Research and Communications in Mathematics and Mathematical Sciences
Vol. 5, Issue 2, 2015, Pages 105-119
ISSN 2319-6939
Published Online on September 8, 2015

http://jyotiacademicpress.net

AN IDEAL-BASED δ-ZERO-DIVISOR GRAPH OF COMMUTATIVE RINGS

S. EBRAHIMI ATANI and Z. EBRAHIMI SARVANDI

Faculty of Mathematical Sciences
University of Guilan
P. O. Box 1914
Rasht
Iran
e-mail: ebrahimi@guilan.ac.ir
zahra_2006_ebrahimi@yahoo.com

Abstract

Let I be an ideal of a commutative ring R with an ideal expansion δ and $Z_{\delta}(I) = \{x \in R \setminus \delta(I) : xy \in \delta(I)$ for some $y \in R \setminus \delta(I)\}$. In this paper, we introduce and investigate an ideal-based δ-zero-divisor graph of R, denoted by $\Gamma_{\delta}(I)$. It is the (undirected) graph with vertices $Z_{\delta}(I)$, and for distinct $x, y \in Z_{\delta}(I)$, the vertices x and y are adjacent if and only if $xy \in \delta(I)$. An ideal-based zero-divisor graph of a commutative ring, denoted by $\Gamma_I(R)$, is the undirected graph with vertices $\{x \in R \setminus I : xy \in I$ for some $y \in R \setminus I\}$, and distinct vertices x and y are adjacent if and only if $xy \in I$. This is due to Redmond [9]. In the case $\delta =$ the identity function, $\Gamma_{\delta}(I) = \Gamma_I(R)$.

2010 Mathematics Subject Classification: 05C40, 05C45, 16P10.
Keywords and phrases: δ-zero-divisor, graph, ideal-based.
Communicated by Bao-Xuan Zhu.
Received May 11, 2015

1. Introduction

In the literature, there are many papers on assigning a graph to a ring (see, for example, [1-6, 9]). Among the most interesting graphs are the zero-divisor graphs, because these involve both ring theory and graph theory. By studying these graphs, we can gain a broader insight into the concepts and properties that involve both graphs and rings. The concept of the zero-divisor graph of commutative ring R was first introduced by Beck [6], where he was mainly interested in colourings. In his work, all elements of the ring were vertices of the graph. This investigation of colorings of a commutative ring was then continued by Anderson and Naseer in [1]. Let $Z(R)$ be the set of zero-divisors of R. In [2], Anderson and Livingston associated a graph $\Gamma(R)$ to R, with vertices $Z(R)\setminus\{0\}$, the set of nonzero-divisors of R, and for distinct $x, y \in Z(R)\setminus\{0\}$, the vertices x and y are adjacent if and only if $xy = 0$.

Throughout this work, all rings are assumed to be commutative with nonzero identity. Let R be a commutative ring with $Id(R)$ its set of ideals. An ideal expansion is a function δ which assigns to each ideal I of R another ideal $\delta(I)$ of R, such that $I \subseteq \delta(I)$, and $J \subseteq L$ implies $\delta(J) \subseteq \delta(L)$ for all ideals I, J, and L of R (so $\delta_0(I) = I$ and $\delta_1(I) = I$ for every ideal I of R are ideal expansions [8]). Let $Z_\delta(R)^*$ be the set of δ-zero-divisor elements in R that are not elements of $\delta(\{0\})$. We say that $x \in R$ is δ-zero-divisor if $xy \in \delta(\{0\})$ for some $y \notin \delta(\{0\})$ [7]. In [7], the present authors introduced the δ-zero-divisor graph of a commutative ring R, denoted by $\Gamma_\delta(I)$. It is the graph with vertices all elements of $Z_\delta(R)^*$, and two distinct vertices $x, y \in Z_\delta(R)^*$ are adjacent if and only if $xy \in \delta(\{0\})$. Clearly, if δ is the identity function, then δ-zero-divisor elements are exactly the ordinary zero-divisor elements, then $\Gamma_\delta(I) = \Gamma(R)$. In this paper, we will generalize this notion by replacing elements whose product lies in $\delta(\{0\})$ with elements whose product lies in some ideal $\delta(I)$ of R. Indeed, we define an undirected graph $\Gamma_\delta(I)$ with vertices

$Z_{\delta}(I) = \{x \in R \setminus \delta(I) : xy \in \delta(I)$ for some $y \in R \setminus \delta(I)\}$, where distinct vertices x and y are adjacent if and only if $xy \in \delta(I)$. This definition was motivated from [8]. Here is a brief summary of our paper. A number of basic results concerning the ideal-based δ-zero-divisor graph of commutative rings are given. For example, we show that $\Gamma_{\delta}(I)$ is connected with $\text{diam}(\Gamma_{\delta}(I)) \leq 3$. Furthermore, if $\Gamma_{\delta}(I)$ contains a cycle, then $\text{gr}(\Gamma_{\delta}(I)) \leq 4$. Also, we study $\Gamma_{\delta}(I)$ for some classes of rings which generalize valuation domains to the context of rings with δ-zero-divisors. In these cases, we completely characterize the diameter and girth of the graph $\Gamma_{\delta}(I)$ of such rings (see Sections 2, 3, 4).

