [4], [7], [14]). Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f in R^n, the sharp maximal function f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. We say that f belongs to $BMO(R^n)$ if $M^{\#}(f)$ belongs to $L^\infty(R^n)$ and $\|f\|_{BMO} = \|M^{\#}(f)\|_{L^\infty}$. It has been known that (see [14])

$$\|b - b_{2^k Q}\|_{BMO} \leq Ck\|b\|_{BMO} \text{ for } k \geq 1.$$

Let M be the Hardy-Littlewood maximal operator, that is,

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

We write that $M_p(f)(x) = (M|f|^p)^{1/p}(x)$, for $0 < p < \infty$.

For $b_j \in BMO(R^n)(j = 1, \cdots, m)$, set

$$\|\vec{b}\|_{BMO} = \prod_{j=1}^{m} \|b_j\|_{BMO}.$$

Given some functions b_j $(j = 1, \cdots, m)$ and a positive integer m and $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $\|\vec{b}_\sigma\|_{BMO} = \|b_{\sigma(1)}\|_{BMO} \cdots \|b_{\sigma(j)}\|_{BMO}$.

Definition 1. Let $\Phi = \{\phi_1, \cdots, \phi_m\}$ be a finite family of bounded functions in R^n. For any locally integrable function f, the Φ sharp maximal function of f is defined by

$$M_\Phi^\# = \sup_{Q \ni x} \inf_{\{c_1, \cdots, c_m\}} \frac{1}{|Q|} \int_Q \left| f(y) - \sum_{j=1}^{m} c_j \phi_j (x_Q - y) \right| dy,$$

where the infimum is taken over all m-tuples $\{c_1, \cdots, c_m\}$ of complex numbers and x_Q is the center of Q.

Remark. We note that $M_\Phi^\# \approx M^\#$ if $m = 1$ and $\phi_1 = 1$.

Definition 2. Given a positive and locally integrable function f in R^n, we say that f satisfies the reverse Hörmander's condition (write this as $f \in RH_\infty(R^n)$), if for any cube Q centered at origin we have

$$0 < \sup_{x \in Q} f(x) \leq C \frac{1}{|Q|} \int_Q f(y) dy.$$

In this paper, we will study some integral operators as following (see [8]).

Definition 3. Let $K \in L^2(R^n)$ and satisfy

$$\|K\|_{L^\infty} \le C,$$

$$|K(x)| \le C|x|^{-n},$$

there exist functions $B_1, \cdots, B_m \in L^1_{\text{loc}}(R^n - \{0\})$ and $\Phi = \{\phi_1, \cdots, \phi_m\} \subset L^\infty(R^n)$ such that $|\det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nm})$, and for a fixed $\delta > 0$ and any $|x| > 2|y| > 0$,

$$|K(x-y) - \sum_{j=1}^{m} B_j(x)\phi_j(y)| \le C \frac{|y|^\delta}{|x-y|^{n+\delta}}.$$

For $f \in C_0^\infty$, we define the singular integral operator related to the kernel K by

$$T(f)(x) = \int_{R^n} K(x-y)f(y)dy.$$

Give some locally integrable function $b_j (j = 1, \cdots, m)$. The iterated operator associated to T is defined by

$$T_{\vec{b}}(f)(x) = \int_{R^n} \left[\prod_{j=1}^{m} (b_j(x) - b_j(y)) \right] K(x-y)f(y)dy.$$

Remark. Note that the classical Calderón-Zygmund singular integral operator satisfies Definition 3. And when $b_1, \cdots, b_m$, T_b is just m-order commutator.

Definition 4. Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \le D\varphi(t) \text{ for all } t \ge 0.$$

Let f be an integrable function on R^n. Set, for $1 \le p \le \infty$,

$$\|f\|_{L^{p,\varphi}} = \sup_{x \in R^n, d > 0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} |f(y)|^p dy \right)^{\frac{1}{p}},$$

where $Q(x, d) = \{y \in R^n : |x - y| < d\}$. The generalized Morrey space is defined by

$$L^{p,\varphi}(R^n) = \{f \in L^1_{\text{loc}}(R^n) : \|f\|_{L^{p,\varphi}} < \infty\}.$$

If $\varphi(d) = d^\eta$, $\eta > 0$, then $L^{p,\varphi}(R^n) = L^{p,\eta}(R^n)$, which is the classical Morrey spaces (see [11], [12]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n) = L^p(R^n)$, which is the Lebesgue space.

As Morrey space may be considered as an extension of Lebesgue space, it is natural and important to study the boundedness of operator on Morrey space (see [3], [5], [6], [9], [17]).

