Research and Communications in Mathematics and Mathematical Sciences
Vol. 6, Issue 1, 2016, Pages 53-87
ISSN 2319-6939
Published Online on March 21, 2016

http://jyotiacademicpress.net

BLOW-UP OF POSITIVE SOLUTIONS FOR A LOCALIZED SEMILINEAR HEAT EQUATIONS WITH DIRICHLET BOUNDARY CONDITIONS

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2010 Mathematics Subject Classification: 35B40, 35B50, 35K60, 65M06.

Keywords and phrases: discretization, localized semilinear parabolic equation, numerical blow-up time, convergence.

Communicated by S. Ebrahimi Atani.

Received November 20, 2015

Abstract

This paper concerns the study of the numerical approximation for the following initial-boundary value problem:

$$\begin{cases} u_t(x, t) - u_{xx}(x, t) = \lambda f(u(x_0, t)), & (x, t) \in (0, 1) \times (0, T), \\ u(0, t) = u(1, t) = 0, & t \in (0, T), \\ u(x, 0) = u_0(x) \geq 0, & x \in (0, 1), \end{cases}$$

where $f : [0, \infty) \to [0, \infty)$ is a C^1 convex, nondecreasing function, $\int^{\infty} \frac{d\sigma}{f(\sigma)} < \infty$, x_0 is a fixed point in the domain, $x_0 = \frac{1}{2}$, and λ is a positive parameter. Under some assumptions, we prove that the solution of a discrete form of the above problem blows up in a finite time and estimate its numerical blow-up time. We also show that the numerical blow-up time in certain cases converges to the real one when the mesh size tends to zero. Finally, we give some numerical experiments to illustrate our analysis.

1. Introduction

We consider the following initial-boundary value problem for a semilinear heat equation of the form:

$$u_t(x, t) - u_{xx}(x, t) = \lambda f(u(x_0, t)), \quad (x, t) \in (0, 1) \times (0, T), \tag{1}$$

$$u(0, t) = u(1, t) = 0, \quad t \in (0, T), \tag{2}$$

$$u(x, 0) = u_0(x) \geq 0, \quad x \in (0, 1), \tag{3}$$

which models the temperature distribution of a large number of physical phenomena from physics, chemistry, and biology. The particularity of the problem describes in (1)-(3) is that it represents a model in physical phenomena where the reaction is driven by the temperature at a single site. This kind of phenomena is observed in biological systems and in chemical reaction diffusion processes in which the reaction takes place only at some local sites. For instance, the above model is appropriate to describe:

(i) The influence of defect structures on a catalytic surface.

(ii) The temperature in a solid-fuel combustion scenario where the heat that is input into the system is localized, say as in a laser focused on one spot in the domain.

(iii) Chemical reaction-diffusion processes in which, due to effect of catalyst, the reaction takes place only at a single site.

(iv) A heat stationary source which can support an explosive reaction. A stationary source provides a continuous supply of heat to the same environment.

(v) The ignition of a combustible medium with damping, where either a heated wire or a pair of small electrodes supplies a large amount of energy to every confined area.

For more physical motivation, see [4], [5], and [25]. Here $f : [0, \infty) \to [0, \infty)$ is a C^1 convex, nondecreasing function, $\int^{\infty} \frac{d\sigma}{f(\sigma)} < \infty$, λ is a positive parameter (which is called the scaled Damköhler number in the combustion theory). The initial data u_0 is a function which is bounded and symmetric. In addition, $u_0''(x) + \lambda f(u_0(\frac{1}{2})) \geq 0$ in $(0, 1)$. The interval $(0, T)$ is the maximal time interval of existence of the solution u. The time T may be finite or infinite. When T is infinite, then we say that the solution u exists globally. When T is finite, then the solution u develops a singularity in a finite time, namely,

$$\lim_{t \to T} \|u(\cdot, t)\|_{\infty} = \infty,$$

where $\|u(\cdot, t)\|_{\infty} = \max_{0 \leq x \leq 1} |u(x, t)|$. In this last case, we say that the solution u blows up in a finite time, and the time T is called the blow-up time of the solution u. The local in time existence and uniqueness of the solution u have been proved (see [8], [9], [27]).

The theoretical study of blow-up of solutions for localized semilinear heat equations has been the subject of investigations of many authors (see [8], [9], [12], [22]-[25], and the references cited therein). Under the assumptions given in the Introduction of the present paper, the authors have proved that the solution u of (1)-(3) blows up globally in a finite time on the whole interval $(0, 1)$, and the blow-up time is estimated (see [9], [25]). In the present paper, we are interested in the numerical study using the discrete form of (1)-(3). We give some assumptions under which the solution of the discrete problem blows up in a finite time and estimate its numerical blow-up time. We also show that the numerical blow-up time converges to the theoretical one when the mesh size goes to zero. Previously, some authors have used semidiscrete and discrete schemes to study the phenomenon of blow-up, but only the case where the reaction term $\lambda f(u(x_0, t))$ is replaced by $f(u(x, t))$ has been taken into account (see [7], [10], [11], [17]).

In this paper, we are interested in the numerical study of the above problem. Our aim is to build an explicit scheme in which the discrete solution reproduces the properties of the continuous one.

2. Full Discretization of the Blowing-up Solutions

We start by the construction of an adaptive scheme as follows. Let I be a positive integer, and consider the grid $x_i = ih, 0 \leq i \leq I,$ where $h = 1/I.$ Approximate the solution u of (1)-(3) by the solution $U_h^{(n)} = (U_0^{(n)}, U_1^{(n)}, \dots, U_I^{(n)})^T$ of the following discrete equations:

$$\delta_t U_i^{(n)} - \delta^2 U_i^{(n)} = \lambda f(U_k^{(n)}), \quad 1 \leq i \leq I - 1, \tag{4}$$

$$U_0^{(n)} = 0, \quad U_I^{(n)} = 0, \tag{5}$$

$$U_i^{(0)} = \varphi_i \geq 0, \quad 1 \leq i \leq I - 1, \tag{6}$$

where k is the integer part of the number $I/2$,

$$\delta^2 U_i^{(n)} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2}, \quad 1 \le i \le I-1,$$

$$\delta_t U_i^{(n)} = \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n}, \quad 1 \le i \le I-1,$$

$$\varphi_0 = 0, \quad \varphi_I = 0, \quad \varphi_i = \varphi_{I-i}, \quad 0 \le i \le I, \quad \delta^+\varphi_i > 0, \quad 0 \le i \le k-1,$$

$$\delta^+\varphi_i = \frac{\varphi_{i+1} - \varphi_i}{h}.$$

In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the blow-up time T, we need to adapt the size of the time step so that we take

$$\Delta t_n = \min\{\frac{h^2}{3}, \frac{\tau}{f(\|U_h^{(n)}\|_\infty)}\},$$

with $\tau \in (0, 1)$.

Let us notice that the restriction on the time step ensures the nonnegativity of the discrete solution.

To facilitate our discussion, we need to define the notion of numerical blow-up.

Definition 2.1. We say that the solution $U_h^{(n)}$ of the explicit scheme blows up in a finite time if $\lim_{n\to\infty}\|U_h^{(n)}\|_\infty = \infty$, and the series $\sum_{n=0}^{\infty}\Delta t_n$ converges. The quantity $\sum_{n=0}^{\infty}\Delta t_n$ is called the numerical blow-up time of the discrete solution.

Our paper is organized in the following manner. In the next section, we prove some results about the discrete maximum principle for localized parabolic problems. In the fourth section, we prove that the solution of the discrete problem blows up in a finite time and estimate its numerical blow-up time. In the fifth section, we give a result about the convergence of numerical blow-up times in some cases where the blow-up occurs. Finally, in the last section, we give some numerical results to illustrate our analysis.

3. Properties of the Discrete Solution

In this section, we give some lemmas about the discrete maximum principle for localized parabolic problems and reveal certain properties concerning the discrete solution.