In order to make this paper easier to follow, we recall in this section various notions which will be used in the sequel. For a graph Γ, let $E(T)$ and $V(T)$ denote the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two distinct vertices. At the other extreme, we say that Γ is totally disconnected if no two vertices of Γ are adjacent. The distance between two distinct vertices a and b, denoted by $d(a, b)$, is the length of a shortest path connecting them ($d(a, a) = 0$ and $d(a, b) = \infty$ if there is no such path). The diameter of graph Γ, denoted by $\text{diam}(\Gamma)$, is equal to $\sup\{d(a, b) : a, b \in V(T)\}$. A graph is complete if it is connected with diameter less than or equal to one. The girth of graph Γ, denoted by $\text{gr}(\Gamma)$ is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise, $\text{gr}(\Gamma) = \infty$.

2. Some Basic Properties of $\Gamma_{\delta}(I)$

A commutative ring R with an ideal expansion δ is called a δ-domain if whenever $ab \in \delta(\{0\})$ $(a, b \in R)$, then either $a \in \delta(\{0\})$ or $b \in \delta(\{0\})$ [7]. Given an expansion δ of a ideals, a proper ideal I of a ring R is called δ-primary if whenever $ab \in I$ and $a \notin I$, then $b \in \delta(\{0\})$ [8].

Remark 2.1. Let R be a commutative ring with identity. Clearly, the function δ, where $\delta(I) = R$ for every ideal I of R, is an expansion of ideals. Then $\Gamma_\delta(I) = \emptyset$. So throughout this paper, we shall assume unless otherwise stated, that $\delta(I) \neq R$ for every proper ideal I of R.

Theorem 2.2. *Let I be an ideal of a commutative ring R with an ideal expansion* δ. *Then the following hold*:

(1) *If* $I = \{0\}$, *then* $\Gamma_\delta(I) = \Gamma(R)$.

(2) *If I is a* δ-*primary ideal of R, then* $\Gamma_\delta(I) = \emptyset$.

(3) $\Gamma_\delta(I) = \emptyset$ *if and only if* $\delta(I)$ *is a prime ideal of R.*

Proof. (1) It is clear.

(2) Let I be a δ-primary ideal of R. Then $xy \in I$ implies $x \in I \subseteq \delta(I)$ or $y \in \delta(I)$. Hence, the vertex set of $\Gamma_\delta(I)$ is empty.

(3) Let $\delta(I)$ be a prime ideal of R. Then $xy \in \delta(I)$ implies $x \in \delta(I)$ or $y \in \delta(I)$. Hence, the vertex set of $\Gamma_\delta(I)$ is empty. Conversely, suppose that $\Gamma_\delta(I) = \emptyset$. Therefore, if $x \in R \setminus \delta(I)$ and $xy \in \delta(I)$ for some $y \in R$, we must have $y \in \delta(I)$ (otherwise, x is a vertex of $\Gamma_\delta(I)$). Thus $\delta(I)$ is a prime ideal of R. □

By an argument like that in [9, Theorem 2.4], we have the following theorem:

Theorem 2.3. *Let R be a commutative ring R with an ideal expansion* δ. *Then* $\Gamma_\delta(I)$ *is connected and* $\operatorname{diam}(\Gamma_\delta(I)) \leq 3$.

Proof. Let $x, y \in Z_\delta(I)$ be distinct. If $xy \in \delta(I)$, then $d_\delta(x, y) = 1$. So we may assume that $xy \notin \delta(I)$. If $x^2, y^2 \in \delta(I)$, then $x - xy - y$ is a path of length 2 since $\delta(I)$ is an ideal; thus $d_\delta(x, y) = 2$. If $x^2 \in \delta(I)$ and $y^2 \notin \delta(I)$, then there exists $b \in Z_\delta(I) \setminus \{x, y\}$ with $by \in \delta(I)$. If $bx \in \delta(I)$, then $x - b - y$ is a path of length 2. If $bx \notin \delta(I)$, then $x - bx - y$ is a path of length 2. In either case, $d_\delta(x, y) = 2$. A similar argument holds if $y^2 \in \delta(I)$ and $x^2 \notin \delta(I)$. Thus, we may assume that $xy, y^2, x^2 \notin \delta(I)$. Hence there exist $a, b \in Z_\delta(I) \setminus \{x, y\}$ with $ax, by \in \delta(I)$. If $a = b$, then $x - a - y$ is a path of length 2. So we may assume that $a \neq b$. If $ab \in \delta(I)$, then $x - a - b - y$ is a path of length 3, and hence $d_\delta(x, y) \leq 3$. If $ab \notin \delta(I)$, then $x - ab - y$ is a path of length 2; thus $d_\delta(x, y) = 2$. Hence $d_\delta(x, y) \leq 3$, and thus $\operatorname{diam}(\Gamma_\delta(I)) \leq 3$. □

Theorem 2.4. *Let R be a commutative ring R with an ideal expansion* δ. *If* $\Gamma_\delta(I)$ *contains a cycle, then* $\operatorname{gr}(\Gamma_\delta(I)) \leq 4$.