It is well-known that commutators are great interest in harmonic analysis and have been widely studied by many authors. In this paper, our main purpose is to establish the sharp maximal inequality for the iterated commutator.

Now we state our theorems as following:

Theorem 1. *Let T be the singular integral operator as Definition* 3 *and $b_j \in BMO(R^n)$ for $j = 1, \cdots, m$. Then for any $1 < r < \infty$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $\tilde{x} \in R^n$,*

$$M_\Phi^\#(T_{\vec{b}}(f))(\tilde{x}) \le C\|\vec{b}\|_{BMO} \left(M_r(f)(\tilde{x}) + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} M_r(T_{\vec{b}_{\sigma^c}}(f))(\tilde{x}) \right).$$

Theorem 2. *Let T be the singular integral operator as Definition* 3 *and $b_j \in BMO(R^n)$ for $j = 1, \cdots, m$. Then $T_{\vec{b}}$ is bounded on $L^p(R^n)$ for any $1 < p < \infty$, that is,*

$$\|T_{\vec{b}}(f)\|_{L^p} \leq C\|f\|_{L^p}.$$

Theorem 3. *Let T be the singular integral operator as Definition* 3 *and $b_j \in BMO(R^n)$ for $j = 1, \cdots, m$, $0 < D < 2^n$. Then $T_{\vec{b}}$ is bounded on $L^{p,\varphi}(R^n)$ for any $1 < p < \infty$, that is,*

$$\|T_{\vec{b}}(f)\|_{L^{p,\varphi}} \leq C\|f\|_{L^{p,\varphi}}.$$

3. Proofs of Theorems

To prove the theorems, we need the following lemmas:

Lemma 1 (see [8]). *Let T be the singular integral operator as Definition* 3 *and $1 < p < \infty$. Then, we have*

$$\|T(f)\|_{L^p} \leq C\|f\|_{L^p}.$$

Lemma 2 (see [14]). *Let $1 < r < \infty$, $b_j \in BMO(R^n)$ for $j = 1, \cdots, k$ and $k \in N$. Then, we have*

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^{k} |b_j(y) - (b_j)_Q| dy \leq C\prod_{j=1}^{k} \|b_j\|_{BMO},$$

and

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^{k} |b_j(y) - (b_j)_Q|^r dy\right)^{1/r} \leq C\prod_{j=1}^{k} \|b_j\|_{BMO}.$$

Lemma 3 (see [8]). *Let $1 < r < \infty$, $w \in A_\infty$, and $\Phi = \{\phi_1, \cdots, \phi_m\} \subset L^\infty(R^n)$ such that $|\det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nm})$. Then*

$$\int_{R^n} M(f)(x)^p w(x) \leq C\int M_\Phi^{\#}(f)(x)^p w(x)dx.$$

Lemma 4. *Let* $1 < r < \infty, 0 < D < 2^n$, *and* $\Phi = \{\phi_1, \cdots, \phi_m\} \subset L^\infty(R^n)$ *such that* $|\det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nm})$. *Then for any smooth function f for which the left-hand side is finite*,

$$\|M(f)\|_{L^{p,\varphi}} \leq C\|M_\Phi^\#(f)\|_{L^{p,\varphi}}.$$

Proof. For any cube $Q = Q(x_0, d)$ in R^n, we know $(M(\chi_Q))^\delta \in A_1$ for any cube $Q = Q(x, d)$ and some $\delta \in (0, 1)$ by [4]. Noticing that $M(\chi_Q) \leq 1$ and $M(\chi_Q)(x) \leq d^n / (|x - x_0| - d)^n$ if $x \in Q^c$, by Lemma 3, we have, for $f \in L^{p,\varphi}(R^n)$,

$$\int_Q M(f)(x)^p dx$$

$$= \int_{R^n} M(f)(x)^p \chi_Q(x) dx$$

$$\leq \int_{R^n} M(f)(x)^p M(\chi_Q)(x) dx$$

$$\leq C\int_{R^n} M_\Phi^\#(f)(x)^p M(\chi_Q)(x) dx$$

$$= C\left(\int_Q M_\Phi^\#(f)(x)^p M(\chi_Q)(x) dx + \sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^k Q} M_\Phi^\#(f)(x)^p M(\chi_Q)(x) dx\right)$$

$$\leq C\left(\int_Q M_\Phi^\#(f)(x)^p dx + \sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^k Q} M_\Phi^\#(f)(x)^p \frac{|Q|}{|2^k Q|} dx\right)$$