The following lemma is a discrete form of the maximum principle for localized parabolic problems.

Lemma 3.1. *Let* $a^{(n)}$ *and* $V_h^{(n)}$ *be two sequences such that* $a^{(n)}$ *is nonnegative and*

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} - a^{(n)} V_k^{(n)} \geq 0, \quad 1 \leq i \leq I-1, \quad n \geq 0, \tag{7}$$

$$V_0^{(n)} \geq 0, \quad V_I^{(n)} \geq 0, \quad n \geq 0, \tag{8}$$

$$V_i^{(0)} \geq 0, \quad 0 \leq i \leq I. \tag{9}$$

Then $V_i^{(n)} \geq 0,\ 0 \leq i \leq I,\ n > 0,$ *when* $\Delta t_n \leq \frac{h^2}{2}$.

Proof. A straightforward computation shows that

$$V_i^{(n+1)} \geq \frac{\Delta t_n}{h^2} V_{i-1}^{(n)} + (1 - 2\frac{\Delta t_n}{h^2})V_i^{(n)} + \frac{\Delta t_n}{h^2} V_{i+1}^{(n)} + \Delta t_n a^{(n)} V_k^{(n)},\ 1 \leq i \leq I-1.$$

The choice of Δt_n implies that $1 - 2\frac{\Delta t_n}{h^2} \geq 0$. If $V_h^{(n)} \geq 0$, then using an argument of recursion, we easily see that $V_h^{(n+1)} \geq 0$. This ends the proof. □

An immediate consequence of the above result is the following comparison lemma. Its proof is straightforward.

Lemma 3.2. *Let* $V_h^{(n)}, W_h^{(n)},$ *and* $a^{(n)}$ *be three sequences such that* $a^{(n)}$ *is nonnegative and*

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} - a^{(n)} V_k^{(n)} \le \delta_t W_i^{(n)} - \delta^2 W_i^{(n)} - a^{(n)} W_k^{(n)},$$

$$1 \le i \le I-1, \quad n \ge 0,$$

$$V_0^{(n)} \le W_0^{(n)}, \quad V_I^{(n)} \le W_I^{(n)},$$

$$V_i^{(0)} \le W_i^{(0)}, \quad 0 \le i \le I.$$

Then $V_i^{(n)} \le W_i^{(n)}$, $0 \le i \le I$, $n > 0$ *when* $\Delta t_n \le \frac{h^2}{2}$.

The lemma below reveals some properties of the discrete solution.

Lemma 3.3. *The discrete solution* $U_h^{(n)}$ *of* (4)-(6) *obeys the following relations*:

$$U_i^{(n)} = U_{I-i}^{(n)}, \quad 0 \le i \le I, \quad \delta^+ U_i^{(n)} \ge 0, \quad 0 \le i \le k-1. \tag{10}$$

Proof. Introduce the vector $V_h^{(n)}$ defined as follows:

$$V_i^{(n)} = U_i^{(n)} - U_{I-i}^{(n)}, \quad 0 \le i \le I, \quad n \ge 0.$$

A routine calculation reveals that

$$V_i^{(n+1)} = \frac{\Delta t_n}{h^2} V_{i-1}^{(n)} + (1 - 2\frac{\Delta t_n}{h^2}) V_i^{(n)} + \frac{\Delta t_n}{h^2} V_{i+1}^{(n)}, \quad 1 \le i \le I-1, \quad n \ge 0,$$

$$V_0^{(n)} = 0, \quad V_I^{(n)} = 0, \quad n \ge 0,$$

$$V_i^{(0)} = 0, \quad 0 \le i \le I.$$

Using an argument of recursion, we easily note that $V_h^{(n)} = 0,\ n \geq 0,$ and the first part of the lemma is proved. In order to prove the second one, we proceed as follows. Set

$$W_i^{(n)} = U_{i+1}^{(n)} - U_i^{(n)}, \quad 0 \leq i \leq k-1.$$

We remark that

$$W_0^{(n)} = U_1^{(n)} \geq 0. \tag{11}$$

On the other hand, it is easy to check that $U_{k+1}^{(n)} = U_k^{(n)}$ if I is odd, and $U_{k+1}^{(n)} = U_{k-1}^{(n)}$ if I is even. This implies that

$$\delta^2 W_{k-1}^{(n)} = \begin{cases} \dfrac{-2W_{k-1}^{(n)} + W_{k-2}^{(n)}}{h^2}, & \text{if } I \text{ is odd,} \\ \dfrac{-3W_{k-1}^{(n)} + W_{k-2}^{(n)}}{h^2}, & \text{if } I \text{ is even.} \end{cases}$$

Obviously

$$\delta_t W_i^{(n)} = \delta^2 W_i^{(n)}, \quad 0 \leq i \leq k-2, \quad n \geq 0. \tag{12}$$

Making use of the above relations, we arrive at

$$W_0^{(n)} \geq 0, \quad n \geq 0,$$

$$W_i^{(n+1)} = \frac{\Delta t_n}{h^2} W_{i-1}^{(n)} + (1 - 2\frac{\Delta t_n}{h^2})W_i^{(n)} + \frac{\Delta t_n}{h^2} W_{i+1}^{(n)}, \quad 1 \leq i \leq k-2, \quad n \geq 0,$$

$$W_{k-1}^{(n+1)} = \frac{\Delta t_n}{h^2} W_{k-2}^{(n)} + (1 - 3\frac{\Delta t_n}{h^2})W_{k-1}^{(n)}, \quad n \geq 0 \text{ if } I \text{ is even,}$$

$$W_{k-1}^{(n+1)} = \frac{\Delta t_n}{h^2} W_{k-2}^{(n)} + (1 - 2\frac{\Delta t_n}{h^2})W_{k-1}^{(n)}, \quad n \geq 0 \text{ if } I \text{ is odd,}$$

$$W_i^{(0)} \geq 0, \quad 1 \leq i \leq k-1.$$

We deduce by induction that

$$W_i^{(n)} \geq 0, \quad 1 \leq i \leq k-1, \quad n \geq 0.$$

This completes the proof. □

The above lemma says that, if the initial data of the discrete solution is symmetric in space, then the discrete solution also obeys this property. In addition, if the initial data is nondecreasing in space, then the discrete solution also verifies this assertion. These properties imply that the discrete solution attains its maximum at the node x_k.

The following lemma is a discrete version of Green's formula.

Lemma 3.4. *Let* U_h *and* $V_h \in \mathbb{R}^{I+1}$ *such that* $U_0 = 0$, $U_I = 0$, $V_0 = 0$, $V_I = 0$. *Then, we have*

$$\sum_{i=1}^{I-1} U_i \delta^2 V_i = \sum_{i=1}^{I-1} V_i \delta^2 U_i.$$

Proof. A straightforward computation reveals that

$$\sum_{i=1}^{I-1} U_i \delta^2 V_i = \sum_{i=1}^{I-1} V_i \delta^2 U_i + \frac{U_1 V_0 - U_0 V_1}{h^2} + \frac{U_{I-1} V_I - U_I V_{I-1}}{h^2},$$

and the result follows using the assumptions of the lemma. □

4. The Blow-up Solutions

In this section, under some assumptions, we show that the solution of the discrete problem blows up in a finite time and estimate its numerical blow-up time. We need the following lemmas:

Lemma 4.1. *Let a and b be two positive numbers. Then, we have*

$$\sum_{n=0}^{\infty} \frac{1}{f(a+bn)} \leq \frac{1}{f(a)} + \frac{1}{b} \int_a^{\infty} \frac{d\sigma}{f(\sigma)}.$$

Proof. We observe that

$$\int_0^{\infty} \frac{dx}{f(a+bx)} = \sum_{n=0}^{\infty} \int_n^{n+1} \frac{dx}{f(a+bx)} \geq \sum_{n=0}^{\infty} \int_n^{n+1} \frac{dx}{f(a+b(n+1))},$$

because $f(s)$ is nondecreasing for $s \geq 0$. We deduce that

$$\int_0^{\infty} \frac{dx}{f(a+bx)} \geq \sum_{n=0}^{\infty} \frac{1}{f(a+b(n+1))} = -\frac{1}{f(a)} + \sum_{n=0}^{\infty} \frac{1}{f(a+bn)}.$$