Proof. Suppose not. Assume that $\Gamma_\delta(I)$ contains a cycle $x_0 - x_1 - \cdots - x_n - x_0$ such that $\operatorname{gr}(\Gamma_\delta(I)) > 4$ (so $n \geq 4$), $x_i x_j \notin \delta(I)$ for all $i, j \in \{0, 1, \cdots, n\}$ with $|i - j| \geq 2$ and $x_i x_{i+1} \in \delta(I)$. We split the proof into three cases.

Case 1. $x_1 x_{n-1} \neq x_0$ and $x_1 x_{n-1} \neq x_n$. Then $x_0 x_n \in \delta(I)$ and $x_1 x_{n-1} \notin \delta(I)$ since $|n - 2| \geq 2$, and we have $x_0 x_1 x_{n-1} \in \delta(I)$ since $\delta(I)$ is an ideal of R. Similarly, $x_1 x_{n-1} x_n \in \delta(I)$. So $x_0 - x_1 x_{n-1} - x_n - x_0$ is a cycle of length 3.

Case 2. $x_1 x_{n-1} = x_0$. Then $x_0^2 = x_0 x_1 x_{n-1} \in \delta(I)$. We claim that there is an element y of R such that $x_0 y \notin \delta(I)$ and $x_0 y \neq x_0$. Suppose not. Then for every $y \in R$, either $x_0 y \in \delta(I)$ or $x_0 y = x_0$. Take $y = x_3$. Then by assumption, $x_0 x_3 \notin \delta(I)$ and $x_0 x_3 \neq x_0$ (if $x_0 x_3 = x_0$, then $x_0 x_2 \in \delta(I)$, a contradiction), which is a contradiction. So, there is a $y \in R$ such that $x_0 y \notin \delta(I)$ and $x_0 y \neq x_0$. If $x_0 y \neq x_1$, then $x_0 x_1 y \in \delta(I)$; thus we have $x_0 - x_1 - x_0 y - x_0$ is a cycle. Similarly, if $x_0 y = x_1$, then $x_0 y \neq x_n$ and $x_n x_0 y = x_0 y x_0 \in \delta(I)$. Thus $x_0 - x_n - x_0 y - x_0$ is a 3-cycle in $\Gamma_\delta(I)$.

Case 3. $x_1x_{n-1} = x_n$. This necessarily forces $x_n^2 \in \delta(I)$ and there exists an element y of R such that $x_ny \notin \delta(I)$ and $x_ny \neq x_n$. If $x_ny \neq x_n$, then $x_n - x_ny - x_{n-1} - x_n$ is a cycle of length 3, and if $x_ny = x_n$, then $x_n - x_0 - x_ny - x_n$ is a 3-cycle in $\Gamma_\delta(I)$. Thus every case leads to a contradiction. □

3. Chained Rings

In this section, we continue the investigation of $\Gamma_\delta(I)$ when R a commutative chained ring with an ideal expansion δ. We say that a ring R is a chained ring if the (principal) ideals of R are linearly ordered (by inclusion), equivalently, if either $x|y$ or $y|x$ for all $x, y \in R$.

Definition 3.1. Let I be an ideal of a commutative ring R with an ideal expansion δ. An $x \in R$ is said to be δ_I-potent if there exists a positive integer n such that $a^n \in \delta(I)$.

One can easily show that if δ is the identity mapping and $I = \{0\}$, then δ_I-potent elements are exactly the ordinary nilpotent elements. The set of all δ_I-potent elements of R is denoted by $\text{nil}_\delta(I)$. Clearly, $I \subseteq \delta(I) \subseteq \text{nil}_\delta(I)$. Set $\text{nil}_\delta(I)^* = \text{nil}_\delta(I) \setminus \delta(I)$.

Proposition 3.2. *Let I be an ideal of a commutative ring R with an ideal expansion δ. Then the following hold*:

(1) $\text{nil}_\delta(I)$ *is an ideal of R with* $\text{nil}_\delta(I)^* \subseteq Z_\delta(I)$.

(2) *If $Z_\delta(I)$ is an ideal of R, then $Z_\delta(I)$ is prime.*

Proof. (1) Let $x, y \in \text{nil}_\delta(I)$ and $r \in R$. Then $x^n, y^m \in \delta(I)$ for some positive integers n, m. So, there are integers $a_0, a_1, \ldots, a_{n+m}$ such that

$$(x - y)^{n+m} = a_0x^{n+m} + \cdots + a_mx^ny^m + \cdots + a_{n+m}y^{n+m} \in \delta(I),$$

and hence $x - y \in \mathrm{nil}_\delta(I)$. Since $(rx)^n = r^n x^n \in \delta(I)$, we conclude that $rx \in \mathrm{nil}_\delta(I)$. Thus $\mathrm{nil}_\delta(I)$ is an ideal of *R*. Finally, let $x \in \mathrm{nil}_\delta(I)^*$. Let n $(n \geq 2)$ be the least positive integer such that $x^n \in \delta(I)$. As $x^{n-1} \notin \delta(I)$, $x \notin \delta(I)$ and $xx^{n-1} \in \delta(I)$, we conclude that $x^n \in Z_\delta(I)$, as required.