$$\leq C\left(\int_Q M_\Phi^\#(f)(x)^p dx + \sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^k Q} M_\Phi^\#(f)(x)^p 2^{-kn} dx\right)$$

$$\leq C\|M_\Phi^\#(f)\|_{L^{p,\varphi}}^p \sum_{k=0}^\infty 2^{-kn} \varphi(2^{k+1}d)$$

$$\leq C\|M_\Phi^\#(f)\|_{L^{p,\varphi}}^p \sum_{k=0}^{\infty}(2^{-n}D)^k\varphi(d)$$

$$\leq C\|M_\Phi^\#(f)\|_{L^{p,\varphi}}^p \varphi(d),$$

thus

$$\left(\frac{1}{\varphi(d)}\int_Q M(f)(x)^p\,dx\right)^{1/p} \leq C\left(\frac{1}{\varphi(d)}\int_Q M_\Phi^\#(f)(x)^p\,dx\right)^{1/p},$$

and

$$\|M(f)\|_{L^{p,\varphi}}^p \leq C\|M_\Phi^\#(f)\|_{L^{p,\varphi}}^p.$$

This finishes the proof.

Lemma 5. *Let T be the singular integral operator as Definition* 3, $1 < r < \infty$, $0 < D < 2^n$. *Then*

$$\|T(f)\|_{L^{p,\varphi}} \leq C\|f\|_{L^{p,\varphi}}.$$

The proof of the Lemma is similar to that of Lemma 4 by Lemma 1, we omit the details.

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q |T_{\vec{b}}(f)(x) - C_0|\,dx \leq C\|\vec{b}\|_{BMO}\left(M_r(f)(\tilde{x}) + \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} M_r(T_{\vec{b}_{\sigma^c}}(f)(\tilde{x}))\right).$$

When $m = 1$, it has been done (see [17], [18]). Now we consider the case $m \geq 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Set $C_0 = \sum_{j=1}^m g_j\phi_j(x_0 - x)$ and $g_j = \int_{R^n} B_j(x_0 - y)\prod_{j=1}^m (b_j(y) - (b_j)_{2Q})f_2(y)dy$. Write for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{2Q^c}$. We have know that, for $b = (b_1, \cdots, b_m)$,

$$T_{\vec{b}}(f)(x) = \int_{R^n}\left[\prod_{j=1}^{m}(b_j(x) - b_j(y))\right]K(x-y)f(y)dy$$

$$= \int_{R^n}\prod_{j=1}^{m}[(b_j(x) - (b_j)_{2Q}) - (b_j(y) - (b_j)_{2Q})]K(x-y)f(y)dy$$

$$= \prod_{j=1}^{m}(b_j(x) - (b_j)_{2Q})\int_{R^n}K(x-y)f(y)dy$$

$$+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x) - (b)_{2Q})_\sigma\int_{R^n}(b(y) - (b)_{2Q})_{\sigma^c}$$

$$\times\ K(x-y)f(y)dy$$

$$+ (-1)^m\int_{R^n}\prod_{j=1}^{m}(b_j(y) - (b_j)_{2Q})K(x-y)f(y)dy$$

$$= (b_1(x) - (b_1)_{2Q})\cdots(b_m(x) - (b_m)_{2Q})T(f)(x)$$

$$+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x) - (b)_{2Q})_\sigma T_{\vec{b}_{\sigma^c}}(f)(x)$$

$$+ (-1)^m T((b_1 - (b_1)_{2Q})\cdots(b_m - (b_m)_{2Q})f)(x),$$

then,

$$|T_{\vec{b}}(f)(x) - C_0|$$

$$= \left|T_{\vec{b}}(f)(x) - \sum_{j=1}^{m}\int_{R^n}B_j(x_0 - y)\phi_j(x_0 - x)\prod_{j=1}^{m}(b_j(y) - (b_j)_{2Q})f_2(y)dy\right|$$

$$\le |(b_1(x) - (b_1)_{2Q})\cdots(b_m(x) - (b_m)_{2Q})T(f)(x)|$$

$$+ \left|\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x) - (b)_{2Q})_\sigma T_{\vec{b}_{\sigma^c}}(f)(x)\right|$$

$$+ |T((b_1 - (b_1)_{2Q})\cdots(b_m - (b_m)_{2Q})f_1)(x)|$$

$$+ |T((b_1 - (b_1)_{2Q})\cdots(b_m - (b_m)_{2Q})f_2)(x) - C_0|$$

$$= I_1(x) + I_2(x) + I_3(x) + I_4(x),$$

thus,

$$\frac{1}{|Q|}\int_Q |T_{\vec{b}}(f)(x) - C_0|dx$$

$$\leq \frac{1}{|Q|}\int_Q I_1(x)dx + \frac{1}{|Q|}\int_Q I_2(x)dx + \frac{1}{|Q|}\int_Q I_3(x)dx + \frac{1}{|Q|}\int_Q I_4(x)dx$$