On the other hand, by a change of variables, we see that $\int_0^{\infty} \frac{dx}{f(a+bx)}$ $= \frac{1}{b} \int_a^{\infty} \frac{d\sigma}{f(\sigma)}$, which implies that

$$\sum_{n=0}^{\infty} \frac{1}{f(a+bn)} \leq \frac{1}{f(a)} + \frac{1}{b} \int_a^{\infty} \frac{d\sigma}{f(\sigma)}.$$

This ends the proof. □

Lemma 4.2. *We have*

$$\sum_{i=1}^{I-1} \tan\left(\frac{\pi h}{2}\right) \sin(i\pi h) = 1.$$

Proof. We observe that

$$\sum_{i=1}^{I-1} \sin(i\pi h) = \mathrm{Im}\left(\sum_{i=1}^{I-1} e^{ji\pi h}\right) = \mathrm{Im}\left(e^{j\pi h} \frac{(e^{j\pi h(I-1)} - 1)}{e^{j\pi h} - 1}\right),$$

where $j = \sqrt{-1}$. Using the fact that $Ih = 1$, we deduce that

$$\sum_{i=1}^{I-1} \sin(i\pi h) = \mathrm{Im}\left(\frac{e^{j\pi} - e^{j\pi h}}{e^{j\pi h} - 1}\right) = \mathrm{Im}\left(\frac{e^{j\pi h} + 1}{1 - e^{j\pi h}}\right),$$

or equivalently,

$$\sum_{i=1}^{I-1} \sin(i\pi h) = \mathrm{Im}\left(\frac{e^{\frac{j\pi h}{2}} + e^{-\frac{j\pi h}{2}}}{e^{-\frac{j\pi h}{2}} - e^{\frac{j\pi h}{2}}}\right) = \mathrm{cotan}\left(\frac{\pi h}{2}\right).$$

This implies that

$$\sum_{i=1}^{I-1} \tan\left(\frac{\pi h}{2}\right) \sin(i\pi h) = 1,$$

and the proof is complete. □

The statement of our first result on blow-up is the following.

Theorem 4.1. *Suppose that* $f(0) > 0$ *and* $A = \int_0^\infty \frac{d\sigma}{f(\sigma)}$. *Let* $\rho_h = \frac{2 - 2\cos(\pi h)}{h^2}$. *If* $\lambda > \rho_h A$, *then the solution* $U_h^{(n)}$ *of* (4)-(6) *blows up in a finite time, and its numerical blow-up time* $T_h^{\Delta t}$ *is estimated as follows*:

$$T_h^{\Delta t} \leq \frac{\tau}{f(B)} + \frac{\tau}{(\lambda - \rho_h A)\tau'} \int_B^\infty \frac{d\sigma}{f(\sigma)},$$

where $B = \sum_{i=1}^{I-1} \tan\left(\frac{\pi}{2} h\right) \sin(i\pi h)\varphi_i$ *and* $\tau' = \min\{\frac{h^2}{3} f(B), \tau\}$.

Proof. Introduce the sequence v^n defined as follows:

$$v^n = \sum_{i=1}^{I-1} \tan\left(\frac{\pi h}{2}\right) \sin(i\pi h) U_i^{(n)}, \quad n \geq 0.$$

A straightforward computation reveals that

$$\delta_t v^n = \sum_{i=1}^{I-1} \tan\left(\frac{\pi h}{2}\right) \sin(i\pi h) \delta_t U_i^{(n)}, \quad n \geq 0.$$

Making use of (4), we arrive at

$$\delta_t v^n = \sum_{i=1}^{I-1} \tan\left(\frac{\pi h}{2}\right) \sin(i\pi h) \delta^2 U_i^{(n)} + \lambda \sum_{i=1}^{I-1} \tan\left(\frac{\pi h}{2}\right) \sin(i\pi h) f(U_k^{(n)}),\ n \geq 0.$$

We observe that

$$\delta^2 \sin(i\pi h) = -\rho_h \sin(i\pi h).$$

Exploiting Lemma 2.4, we derive the following equality:

$$\delta_t v^n = -\rho_h v^n + \lambda f(U_k^{(n)}) \sum_{i=1}^{I-1} \tan(\frac{\pi h}{2}) \sin(i\pi h), \quad n \geq 0. \tag{13}$$

With the help of Lemma 3.2, we see that

$$\delta_t v^n = -\rho_h v^n + \lambda f(U_k^{(n)}), \quad n \geq 0. \tag{14}$$

Invoking Lemma 2.3, we note that $\|U_h^{(n)}\|_\infty = U_k^{(n)} \geq v^n$, $n \geq 0$. We infer from (14) that

$$\delta_t v^n \geq -\rho_h U_k^{(n)} + \lambda f(U_k^{(n)}), \quad n \geq 0,$$

which implies that

$$\delta_t v^n \geq \lambda f(U_k^{(n)})(1 - \frac{\rho_h U_k^{(n)}}{\lambda f(U_k^{(n)})}), \quad n \geq 0. \tag{15}$$

We observe that

$$\int_0^\infty \frac{d\sigma}{f(\sigma)} \geq \sup_{t \geq 0} \int_0^t \frac{d\sigma}{f(\sigma)} \geq \sup_{t \geq 0} \frac{t}{f(t)},$$

because $f(s)$ is nondecreasing for $s \geq 0$. According to (15), we get

$$\delta_t v^n \geq \lambda f(U_k^{(n)})(1 - \frac{\rho_h A}{\lambda}), \quad n \geq 0,$$

or equivalently,

$$v^{n+1} \geq v^n + (\lambda - \rho_h A)\Delta t_n f(U_k^{(n)}), \quad n \geq 0. \tag{17}$$

Recalling that $\|U_h^{(n)}\|_\infty = U_k^{(n)}$, we note that

$$\Delta t_n f(U_k^{(n)}) = \min\{\frac{h^2}{3} f(U_k^{(n)}), \tau\}.$$

Due to (17), we get $v^{n+1} \geq v^n$, $n \geq 0$, and by induction, we arrive at

$$v^n \geq v^0, \quad n \geq 0.$$

Since $U_k^{(n)} \geq v^n \geq v^0$, we deduce that

$$\Delta t_n f(U_k^{(n)}) \geq \min\{\frac{h^2}{3} f(v^0), \tau\} = \tau'.$$

Exploiting (17), we derive the following estimate:

$$v^{n+1} \geq v^n + (\lambda - \rho_h A)\tau', \quad n \geq 0, \tag{18}$$

and by induction, we see that

$$v^n \geq v^0 + (\lambda - \rho_h A)n\tau', \quad n \geq 0. \tag{19}$$

This implies that $\|U_h^{(n)}\|_\infty$ goes to infinity as n approaches infinity because $\|U_h^{(n)}\|_\infty \geq v^n$. Now, let us estimate the numerical blow-up time of $U_h^{(n)}$. The restriction on the time step ensures that $\sum_{n=0}^\infty \Delta t_n \leq \sum_{n=0}^\infty \frac{\tau}{f(\|U_h^{(n)}\|_\infty)}$. Due to (19) and the fact that $\|U_h^{(n)}\|_\infty \geq v^n$, we get

$$\sum_{n=0}^\infty \Delta t_n \leq \sum_{n=0}^\infty \frac{\tau}{f(v^0 + (\lambda - \rho_h A)n\tau')}.$$

Invoking Lemma 3.1, we discover that

$$\sum_{n=0}^\infty \Delta t_n \leq \frac{\tau}{f(v^0)} + \frac{\tau}{(\lambda - \rho_h A)\tau'} \int_{v^0}^\infty \frac{d\sigma}{f(\sigma)}.$$

Since $B = v^0$, then the above estimate may be rewritten in the following manner:

$$\sum_{n=0}^{\infty} \Delta t_n \leq \frac{\tau}{f(B)} + \frac{\tau}{(\lambda - \rho_h A)\tau'} \int_B^{\infty} \frac{d\sigma}{f(\sigma)}.$$

Use the fact that the quantity on the right hand side of the above inequality is finite to complete the rest of the proof. □

If $f(0) = 0$ and $B > 0$, then Theorem 3.1 remains valid when A is replaced by $\frac{B}{f(B)}$. In fact, we observe that $\delta_t v^0 > 0$, and we claim that $\delta_t v^n > 0$ for $n > 0$. To prove the claim, we argue by contradiction. Assume that there exists $N \geq 1$ such that $\delta_t v^n > 0$ for $0 \leq n < N$, but $\delta_t v^N \leq 0$.