(2) Let $x, y \in R$ such that $xy \in Z_\delta(I)$. Then there exists $z \in R$ such that $z \notin \delta(I)$ and $xyz \in \delta(I)$. Therefore, if $yz \in \delta(I)$, then $y \in Z_\delta(I)$. If $yz \notin \delta(I)$, then $x \in Z_\delta(I)$. Thus $Z_\delta(I)$ is a prime ideal of *R*. □

Theorem 3.3. *Let I be an ideal of a commutative ring R with an ideal expansion* δ. *Then the following hold*:

(1) *If* $x \in \mathrm{nil}^*_\delta(I)$ *and* $y \in Z_\delta(I)$, *then* $d_\delta(x, y) = 1$ *in* $\Gamma_\delta(I)$.

(2) *Let* $x \in Z_\delta(I) \setminus \mathrm{nil}_\delta(I)$, *and let* $y \in \mathrm{nil}_\delta(I)^*$ *such that* $x|zy^n$ *for some positive integers n and* $z \in R \setminus Z_\delta(I)$. *Then* $d_\delta(x, y) \leq 2$ *in* $\Gamma_\delta(I)$.

(3) *If* $\mathrm{nil}_\delta(I)$ *is a prime ideal of R*, *then* $V(\Gamma_\delta(I)) \setminus \mathrm{nil}_\delta(I)$ *is totally disconnected*.

Proof. (1) We may assume that $x \neq y$ and $xy \notin \delta(I)$. Since $y \in Z_\delta(I)$ and $xy \notin \delta(I)$, there is a $z \in Z_\delta(I) \setminus \{x\}$ such that $zy \in \delta(I)$. Let n be the least positive integer such that $x^n z \in \delta(I)$ since $x \in \mathrm{nil}_\delta(I)^*$. If $n = 1$, then $x - z - y$ is a path of length 2 from *x* to *y*. If $n \geq 2$, then $x - x^{n-1}z - y$ is a path between *x* and *y*. Thus $d_\delta(x, y) \leq 2$.

(2) We may assume that $x \neq y$ and $xy \notin \delta(I)$. Since $x \in Z_\delta(I) \setminus \mathrm{nil}_\delta(I)$ and $xy \notin \delta(I)$, there is a $w \in Z_\delta(I) \setminus \{x, y\}$ such that $xw \in \delta(I)$. Since $x|zy^n$ with $z \notin \delta(I)$, we get $zy^n w \in \delta(I)$. If $y^n w \notin \delta(I)$, then $z \in Z_\delta(I)$, a contradiction. So, we conclude that $y^n w \in \delta(I)$. Let *m* be

the least positive integer such that $wy^m \in \delta(I)$. If $m = 1$, then $x - w - y$ is a path of length 2 from x to y. If $m \geq 2$, then $x - y^{m-1}w - y$ is a path between x and y. Thus $d_\delta(x, y) \leq 2$ in $\Gamma_\delta(I)$.

(3) Assume that $\mathrm{nil}_\delta(I)$ is a prime ideal of R, and let x and y be two distinct elements of $V(\Gamma_\delta(I)) \setminus \mathrm{nil}_\delta(I)$. Suppose that $xy \in \delta(I)$; hence either x or y belong to $\mathrm{nil}_\delta(I)$, which is a contradiction. □

Compare the next result with [3, Lemma 4.2].

Proposition 3.4. *Let I be an ideal of a commutative chained ring R with an ideal expansion δ, $N_\delta(I) = \{x \in R : x^2 \in \delta(I)\}$, and $x, y \in R$.*

(1) *If $xy \in \delta(I)$, then either $x \in N_\delta(I)$ or $y \in N_\delta(I)$.*

(2) *If $x, y \in N_\delta(I)$, then $xy \in \delta(I)$.*

(3) *If $x, y \in Z_\delta(I) \setminus N_\delta(I)$, then $xy \notin \delta(I)$.*

(4) *If $x \in Z_\delta(I)$, then $xy \in \delta(I)$ for some $y \in N_\delta(I)^*$, where $N_\delta(I)^* = N_\delta(I) \setminus \delta(I)$.*

(5) *If $x_1, x_2, \cdots, x_n \in \delta(I)$, then there is a $y \in N_\delta(I)^*$ such that $x_i y \in \delta(I)$ for every integer i, $1 < i < n$.*

(6) *$N_\delta(I)$ is an ideal of R.*

(7) *$N_\delta(I)$ is a prime ideal of R if and only if $N_\delta(I) = \mathrm{nil}_\delta(I)$.*

Proof. (1) We may assume that $x | y$. Then $y = ax$ for some $a \in R$; hence $y^2 = axy \in \delta(I)$. Thus $y \in N_\delta(I)$.

(2) We may assume that $x | y$. Then $y = ax$ for some $a \in R$; hence $xy = ax^2 \in \delta(I)$.