$$= I_1 + I_2 + I_3 + I_4.$$

For I_1, by Hölder's inequality with exponent $1/p_1 + \cdots + 1/p_m + 1/r = 1$, when $1 < p_i < \infty$, $i = 1, \cdots, m$, we get

$$I_1 = \frac{1}{|Q|}\int_Q |(b_1(x) - (b_1)_{2Q})\cdots(b_m(x) - (b_m)_{2Q})T(f)(x)|dx$$

$$\leq \frac{1}{|Q|}\int_Q |b_1(x) - (b_1)_{2Q}|\cdots|b_m(x) - (b_m)_{2Q}||T(f)(x)|dx$$

$$\leq \left(\frac{1}{|Q|}\int_Q |b_1(x) - (b_1)_{2Q}|^{p_1}dx\right)^{1/p_1}\cdots\left(\frac{1}{|Q|}\int_Q |b_m(x) - (b_m)_{2Q}|^{p_m}dx\right)^{1/p_m}$$

$$\times\left(\frac{1}{|Q|}\int_Q |T(f)(x)|^r dx\right)^{1/r}$$

$$\leq C\|\vec{b}\|_{BMO} M_r(T(f))(\tilde{x})$$

$$\leq C\|\vec{b}\|_{BMO} M_r(f)(\tilde{x}).$$

For I_2, by the Minkowski's and by Hölder's inequality, we get

$$I_2 = \frac{1}{|Q|}\int_Q \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} |(b(x) - (b)_{2Q})_\sigma T_{\vec{b}_{\sigma^c}}(f)(x)|dx$$

$$\le \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|} \int_Q |(b(x) - (b)_{2Q})_\sigma| |T_{\vec{b}_{\sigma^c}}(f)(x)| dx$$

$$\le C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left(\frac{1}{|2Q|} \int_{2Q} |(b(x) - (b)_{2Q})_\sigma|^{r'} dx \right)^{1/r'} \left(\frac{1}{|Q|} \int_Q |T_{\vec{b}_{\sigma^c}}(f)(x)|^r dx \right)^{1/r}$$

$$\le C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{BMO} M_r(T_{\vec{b}_{\sigma^c}}(f))(\tilde{x}).$$

For I_3, choose $1 < p < r, 1 < q_j < \infty$, $j = 1, \cdots, m$ such that $1/q_1 + \cdots + 1/q_m + p/r = 1$, by the boundedness of T on $L^p(R^n)$ and Hölder's inequality, we get

$$I_3 = \frac{1}{|Q|} \int_Q |T((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f_1)(x)| dx$$

$$\le \left(\frac{1}{|Q|} \int_{R^n} |T((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f \chi_{2Q})(x)|^p dx \right)^{1/p}$$

$$\le C \left(\frac{1}{|Q|} \int_{R^n} |b_1 - (b_1)_{2Q}|^p \cdots |b_m - (b_m)_{2Q}|^p |f(x) \chi_{2Q}(x)|^p dx \right)^{1/p}$$

$$\le C \left(\frac{1}{|2Q|} \int_{2Q} |f(x)|^r dx \right)^{1/r}$$

$$\times C \left(\frac{1}{|2Q|} \int_{2Q} |b_1 - (b_1)_{2Q}|^{pq_1} dx \right)^{1/pq_1}$$

$$\cdots \left(\frac{1}{|2Q|} \int_{2Q} |b_m - (b_m)_{2Q}|^{pq_m} dx \right)^{1/pq_m}$$

$$\le C \|\vec{b}\|_{BMO} M_r(f)(\tilde{x}).$$

For I_4, choose $1 < p_j < \infty$, $j = 1, \cdots, m$ such that $1/p_1 + \cdots + 1/p_m + 1/r = 1$, then

$$I_4 = \frac{1}{|Q|}\int_Q \left|\int_{R^n} T((b_1 - (b_1)_{2Q})\cdots(b_m - (b_m)_{2Q})f_2)(y) - C_0\right| dx$$

$$= \frac{1}{|Q|}\int_Q \left|\int_{R^n}\left[T((b_1 - (b_1)_{2Q})\cdots(b_m - (b_m)_{2Q})f_2)(y)\right.\right.$$