This implies that $v^N \geq v^0$, and $\frac{v^N}{f(v^N)} \leq \frac{v^0}{f(v^0)}$ because $\frac{s}{f(s)}$ is nonincreasing for $s > 0$.

Consequently, we get $0 \geq \delta_t v^N \geq \lambda f(v^N)(1 - \frac{\rho_h v^0}{\lambda f(v^0)}) > 0$, which is a contradiction and the claim is proved. Since $\delta_t v^n > 0$ for $n > 0$, we deduce that $U_k^{(n)} \geq v^n \geq v^0$ for $n > 0$, and $\frac{U_k^{(n)}}{f(U_k^{(n)})} \leq \frac{v^0}{f(v^{(0)})} = \frac{B}{f(B)}$. This implies that

$$\delta_t v^n \geq \lambda f(U_k^{(n)})(1 - \frac{\rho_h B}{\lambda f(B)}) \quad \text{for} \quad n > 0,$$

or equivalently,

$$\delta_t v^n \geq \lambda f(U_k^{(n)})(1 - \frac{\rho_h A}{\lambda}) \quad \text{for} \quad n > 0.$$

Now, reasoning as in the proof of Theorem 3.1, we arrive at the desired result.

Remark 4.1. Using (18), we deduce by induction that

$$v^n \geq v^q + (\lambda - \rho_h A)(n-q)\tau', \quad n \geq q. \tag{20}$$

Thanks to (20), the restriction on the time step leads us to

$$T_h^{\Delta t} - t_q = \sum_{n=q}^{\infty} \Delta t_n \leq \sum_{n=q}^{\infty} \frac{\tau}{f(v^q + (\lambda - \rho_h A)(n-q)\tau')}.$$

It follows from Lemma 3.1 that

$$T_h^{\Delta t} - t_q \leq \frac{\tau}{f(v^q)} + \frac{\tau}{(\lambda - \rho_h A)\tau'} \int_{v^q}^{\infty} \frac{d\sigma}{f(\sigma)}.$$

If we pick $\tau = h^2$, then we note that $\frac{\tau'}{\tau} = \min\{\frac{f(B)}{3}, 1\}$, which implies $\frac{\tau}{\tau'} = O(1)$.

In the sequel, we choose $\tau = h^2$.

The following theorem renders an upper bound of the numerical blow-up time when blow-up occurs.

Theorem 4.2. *Assume that the discrete solution* $U_h^{(n)}$ *of* (4)-(6) *blows up in a finite time. Then its numerical blow-up time* $T_h^{\Delta t}$ *is estimated as follows*

$$T_h^{\Delta t} \geq \frac{Nh^2}{3} + \frac{\tau}{f(\|\varphi_h\|_\infty + (N+1)\lambda\tau)} + \frac{1}{\lambda}\int_{\|\varphi_h\|_\infty + (N+1)\lambda\tau}^{\infty} \frac{d\sigma}{f(\sigma)},$$

where N is the first integer such that

$$\frac{\tau}{f(\|\varphi_h\|_\infty + N\lambda\tau)} \leq \frac{h^2}{3}.$$

Proof. We observe that

$$\delta^2 U_k^{(n)} = \frac{U_{k+1}^{(n)} - 2U_k^{(n)} + U_{k-1}^{(n)}}{h^2} \leq 0, \quad n \geq 0,$$

and making use of (4), we deduce that

$$\frac{U_k^{(n+1)} - U_k^{(n)}}{\Delta t_n} \leq \lambda f(U_k^{(n)}), \quad n \geq 0.$$

The above inequality may be rewritten as follows:

$$U_k^{(n+1)} \leq U_k^{(n)} + \lambda \Delta t_n f(U_k^{(n)}), \quad n \geq 0.$$

Since $\Delta t_n = \min\{\frac{h^2}{3}, \frac{\tau}{f(\|U_h^{(n)}\|_\infty)}\}$, we deduce that

$$U_k^{(n+1)} \leq U_k^{(n)} + \lambda\tau, \quad n \geq 0,$$

and by induction, we arrive at

$$U_k^{(n)} \leq U_k^{(0)} + n\lambda\tau = \|\varphi_h\|_\infty + n\lambda\tau, \quad n \geq 0.$$

Now, let us estimate the numerical blow-up time. We have

$$\sum_{n=0}^{\infty} \Delta t_n \geq \sum_{n=0}^{\infty} \min\{\frac{h^2}{3}, \frac{\tau}{f(\|\varphi_h\|_\infty + n\lambda\tau)}\},$$

which implies that

$$\sum_{n=0}^{\infty} \Delta t_n \geq \frac{Nh^2}{3} + \sum_{n=N+1}^{\infty} \frac{\tau}{f(\|\varphi_h\|_\infty + n\lambda\tau)}.$$

Since

$$\sum_{n=N+1}^{\infty} \frac{\tau}{f(\|\varphi_h\|_\infty + n\gamma\tau)} = \sum_{n=0}^{\infty} \frac{\tau}{f(\|\varphi_h\|_\infty + (N+1)\lambda\tau + n\lambda\tau)},$$

then employing Lemma 3.1, we arrive at the desired result. □

When $\|\varphi_h\|_\infty = o(h)$, then using Theorems 3.1 and 3.2, we easily derive the following estimates:

$$\frac{A}{\lambda} \leq \lim_{h \to 0} T_h^{\Delta t} \leq \frac{A}{\lambda - \pi^2 A} \quad \text{for} \quad \lambda \geq \pi^2 A.$$

Apply Taylor's expansion to obtain

$$\frac{1}{1-\frac{\pi^2 A}{\lambda}} = 1+\frac{\pi^2 A}{\lambda}+o\left(\frac{1}{\lambda}\right) \quad \text{as} \quad \lambda \to \infty,$$

which implies that

$$0 \leq \lim_{h\to 0} T_h^{\Delta t} - \frac{A}{\lambda} \leq \frac{\pi^2 A^2}{\lambda^2}+o\left(\frac{1}{\lambda^2}\right) \quad \text{as} \quad \lambda \to \infty.$$

5. Convergence of the Blow-up Time

In this section, under some conditions, we show that the discrete solution blows up in a finite time and its numerical blow-up time converges to the real one when the mesh size goes to zero. In order to prove this result, we firstly show that the discrete solution approaches the continuous one on any interval $[0, 1]\times[0, T-\tau]$ with $\tau \in (0, T)$ as the parameter h goes to zero.

The result on the convergence of the discrete solution to the theoretical one is stated in the following theorem:

Theorem 5.1. *Suppose that the problem* (1)-(3) *has a solution* $u \in C^{4,2}([0, 1]\times[0, T-\tau])$ *with* $\tau \in (0, T)$. *Assume that the initial data at* (6) *satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \to 0.$$

Then, the problem (4)-(6) *admits a unique solution* $U_h^{(n)}$ *for h sufficiently small*, $0 \leq n \leq J$, *and the following relation holds*:

$$\sup_{0\leq n\leq J} \|U_h^{(n)} - u_h(t_n)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + h^2) \quad as \quad h \to 0,$$

where J is any quantity satisfying the inequality $\sum_{j=0}^{J-1}\Delta t_j \leq T-\tau$ *and* $t_n = \sum_{j=0}^{n-1}\Delta t_j$.