(3) Follows from case (1) above.

(4) If $x \in N_\delta(I)$. then let $y = x$. If $x \in Z_\delta(I) \setminus N_\delta(I)$, then there exists $y \in R$ with $y \notin \delta(I)$ such that $xy \in \delta(I)$. By case (3) above, we must have $y \in N_\delta(I)^*$.

(5) Since R is a chained ring, there is an integer $j, 1 \leq j \leq n$, such that $x_j | x_i$ for all $i, 1 \leq i \leq n$. By case (4) above, there exists $y \in N_\delta(I)^*$ such that $x_j y \in \delta(I)$; hence $x_i y \in \delta(I)$ for all $i, 1 \leq i \leq n$.

(6) Let $x, y \in N_\delta(I)$ and $r \in R$. Then $r^2 x^2 = (rx)^2 \in \delta(I)$; so $rx \in N_\delta(I)$. Now we need only show that $x + y \in N_\delta(I)$. By assumption, $x^2, y^2 \in \delta(I)$, and $xy \in \delta(I)$ by part (2); so $(x + y)^2 = x^2 + y^2 + 2xy \in \delta(I)$. Thus $N_\delta(I)$ is an ideal of R.

(7) Let $N_\delta(I)$ is a prime ideal of R. Since the inclusion $N_\delta(I) \subseteq \text{nil}_\delta(I)$ is clear, we will prove the reverse inclusion. Let $x \in \text{nil}_\delta(I)$. Then $x^n \in \delta(I)$ for some positive integer n. Let m $(m \geq 3)$ be the least positive integer such that $x^m \in \delta(I)$, and let $y = x^m$. Then $y^2 = x^{2m} \in \delta(I)$; hence $N_\delta(I)$ prime gives $x \in N_\delta(I)$, and so we have equality. Conversely, assume that $xy \in N_\delta(I)$ for some $x, y \in R$. Then by part (1) above, either $x^2 \in N_\delta(I) = \text{nil}_\delta(I)$ or $y^2 \in N_\delta(I) = \text{nil}_\delta(I)$; thus either $x \in N_\delta(I)$ or $y \in N_\delta(I)$, as needed. □

Theorem 3.5. *Let I be an ideal of a commutative chained ring R with an ideal expansion* δ *and* $N_\delta(I) = \{x \in R : x^2 \in \delta(I)\}$. *Then* $V(\Gamma_\delta(I)) \setminus N_\delta(I)$ *is totally disconnected.*

Proof. Apply Proposition 3.4. □

Compare the next theorem with [3, Lemma 4.5].

Theorem 3.6. *Let I be an ideal of a commutative chained ring R with an ideal expansion* δ. *Then* $\text{diam}(\Gamma_\delta(I)) \leq 2$.

Proof. If $|Z_\delta(I)| = 1$, then $\mathrm{diam}(\Gamma_\delta(I)) = 0$. So we may assume that $|Z_\delta(I)| \geq 2$. Let $x, y \in Z_\delta(I)$ with $x \neq y$. If $x, y \in N_\delta(I)$, then $xy \in \delta(I)$ by Proposition 3.4 (2), and thus $d_\delta(x, y) = 1$. If $x \in N_\delta(I)$ and $y \notin N_\delta(I)$, then $yz \in \delta(I)$ for some $z \in N_\delta(I)^*$ by Proposition 3.4 (4), and $xz \in \delta(I)$ by Proposition 3.4 (2); hence $x - z - y$ is a path from x to y. Thus $d_\delta(x, y) \leq 2$. Finally, let $x, y \notin N_\delta(I)$. Then $xz, yz \in \delta(I)$ by Proposition 3.4 (5). Thus $d_\delta(x, y) \leq 2$, and hence $\mathrm{diam}(\Gamma_\delta(I)) \leq 2$. □

Theorem 3.7. *Let I be an ideal of a commutative chained ring R with an ideal expansion* δ. *If* $Z_\delta(I) \neq \{0\}$, *then exactly one of the following three cases must occur*:

(1) $|Z_\delta(I)| = 1$. *In this case*, $\mathrm{diam}(\Gamma_\delta(I)) = 0$.

(2) $|Z_\delta(I)| \geq 2$ *and* $N_\delta(I) = Z_\delta(I)$. *In this case*, $\mathrm{diam}(\Gamma_\delta(I)) = 1$.

(3) $|Z_\delta(I)| \geq 2$ *and* $N_\delta(I) \subset Z_\delta(I)$. *In this case*, $\mathrm{diam}(\Gamma_\delta(I)) = 2$.

Proof. This follows directly from Proposition 3.4 and Theorem 3.6. □

Theorem 3.8. *Let I be an ideal of a commutative chained ring R with an ideal expansion* δ. *Then exactly one of the following four cases must occur*:

(1) $|N_\delta(I)| = 1$. *In this case*, $\mathrm{gr}(\Gamma_\delta(I)) = \infty$.

(2) $|N_\delta(I)| = 2$ *and* $N_\delta(I) = Z_\delta(I)$. *In this case*, $\mathrm{gr}(\Gamma_\delta(I)) = \infty$.