$$\left.\left. - \sum_{j=1}^{m}\int_{R^n} B_j(x_0 - y)\phi_j(x_0 - x)\left(\prod_{j=1}^{m}(b_j(y) - (b_j)_{2Q})f_2\right)(y)\right] dy\right| dx$$

$$= \frac{1}{|Q|}\int_Q \left|\int_{(2Q)^c}\left(K(x-y) - \sum_{j=1}^{m} B_j(x_0 - y)\phi_j(x_0 - x)\right)\right.$$

$$\left.\times\left(\prod_{j=1}^{m}(b_j(y) - (b_j)_{2Q})f(y)\right) dy\right| dx$$

$$\le \frac{C}{|Q|}\int_Q\int_{(2Q)^c}\left|\prod_{j=1}^{m}(b_j(y) - (b_j)_{2Q})\right||f(y)|\left(\frac{|x - x_0|^\delta}{|y - x_0|^{n+\delta}}\right) dy dx$$

$$= C\sum_{k=1}^{\infty}\int_{2^k d \le |y - x_0| < 2^{k+1} d}\left|\prod_{j=1}^{m}(b_j(y) - (b_j)_{2Q})\right||f(y)|$$

$$\times\left(\frac{1}{|Q|}\int_Q \frac{|x - x_0|^\delta}{|y - x_0|^{n+\delta}}\, dx\right) dy$$

$$\le C\sum_{k=1}^{\infty}\frac{d^\delta}{(2^k d)^{n+\delta}}|2^{k+1}Q|\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^r dy\right)^{1/r}$$

$$\times\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_1(y) - (b_1)_{2Q}|^{p_1} dy\right)^{1/p_1}$$

$$\cdots\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_m(y) - (b_m)_{2Q}|^{p_m} dy\right)^{1/p_m}$$

$$\leq \sum_{k=1}^{\infty} 2^{-k\delta}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^r\,dy\right)^{1/r}$$

$$\times\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_1(y)-(b_1)_{2Q}|^{p_1}\,dy\right)^{1/p_1}$$

$$\cdots\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_m(y)-(b_m)_{2Q}|^{p_m}\,dy\right)^{1/p_m}$$

$$\leq C\|\vec{b}\|_{BMO}\sum_{k=1}^{\infty} k^m 2^{-k\delta}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^r\,dy\right)^{1/r}$$

$$\leq C\|\vec{b}\|_{BMO} M_r(f)(\tilde{x}).$$

This completes the proof of the theorem.

Proof of Theorem 2. We first consider the case $m = 1$. Choose $1 < r < p$ in Theorem 1 and using Lemmas 1 and 3, we have

$$\|T_{\vec{b}}(f)\|_{L^p} \leq \|M(T_{\vec{b}}(f))\|_{L^p} \leq C\|M_\Phi^{\#}(T_{\vec{b}}(f))\|_{L^p}$$

$$\leq C\|\vec{b}\|_{BMO}\|M_r(f) + M_r(T(f))\|_{L^p}$$

$$\leq C\|\vec{b}\|_{BMO}\|f\|_{L^p} + C\|\vec{b}\|_{BMO}\|T(f)\|_{L^p}$$

$$\leq C\|\vec{b}\|_{BMO}\|f\|_{L^p} + C\|\vec{b}\|_{BMO}\|f\|_{L^p}$$

$$\leq C\|\vec{b}\|_{BMO}\|f\|_{L^p}.$$

When $m \geq 2$, we may get the conclusion of Theorem 2 by induction. This finishes the proof.

Proof of Theorem 3. We first consider the case $m = 1$. Choose $1 < r < p$ in Theorem 1 and using Lemmas 4 and 5, we get

$$\begin{aligned}\|T_{\vec{b}}(f)\|_{L^{p,\varphi}} &\le \|M(T_{\vec{b}}(f))\|_{L^{p,\varphi}} \le C\|M_{\Phi}^{\#}(T_{\vec{b}}(f))\|_{L^{p,\varphi}}\\ &\le C\|\vec{b}\|_{BMO}\|M_r(f)\|_{L^{p,\varphi}} + C\|\vec{b}\|_{BMO}\|M_r(f)\|_{L^{p,\varphi}}\\ &\le C\|\vec{b}\|_{BMO}(\|f\|_{L^{p,\varphi}} + \|T(f)\|_{L^{p,\varphi}})\\ &\le C\|\vec{b}\|_{BMO}\|f\|_{L^{p,\varphi}}.\end{aligned}$$

When $m \geq 2$, we may get the conclusion of Theorem 3 by induction. This finishes the proof.

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