Proof. For each h, the problem (4)-(6) has a solution $U_h^{(n)}$. Let $N \leq J$ be the greatest value of n such that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty < 1 \quad \text{for} \quad n < N. \tag{21}$$

Since $u \in C^{4,2}$, then there exists a positive constant R such that

$$\sup_{t \in [0, T-\tau]} \|u(\cdot, t)\|_\infty \leq R.$$

An application of the triangle inequality gives

$$\|U_h^{(n)}\|_\infty \leq \|u_h(t_n)\|_\infty + \|U_h^{(n)} - u_h(t_n)\|_\infty \leq 1 + R \quad \text{for} \quad n < N. \tag{22}$$

Use Taylor's expansion to obtain

$$\delta_t u(x_i, t_n) - \delta^2 u(x_i, t_n) - \lambda f(u(x_k, t_n)) = -\frac{h^2}{12} u_{xxxx}(\widetilde{x}_i, t_n) + \frac{\Delta t_n}{2} u_{tt}(x_i, \widetilde{t}_n),$$

$$1 \leq i \leq I - 1, \quad n < N.$$

Let $e_h^{(n)} = U_h^{(n)} - u_h(t_n)$ be the error of discretization. From the mean value theorem, we get

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} - \lambda f'(\xi_k^{(n)}) e_k^{(n)} = \frac{h^2}{12} u_{xxxx}(\widetilde{x}_i, t_n) - \frac{\Delta t_n}{2} u_{tt}(x_i, \widetilde{t}_n),$$

$$1 \leq i \leq I - 1, \quad n < N,$$

where $\xi_k^{(n)}$ is an intermediate value between $u(x_k, t_n)$ and $U_k^{(n)}$. Since $u_{xxxx}(x, t)$, $u_{tt}(x, t)$ are bounded and $\Delta t_n = O(h^2)$, then there exists a positive constant M such that

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} - \lambda f'(\xi_k^{(n)}) e_k^{(n)} \leq Mh^2, \quad 1 \leq i \leq I - 1, \quad n < N. \tag{23}$$

Set $L = \lambda f'(R + 1)$ and introduce the vector $V_h^{(n)}$ defined as follows

$$V_i^{(n)} = e^{(L+1)t_n} (\|\varphi_h - u_h(0)\|_\infty + Mh^2), \quad 0 \leq i \leq I, \quad n < N.$$

A straightforward computation gives

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} > \lambda f'(\xi_k^{(n)}) V_k^{(n)} + Mh^2, \quad 1 \leq i \leq I - 1, \quad n < N, \tag{24}$$

$$V_0^{(n)} > e_0^{(n)}, \quad V_I^{(n)} > e_I^{(n)}, \quad n < N, \tag{25}$$

$$V_i^{(0)} > e_i^{(0)}, \quad 0 \leq i \leq I. \tag{26}$$

It follows from Lemma 2.2 that $V_h^{(n)} \geq e_h^{(n)}$. In the same way, we also prove that $V_h^{(n)} \geq -e_h^{(n)}$, which implies that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty \leq e^{(L+1)t_n}(\|\varphi_h - u_h(0)\|_\infty + Mh^2), \quad n < N. \tag{27}$$

Let us show that $N = J$. Suppose that $N < J$. If we replace n by N in (27) and use (21), we find that

$$1 \leq \|U_h^{(N)} - u_h(t_N)\|_\infty \leq e^{(L+1)T}(\|\varphi_h - u_h(0)\|_\infty + Mh^2).$$

Since the term on the right hand side of the second inequality goes to zero as h goes to zero, we deduce that $1 \leq 0$, which is a contradiction and the proof is complete. □

Now, we are in a position to prove the main result of this section.

Theorem 5.2. *Suppose that the problem* (1)-(3) *has a solution u which blows up globally in a finite time T such that* $u \in C^{4,2}([0, 1] \times [0, T])$. *Assume that the initial data at* (6) *satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \to 0.$$

Under the assumption of Theorem 3.1, *the problem* (4)-(6) *admits a unique solution* $U_h^{(n)}$ *which blows up in a finite time* $T_h^{\Delta t}$, *and the following relation holds*:

$$\lim_{h \to 0} T_h^{\Delta t} = T.$$

Proof. We know from Remark 3.1 that $\frac{\tau}{\tau'}$ is bounded. Letting $0 < \varepsilon < T/2$, there exists a positive constant R such that

$$\frac{\tau}{f(R)} + \frac{\tau}{(\lambda - \rho_h A)\tau'} \int_R^{\infty} \frac{d\sigma}{f(\sigma)} < \frac{\varepsilon}{2}. \tag{28}$$

Since u blows up globally at the time T, then we observe that $\sum_{i=1}^{I-1} hu(x_i, t)\varphi_i$ also blows up at the time T. This implies that there exist $T_0 \in (T - \frac{\varepsilon}{2}, T)$ and $h_0(\varepsilon) > 0$ such that $\sum_{i=1}^{I-1} hu(x_i, t)\varphi_i \geq 2R$ for $t \in [T_0, T)$, $h \leq h_0(\varepsilon)$. Let q be a positive integer such that $t_q = \sum_{n=0}^{q-1} \Delta t_n \in [T_0, T)$ for $h \leq h_0(\varepsilon)$. Invoking Theorem 4.1, we see that the problem (4)-(6) has a unique solution $U_h^{(n)}$ which obeys $\|U_h^{(n)} - u_h(t_n)\|_\infty < R$ for $n \leq q$, $h \leq h_0(\varepsilon)$. This implies that

$$v^q \geq \sum_{i=1}^{I-1} hu(x_i, t_q)\varphi_i - \|U_h^{(q)} - u_h(t_q)\|_\infty \geq 2R - R = R, \quad h \leq h_0(\varepsilon).$$

An application of Theorem 3.1 shows that $U_h^{(n)}$ blows up at the time $T_h^{\Delta t}$. It follows from Remark 3.1 and (28) that

$$|T_h^{\Delta t} - t_q| \leq \frac{\tau}{f(v^q)} + \frac{\tau}{(\lambda - \rho_h A)\tau'} \int_{v^q}^{\infty} \frac{d\sigma}{f(\sigma)} \leq \frac{\varepsilon}{2},$$

because $v^q \geq R$ for $h \leq h_0(\varepsilon)$. We deduce that for $h \leq h_0(\varepsilon)$,

$$|T - T_h^{\Delta t}| \leq |T - t_q| + |t_q - T_h^{\Delta t}| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and the proof is complete. □

6. Numerical Experiments

In this section, we give some computational experiments to illustrate our analysis. Firstly, we take the explicit scheme defined in (4)-(6).

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \lambda f(U_k^{(n)}), \quad 1 \le i \le I-1, \tag{29}$$

$$U_0^{(n)} = 0, \quad U_I^{(n)} = 0, \tag{30}$$

$$U_i^{(0)} = \varphi_i \ge 0, \quad 1 \le i \le I-1, \tag{31}$$

where k is the integer part of the number $I/2$, $n \ge 0$, we take

$$\Delta t_n = \min\{\frac{h^2}{3}, \frac{\tau}{f(\|U_h^{(n)}\|_\infty)}\},$$

with $\tau \in (0, 1)$.