(3) $|N_\delta(I)| = 2$ *and* $N_\delta(I) \subset Z_\delta(I)$. *In this case*, $\mathrm{gr}(\Gamma_\delta(I)) = 3$.

(4) $|N_\delta(I)| \geq 3$. *In this case*, $\mathrm{gr}(\Gamma_\delta(I)) = 3$.

Proof. (1) Let $N_\delta(I)^* = \{x\}$. If $N_\delta(I)^* = Z_\delta(I)$, then $\mathrm{gr}(\Gamma_\delta(I)) = \infty$. If $N_\delta(I)^* \subset Z_\delta(I)$, then $\Gamma_\delta(I)$ is a star graph with center x by parts (3) and (4) of Proposition 3.4. Thus $\mathrm{gr}(\Gamma_\delta(I)) = \infty$.

(2) By hypothesis, $|Z_\delta(I)| = 2$; hence $\mathrm{gr}(\Gamma_\delta(I)) = \infty$.

(3) Let $N_\delta(I)^* = \{x, y\}$. If $y \neq -x$, then $(x+y)^2 = x^2 + y^2 + 2xy \in \delta(I)$ (note that by Proposition 3.4 (2), $xy \in \delta(I)$; so $(x+y)^2 \in \delta(I)$. It follows that $x + y \in N_\delta(I)^*$. Thus, either $x + y = x$ or $x + y = y$, a contradiction. So, we may assume that $y \neq -x$. If $z \in Z_\delta(I) \setminus N_\delta(I)^*$, then $x - y - z - x$ is a triangle since by Proposition 3.2 (4), $xz, yz \in \delta(I)$; so $\mathrm{gr}(\Gamma_\delta(I)) = 3$.

(4) If $|N_\delta(I)| \geq 3$, then $\mathrm{gr}(\Gamma_\delta(I)) = 3$ by Proposition 3.4 (2). □

4. δ-Domainlike Rings

In this section, we investigate the properties of $\Gamma_\delta(I)$, where R is a δ-domainlike ring with an ideal expansion δ. We say that a ring R is δ-domainlike rings if $Z_\delta(I) = \mathrm{nil}_\delta(I)^*$.

Proposition 4.1. *Let I be an ideal of a commutative ring R with an ideal expansion δ. If $x, y \in \mathrm{nil}_\delta(I)^*$ are distinct with $xy \notin \delta(I)$, then there is a path of length 2 from x to y in $\mathrm{nil}_\delta(I)^* \subseteq Z_\delta(I)$.*

Proof. Since $xy \notin \delta(I)$ and $x \in \mathrm{nil}_\delta(I)^*$, let $n\,(n \geq 2)$ be the least positive integer such that $x^n y \in \delta(I)$. Also, since $x^{n-1}y \notin \delta(I)$ and $y \in \mathrm{nil}_\delta(I)^*$, let $m\ (m \geq 2)$ be the least positive integer such that $x^{n-1}y^m \in \delta(I)$. Then $x^{n-1}y^{m-1} \in \mathrm{nil}_\delta(I)^*$. Thus $x - x^{n-1}y^{m-1} - y$ is a path of length 2 from x to y in $\mathrm{nil}_\delta(I)^*$. □

Theorem 4.2. *Let I be an ideal of a commutative δ-domainlike ring R with an ideal expansion δ. Then $\mathrm{diam}(\Gamma_\delta(I)) \leq 2$.*

Proof. Apply Proposition 4.1. □

Lemma 4.3. *Let I be an ideal of a commutative ring R with an ideal expansion* δ. *If* $|Z_\delta(I)| \geq 3$ *and there exist* $a, b \in Z_\delta(I)$ *such that* $ab, a^2, b^2 \in \delta(I)$, *then* $\mathrm{gr}(\Gamma_\delta(I)) = 3$.

Proof. By assumption, if $\mathrm{diam}(\Gamma_\delta(I)) = 1$, then there exist x_1, x_2, and x_3 in $Z_\delta(I)$ such that $x_1x_2, x_2x_3, x_3x_1 \in \delta(I)$; hence $x_1 - x_2 - x_3 - x_1$ is a cycle of length 3. So, we may assume that $\mathrm{diam}(\Gamma_\delta(I)) > 1$. Then there exists some $c \in Z_\delta(I) \setminus \{a, b\}$ such that (without loss of generality) $ac \in \delta(I)$ and $bc \notin \delta(I)$. Since $\delta(I)$ is an ideal of R, we have $a(a + b)$, $b(a + b) \in \delta(I)$. Now show that $(a + b) \notin \delta(I)$. Suppose not. Since $c(a + b) \in \delta(I)$, we have $bc = bc + ac - ac = c(a + b) - ac \in \delta(I)$, and this is a contradiction. So $a + b \in Z_\delta(I)$. Thus $a - b - a + b - a$ is a cycle of length 3, as required. □

Lemma 4.4. *Let I be an ideal of a commutative ring R with an ideal expansion* δ, *and let* $a, b \in Z_\delta(I)$ *such that* $ab, a^3, b^3 \in \delta(I)$ *and* $a^2, b^2 \notin \delta(I)$. *Then* $\mathrm{gr}(\Gamma_\delta(I)) = 3$.