Secondly, we use the implicit scheme below

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + \lambda f(U_k^{(n)}), \quad 1 \le i \le I-1,$$

$$U_0^{(n+1)} = 0, \quad U_I^{(n+1)} = 0,$$

$$U_i^{(0)} = \varphi_i, \quad 0 \le i \le I,$$

where $n \ge 0$. As in the case of the explicit scheme, here, we pick

$$\Delta t_n = \frac{\tau}{f(\|U_h^{(n)}\|_\infty)},$$

for the explicit scheme, the problem described in (4)-(6) my be written as follows:

$$U_i^{(n+1)} = U_i^{(n)} + \Delta t_n \delta^2 U_i^{(n+1)} + \lambda \Delta t_n f(U_k^{(n)}),$$

$$U_i^{(n+1)} = U_i^{(n)} + \Delta t_n \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \lambda \Delta t_n f(U_k^{(n)}),$$

$$U_i^{(n+1)} = (1 - \frac{2\Delta t_n}{h^2})U_i^{(n)} + \frac{\Delta t_n}{h^2}\left(U_{i+1}^{(n)} + U_{i-1}^{(n)}\right) + \lambda \Delta t_n f(U_k^{(n)}),$$

$$U_i^{(n+1)} = \frac{\Delta t_n}{h^2}(U_{i+1}^{(n)}) + (1 - \frac{2\Delta t_n}{h^2})U_i^{(n)} + \frac{\Delta t_n}{h^2}(U_{i-1}^{(n)}) + \lambda \Delta t_n f(U_k^{(n)}).$$

For $i = 1, U_1^{(n+1)} = \frac{\Delta t_n}{h^2}(U_2^{(n)}) + (1 - \frac{2\Delta t_n}{h^2})U_1^{(n)} + \frac{\Delta t_n}{h^2}(U_0^{(n)}) + \lambda \Delta t_n f(U_k^{(n)})$.

For $i = 1, U_1^{(n+1)} = (1 - \frac{2\Delta t_n}{h^2})U_1^{(n)} + \frac{\Delta t_n}{h^2}(U_2^{(n)}) + \lambda \Delta t_n f(U_k^{(n)})$,

lead us to the linear system below

$$A_h^{(n)} U_h^{(n)} = F_h^{(n)},$$

where $A_h^{(n)}$ is an $I \times I$ tridiagonal matrix defined as follows:

$$A_h^{(n)} = \begin{pmatrix} 1 - 2\frac{\Delta t_n}{h^2} & \frac{\Delta t_n}{h^2} & 0 & \cdots & 0 \\ \frac{\Delta t_n}{h^2} & 1 - 2\frac{\Delta t_n}{h^2} & \frac{\Delta t_n}{h^2} & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -\frac{\Delta t_n}{h^2} \\ 0 & \cdots & 0 & \frac{\Delta t_n}{h^2} & 1 - 2\frac{\Delta t_n}{h^2} \end{pmatrix}.$$

Let us notice that for the above implicit scheme, existence and nonnegativity of the discrete solution are also guaranteed using standard methods (see [2]). In the following tables, in rows, we present the numerical blow-up times, the numbers of iterations, CPU times, and the orders of the approximations corresponding to meshes of 16, 32, 64, 128,

256. We take for the numerical blow-up time $t_n = \sum_{j=0}^{n-1} \Delta t_j$, which is computed at the first time when $|t_{n+1} - t_n| \leq 10^{-16}$. The order ($s$) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical experiments for $\lambda f(U_k^{(n)}) = \lambda(U_k^{(n)})^2$ and $u(x, 0) = \sin(\pi i h)$

First Case. $\lambda = 50$.

Table 1. Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.023213	179	–	–
32	0.021860	638	–	–
64	0.021510	2390	1	1.95
128	0.021475	9060	1	2.00
256	0.021434	34380	4	1.99

Table 2. Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.023178	177	–	–
32	0.021850	630	–	–
64	0.021681	2361	1	2.01
128	0.021486	8944	1	2.01
256	0.021437	33916	7	2.00

Second Case. $\lambda = 100$.

Table 3. Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.013572	97	–	–
32	0.011178	326	–	–
64	0.010552	1199	1	1.94
128	0.010394	4523	1	1.99
256	0.010354	17140	2	1.99

Table 4. Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.014361	96	–	–
32	0.011347	322	–	–
64	0.010593	1185	1	2.01
128	0.010404	4485	2	2.00
256	0.010356	16924	3	1.98

Numerical experiments for $\lambda f(U_k^{(n)}) = \lambda e^{U_k^{(n)}}$, $u(x, 0) = 0$

First Case. $\lambda = 10$.

Table 5. Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.115474	66	–	–
32	0.111975	247	1	–
64	0.111383	974	2	2.00
128	0.111411	3886	14	2.00
256	0.111512	70333	109	1.97

Table 6. Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.117007	63	–	–
32	0.112997	204	1	–
64	0.111002	930	2	2.00
128	0.110378	3285	14	2.00
256	0.110613	68535	109	1.97

Second Case. $\lambda = 50$.

Table 7. Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.024352	16	–	–
32	0.021216	49	–	–
64	0.020338	181	1	2.01
128	0.020101	707	1	1.99
256	0.020023	1008	10	2.04

Table 8. Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.021664	15	–	–
32	0.020429	45	–	–
64	0.020120	174	1	2.01
128	0.020042	689	1	1.99
256	0.020023	979	10	2.04

Third Case. $\lambda = 100$.

Table 9. Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.013967	10	–	–
32	0.011137	27	–	–
64	0.010311	93	1	2.01
128	0.010083	356	1	1.99
256	0.010022	402	10	2.04

Table 10. Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.012998	9	–	–
32	0.011689	22	–	–
64	0.010299	86	1	2.01
128	0.010025	306	1	1.99
256	0.010012	389	10	2.04

Remark 6.1. The above tables reveal that, when λ increases, then the numerical blow-up time of the discrete solution goes to that of the solution $\alpha(t)$ of the following differential equation:

$$\alpha'(t) = \lambda f(\alpha(t)), \quad t > 0,$$

$$\alpha(0) = \|u_0\|_\infty,$$

as λ goes to infinity. A similar result has been established theoretically by Friedman and Lacey in [13].

In the following, we also give some plots to illustrate our analysis. In Figures 1 to 10, we can appreciate that the discrete solution blows up globally. Let us notice that, theoretically, we know that the continuous solution blows up globally under the assumptions given in the introduction of the present paper (see [9], [30]).

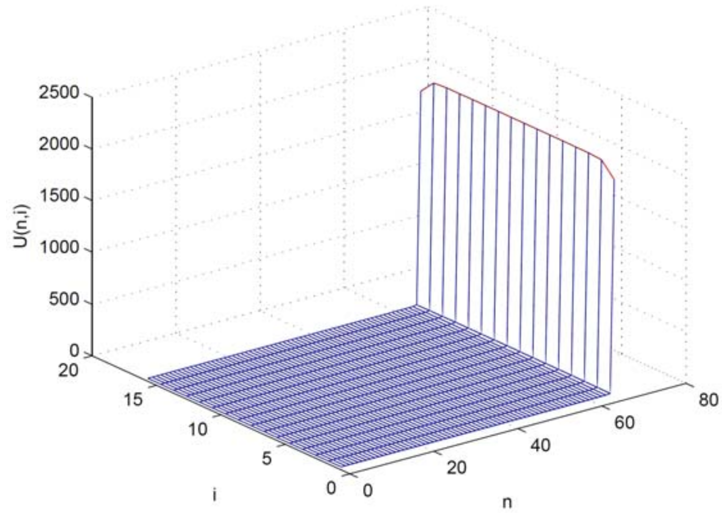

Figure 1. Evolution of the discrete solution, source $f(u) = \lambda e^{u(\frac{1}{2}, t)}$, $\lambda = 50$, $u(x, 0) = 0$, (implicit scheme).

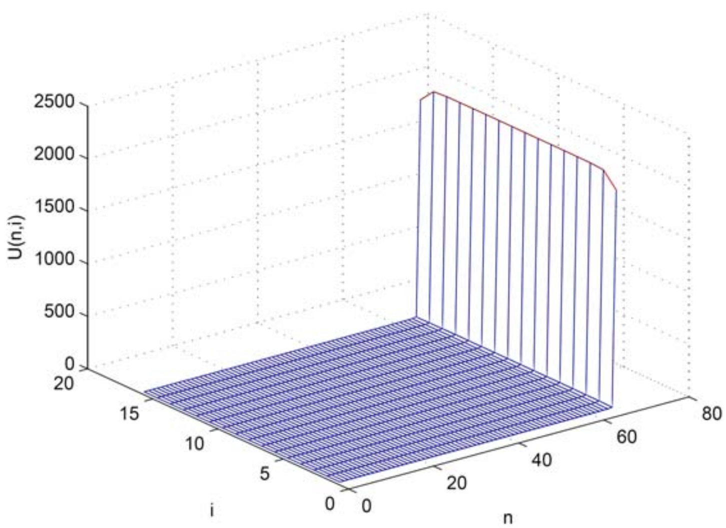

Figure 2. Evolution of the discrete solution, source $f(u) = \lambda e^{u(\frac{1}{2}, t)}$, $\lambda = 50$, $u(x, 0) = 0$, (explicit scheme).

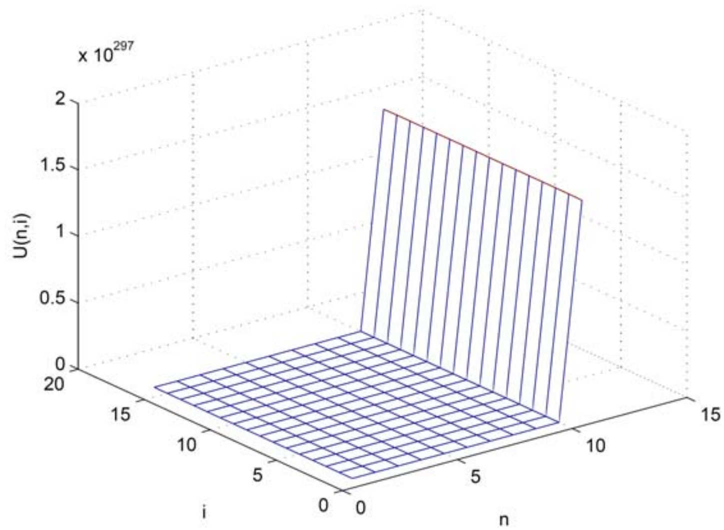

Figure 3. Evolution of the discrete solution, source $f(u) = \lambda e^{u(\frac{1}{2}, t)}$, $\lambda = 100$, $u(x, 0) = 0$, (implicit scheme).

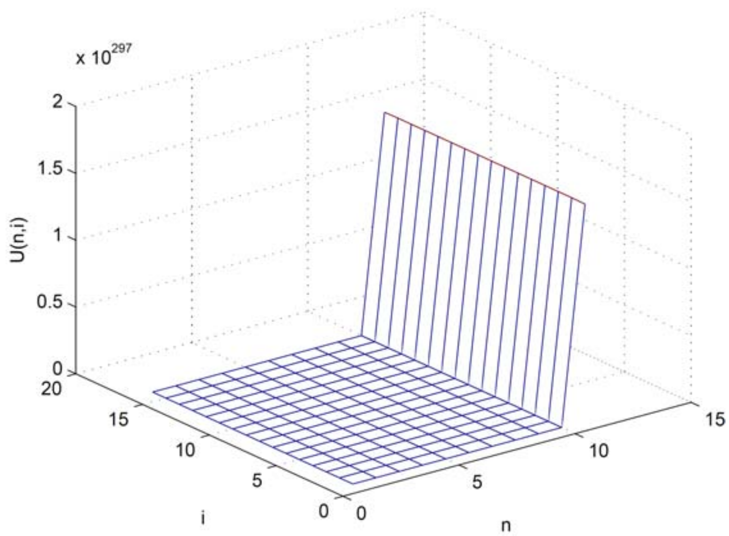

Figure 4. Evolution of the discrete solution, source $f(u) = \lambda e^{u(\frac{1}{2}, t)}$, $\lambda = 100$, $u(x, 0) = 0$, (explicit scheme).

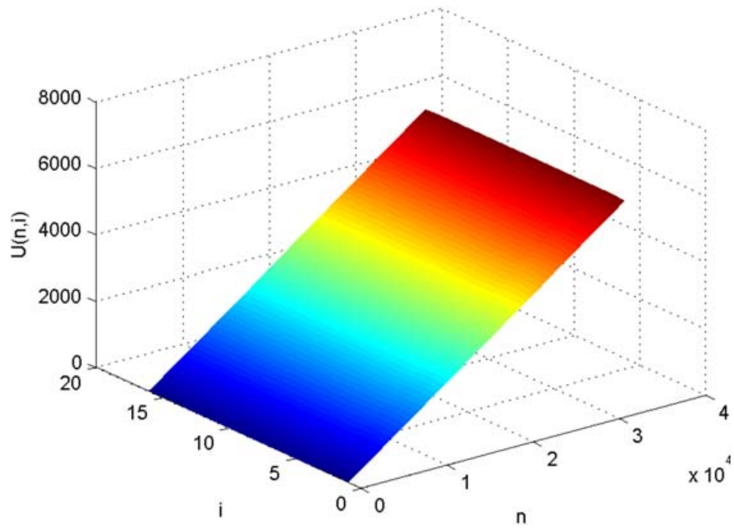

Figure 5. Evolution of the discrete solution, source $f(u) = \lambda(u(\frac{1}{2}, t))^2$, $\lambda = 50$, $u(x, 0) = \sin(\pi x)$ (implicit scheme).

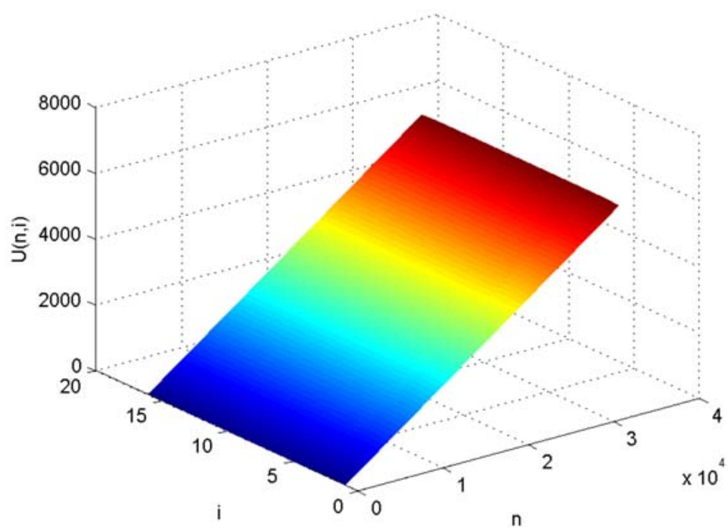

Figure 6. Evolution of the discrete solution, source $f(u) = \lambda(u(\frac{1}{2}, t))^2$, $\lambda = 50$, $u(x, 0) = \sin(\pi x)$ (explicit scheme).

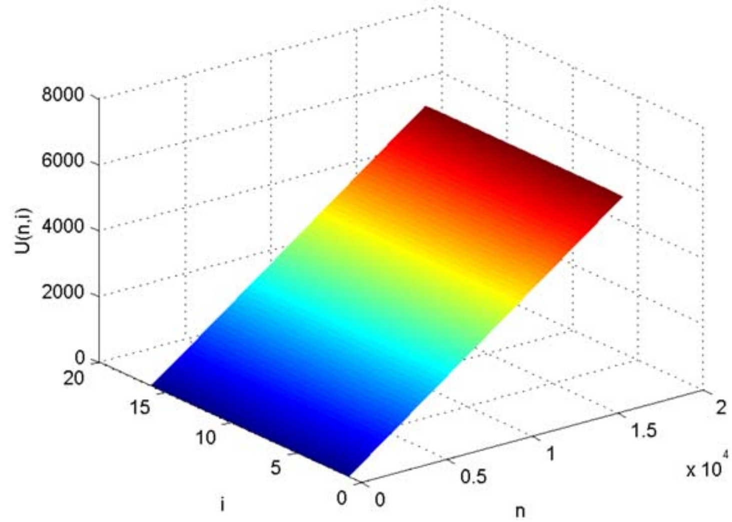

Figure 7. Evolution of the discrete solution, source $f(u) = \lambda(u(\frac{1}{2}, t))^2$, $\lambda = 100$, $u(x, 0) = \sin(\pi x)$ (implicit scheme).