Proof. By hypothesis, $ab^2 \in \delta(I)$, and $b^2 \neq a$ (otherwise, $b^4 = a^2 \in \delta(I)$, a contradiction). Similarly, $b^2 \neq b$ and $b^2 \neq 0$. Thus $b - a - b^2 - b$ is a 3-cycle in $\Gamma_\delta(I)$, and hence $\mathrm{gr}(\Gamma_\delta(I)) = 3$. □

Lemma 4.5. *Let I be an ideal of a commutative ring R with an ideal expansion* δ, *and let* $a \in Z_\delta(I)$ *such that* $a^n \in \delta(I)$ *and* $a^{n-1} \notin \delta(I)$ *for some* $n \geq 4$. *Then* $\mathrm{gr}(\Gamma_\delta(I)) = 3$.

Proof. Let $a \in Z_\delta(I)$ such that $a^n \in \delta(I)$ and $a^{n-1} \notin \delta(I)$ for some $n \geq 5$. If $k > n$, then $a^k \in \delta(I)$. Then $a^{n-3} - a^{n-2} - a^{n-1} - a^{n-3}$ is a 3-cycle in $\Gamma_\delta(I)$, and hence $\mathrm{gr}(\Gamma_\delta(I)) = 3$. If there exists $a \in Z_\delta(I)$ with $a^4 \in \delta(I)$ and $a^3 \notin \delta(I)$, then consider the element $a^2 + a^3$. If $a^2 + a^3 = a^3$, then $a^3 \in \delta(I)$, a contradiction. Thus $a^2 + a^3 \neq a^3$.

Similarly, $a^2 + a^3 \neq a^2$. Clearly, $a^2 + a^3 \neq 0$ and $a^2 \neq 0$. If $a^2 + a^3 \in \delta(I)$, then $a^3 + a^4 \in \delta(I)$; hence $a^3 \in \delta(I)$, a contradiction. So, $a^2 + a^3 \notin \delta(I)$. Thus, we get the cycle $a^2 - a^3 - (a^2 + a^3) - a^2$ with length 3. Thus $\mathrm{gr}(\Gamma_\delta(I)) = 3$. □

Theorem 4.6. *Let I be an ideal of a commutative δ-domainlike ring R with an ideal expansion δ. If $\Gamma_\delta(I)$ contains a cycle, then* $\mathrm{gr}(\Gamma_\delta(I)) = 3$.

Proof. Since $\Gamma_\delta(I)$ contains a cycle, $|Z_\delta(I)| \geq 3$ and $\mathrm{diam}(\Gamma_\delta(I)) \neq 0$. So by Theorem 4.2, either $\mathrm{diam}(\Gamma_\delta(I)) = 1$ or $\mathrm{diam}(\Gamma_\delta(I)) = 2$. If $\mathrm{diam}(\Gamma_\delta(I)) = 1$, then there exist x_1, x_2, and x_3 in $Z_\delta(I)$ such that x_1x_2, x_2x_3, $x_3x_1 \in \delta(I)$; hence $x_1 - x_2 - x_3 - x_1$ is a cycle with length 3, and so $\mathrm{gr}(\Gamma_\delta(I)) = 3$.

For reminder of the proof, we will assume that $\mathrm{diam}(\Gamma_\delta(I)) = 2$. As $\Gamma_\delta(I)$ contains a cycle and $\mathrm{diam}(\Gamma_\delta(I)) = 2$, we may assume that $|Z_\delta(I)| \geq 4$, let $a \in Z_\delta(I)$. Since $Z_\delta(I) = \mathrm{nil}_\delta(I)^*$, there exists a positive integer $n\,(n \geq 2)$ such that $a^n \in \delta(I)$, but $a^{n-1} \notin \delta(I)$. If $n \geq 4$, then $\mathrm{gr}(\Gamma_\delta(I)) = 3$ by Lemma 4.5. Now suppose that $a^3 \in \delta(I)$ for all $a \in Z_\delta(I)$. Since $\mathrm{diam}(\Gamma_\delta(I)) = 2$, there exist a, b, and c in $Z_\delta(I)$ such that $d_\delta(a, b) = 2$ and ac, $bc \in \delta(I)$. We split the proof into three cases.

Case 1. $a^2 \in \delta(I)$ and $b^2 \notin \delta(I)$. If a^2, $c^2 \in \delta(I)$, then by Lemma 4.3, $\mathrm{gr}(\Gamma_\delta(I)) = 3$. So, we may assume that $c^2 \notin \delta(I)$. Since $b^2 \notin \delta(I)$, b^3, $c^3 \in \delta(I)$, Lemma 4.4 gives $\mathrm{gr}(\Gamma_\delta(I)) = 3$.

Case 2. a^2, $b^2 \in \delta(I)$. If $c^2 \in \delta(I)$, then again Lemma 4.3 gives $\mathrm{gr}(\Gamma_\delta(I)) = 3$. So, we may assume that $c^2 \notin \delta(I)$. Since $c^2 \notin \delta(I)$, we get $c \notin \delta(I)$; hence $c^2 \in Z_\delta(I)$ (note that $c^3 \in \delta(I)$). Clearly, either

$c^2 \in Z_\delta(I) \setminus \{a\}$ or $c^2 \in Z_\delta(I) \setminus \{b\}$ (otherwise, $c^2 = a = b$, a contradiction). If $c^2 \in Z_\delta(I) \setminus \{a\}$, then $c - c^2 - a - c$ is a cycle of length 3. If $c^2 \in Z_\delta(I) \setminus \{b\}$, then $c - c^2 - b - c$ is a cycle of length 3. Hence in this case, $\mathrm{gr}(\Gamma_\delta(I)) = 3$.

Case 3. $a^2, b^2 \notin \delta(I)$. If $c^2 \notin \delta(I)$, then $\mathrm{gr}(\Gamma_\delta(I)) = 3$ by Lemma 4.4. So, we may assume that $c^2 \in \delta(I)$. If there exists an $x \in Z_\delta(I)$ such that $c \neq x, x^2 \in \delta(I)$, and $x - a - c$ or $x - b - c$, then by an identical argument as in Case 2, we have $\mathrm{gr}(\Gamma_\delta(I)) = 3$. Since $Z_\delta(I) = \mathrm{nil}_\delta(I)$ is an ideal of R by Proposition 3.2, we have $c + c \in Z_\delta(I)$. Since $c^2 \in \delta(I)$, we have $(c + c)^2 \in \delta(I)$. Clearly if $c + c \neq 0$, let $c + c \neq c$, we get $\mathrm{gr}(\Gamma_\delta(I)) = 3$. Now suppose $c + c = 0$. If either a^2 or b^2 is not equal to c, let $x = a^2$ or $x = b^2$, and again we get $\mathrm{gr}(\Gamma_\delta(I)) = 3$. So, we may assume that $a^2 = b^2 = c$. By hypothesis, $|Z_\delta(I)| \geq 4$, $\mathrm{diam}(\Gamma_\delta(I)) = 2$, and $x^3 \in \delta(I)$ for all $x \in Z_\delta(I)$. So there exists $d \in Z_\delta(I)$ such that either $da \in \delta(I), db \in \delta(I)$, or $dc \in \delta(I)$ (otherwise, $\Gamma_\delta(I)$ is not connected, and this is a contradiction). If $ad \in \delta(I)$, if $dc \in \delta(I)$ and $d^2 \in \delta(I)$ or if $dc \in \delta(I)$ and $d^2 \neq c$, we can appeal to previous cases to obtain $\mathrm{gr}(\Gamma_\delta(I)) = 3$. Now suppose $dc \in \delta(I)$ and $d^2 = c$. If $ab = a$, then $a^2 = a^2b^2 \in \delta(I)$, which is a contradiction. Similarly, $ab \neq b$. Clearly $ab \notin \delta(I)$. Thus, $ab = c$, for otherwise we would let $x = ab$ above and have $\mathrm{gr}(\Gamma_\delta(I)) = 3$. Similarly, $ab = bd = c$. Therefore, $a(b - d) = 0$. If $b - d \neq c$, we again have $\mathrm{gr}(\Gamma_\delta(I)) = 3$. So suppose that $b = d + c$. Similarly, $b(d - a) = 0$. Again, if $d - a \neq c$, we will have $\mathrm{gr}(\Gamma_\delta(I)) = 3$. Now if $d = a + c$, we have $b = d + c = a + c + c = a$, which is a contradiction. Thus, every case leads to $\mathrm{gr}(\Gamma_\delta(I)) = 3$. □

References

[1] D. D. Anderson and M. Naseer, Beck's coloring of commutative rings, J. Algebra 159 (1993), 500-514.

[2] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra 217 (1999), 434-447.

[3] D. F. Anderson and A. Badawi, On the zero-divisor graph of a commutative ring, Comm. Algebra 36 (2008), 3073-3092.

[4] D. F. Anderson and A. Badawi, The total graph of a commutative ring, J. Algebra 20(7) (2008), 2706-2719.

[5] D. F. Anderson, M. C. Axtell and J. A. Stickles, Zero-divisor graphs in commutative rings, in Commutative Algebra, Noetherian and Non-Noetherian Perspectives (M. Fontana, S.-E. Kabbaj, B. Olberding and I. Swanson, Editors), Springer-Verlag, New York, (2011), 23-45.

[6] I. Beck, Coloring of commutative rings, J. Algebra 116 (1988), 208-226.

[7] S. Ebrahimi Atani, Z. Ebrahimi Sarvandi and M. Shajari Kohan, On δ-primary ideals of commutative semirings, Rom. J. Math. and Comput. Sci. 3(1) (2013), 71-81.

[8] Z. Dongsheng, δ-primary ideals of commutative rings, Kyungpook Math. J. 41 (2001), 17-22.

[9] S. P. Redmond, An ideal-based zero-divisor graph of a commutative ring, Comm. Algebra 31 (2003), 4425-4443.

■