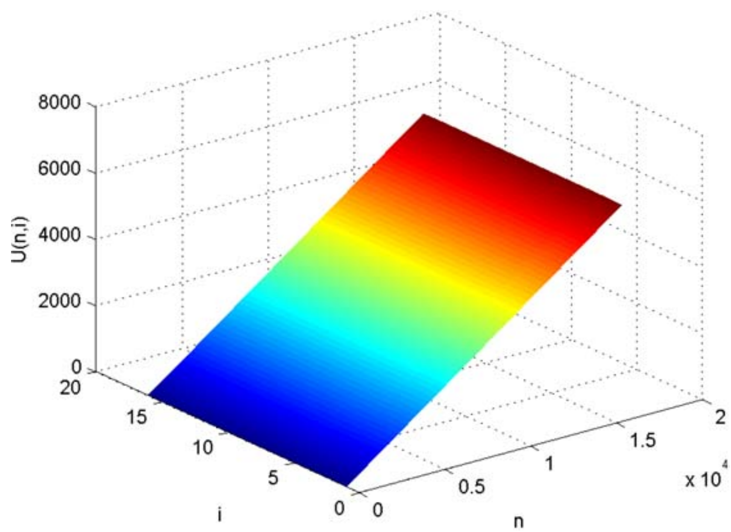

Figure 8. Evolution of the discrete solution, source $f(u) = \lambda(u(\frac{1}{2}, t))^2$, $\lambda = 100$, $u(x, 0) = \sin(\pi x)$ (explicit scheme).

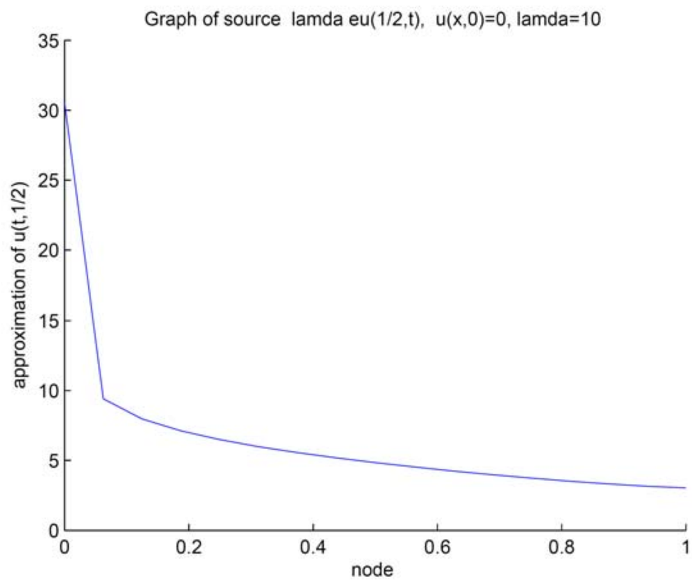

Figure 9. Source $f(u) = \lambda e^{u(\frac{1}{2}, t)}$, $\lambda = 10$, $u(x, 0) = 0$.

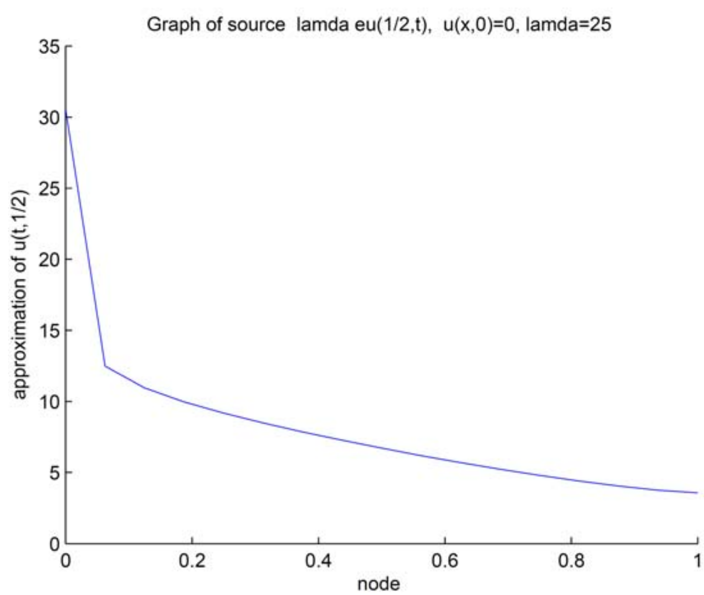

Figure 10. Source $f(u) = \lambda e^{u(\frac{1}{2}, t)}$, $\lambda = 25$, $u(x, 0) = 0$.

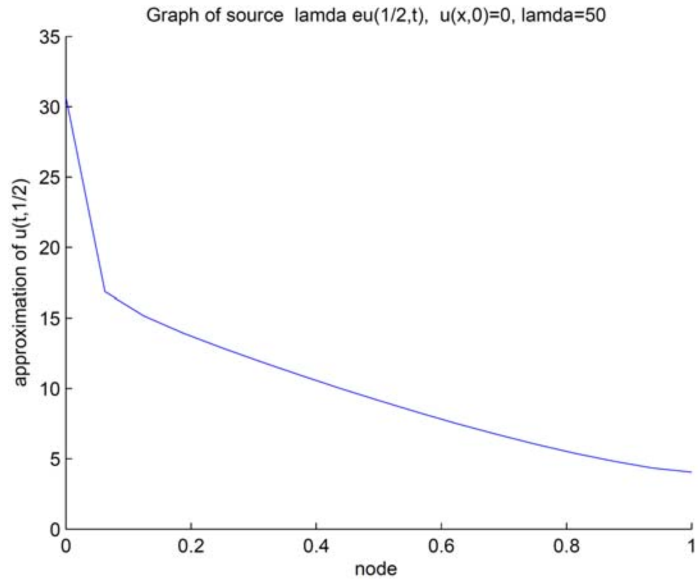

Figure 11. Source $f(u) = \lambda e^{u(\frac{1}{2}, t)}$, $\lambda = 50$, $u(x, 0) = 0$.

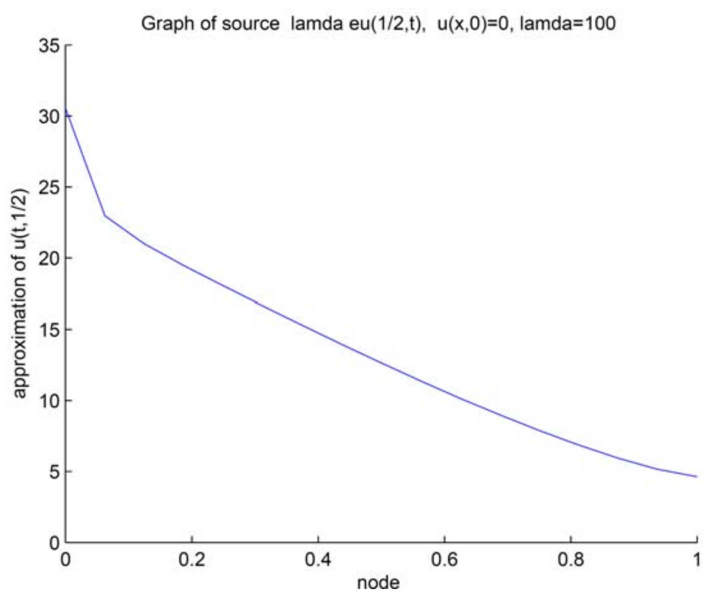

Figure 12. Source $f(u) = \lambda e^{u(\frac{1}{2}, t)}$, $\lambda = 100$, $u(x, 0) = 0$.

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