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GLOBAL BEHAVIOUR OF SOLUTIONS TO A CLASS OF FEEDBACK CONTROL SYSTEMS

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Abstract

In this paper, the method of guiding functions and the topological tools are used to investigate the global bifurcation problem for a class of feedback control systems. We obtain the sufficient conditions, under which there is a connected subset of non-trivial solutions of such systems that bifurcates from $(0, 0)$ and tends to infinity.

1. Introduction

Let $I = [0, T]$ and $k \geq 1$ be a given integer. In this paper, we consider the global bifurcation problem for the following feedback control system:

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$$\begin{cases} x'(t) = a\mu x(t) + f(t, x(t), u_1(t), \cdots, u_k(t), \mu), \text{ for a.e. } t \in I, \\ u'_\tau(t) \in G_\tau(t, x(t), u_\tau(t), \mu), \text{ for a.e. } t \in I, \quad 1 \le \tau \le k, \\ x(0) = x(T), u_\tau(0) = 0, \quad 1 \le \tau \le k, \end{cases} \tag{1.1}$$

where $a > 0$; $f : I \times \mathbb{R}^n \times \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_k} \times \mathbb{R} \to \mathbb{R}^n$ is a continuous map; $G_\tau : I \times \mathbb{R}^n \times \mathbb{R}^{m_\tau} \times \mathbb{R} \to Kv(\mathbb{R}^{m_\tau}), 1 \le \tau \le k,$ be multivalued maps, where $Kv(\mathbb{R}^{m_\tau})$ denotes the collection of all nonempty compact convex subsets of $\mathbb{R}^{m_\tau}$; n is an odd integer. Here $x : I \to \mathbb{R}^n$ is a trajectory of the system, $u_\tau : I \to \mathbb{R}^{m_\tau}, 1 \le \tau \le k,$ be control functions. The first equation describes the dynamics of the system and the differential inclusions represent the feedback.

Applying the method of guiding functions and the topological tools, we obtain the global structure of the solution set of system (1.1). Let us mention that the bifurcations in control systems were studied by many researchers (see, e.g., [5]) and the method of guiding functions was applied to study the global bifurcation problem for differential inclusions in various research papers (see, e.g., [14, 16, 17, 18]).

The paper is organized in the following way. In the next section, we recall some notions and notation from multivalued analysis, theory of Fredholm operators, and bifurcation theory for inclusions. The main result is given in Section 3.

2. Preliminaries

2.1. Multimaps

Let X and Y be metric spaces. Denote by $P(Y)\,[K(Y)]$ the collections of all nonempty [respectively, nonempty compact] subsets of Y.

Definition 1 (see, e.g., [2, 9, 13]). A multivalued map (multimap) $\Sigma : X \to P(Y)$ is said to be

(i) upper semicontinuous (u.s.c.), if for every open subset $V \subset Y$, the set

$$\Sigma_+^{-1}(V) = \{x \in X : \Sigma(x) \subset V\}$$

is open in X;

(ii) compact, if the set $\Sigma(X)$ is relatively compact in Y;

(iii) completely u.s.c., if it maps every bounded subset $U \subset X$ into a relatively compact subset $\Sigma(U)$ of Y.

Definition 2. A set $M \in K(Y)$ is said to be aspheric (or UV^{∞}, or ∞-proximally connected) (see, e.g., [19, 15, 10, 9]), if for every $\varepsilon > 0$, there exists $\delta > 0$ such that each continuous map $\phi : S^n \to O_\delta(M)$, $n = 0, 1, 2, \cdots$, can be extended to a continuous map $\widetilde{\phi} : B^{n+1} \to O_\varepsilon(M)$, where $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ and $B^{n+1} = \{x \in \mathbb{R}^{n+1} : \|x\| \leq 1\}$, and $O_\delta(M)$ $[O_\varepsilon(M)]$ denote the δ-neighbourhood [resp., ε-neighbourhood] of the set M.

Definition 3 (see [11]). A nonempty compact space is said to be an R_δ-set, if it can be represented as the intersection of a decreasing sequence of compact, contractible spaces.

Definition 4 (see [9]). An u.s.c. multimap $\Sigma : X \to K(Y)$ is said to be a J-multimap $(\Sigma \in J(X, Z))$, if every value $\Sigma(x)$, $x \in X$ is an aspheric set.

Now, let us recall (see, e.g., [4]) that a metric space Z is called the *absolute retract* (the *AR-space*) [resp., the *absolute neighbourhood retract* (the *ANR-space*)] provided for each homeomorphism h taking it onto a closed subset of a metric space Z', the set $h(Z)$ is the retract of Z' [resp., of its open neighbourhood $O(h(Z))$ in Z']. Notice that the class of

ANR-spaces is broad enough: In particular, a finite-dimensional compact set is the *ANR*-space, if and only if it is locally contractible. In turn, it means that compact polyhedrons and compact finite-dimensional manifolds are the *ANR*-spaces. The union of a finite number of convex closed subsets in a normed space is also the *ANR*-space.

Proposition 1 (see [9]). *Let Z be an ANR-space. In each of the following cases, an u.s.c. multimap $\Sigma : X \to K(Z)$ is a J-multimap*:

For each $x \in X$, the value $\Sigma(x)$ is

(a) *a convex set*;

(b) *a contractible set*;

(c) *an R_δ-set*;

(d) *an AR-space*.

In particular, every continuous map $\sigma : X \to Z$ is a J-multimap.

Definition 5. Let X and Y be Banach spaces. By $J^c(X, Y)$, we will denote the collection of all multimaps $F : X \to K(Y)$ that may be represented in the form of composition

$$F = \Sigma_q \circ \cdots \circ \Sigma_1,$$

where $\Sigma_i \in J(X_{i-1}, X_i)$, $i = 1 \cdots q$, $X_0 = X$, $X_q = Y$, and $X_i (0 < i < q)$ are normed spaces.

Let us mention that if $U \subset X$ is an open bounded subset and $F : \overline{U} \to K(X)$ is a compact J^c-multimap such that $x \notin F(x)$ for all $x \in \partial U$. Then the topological degree $\deg(i - F, \overline{U})$ is well-defined and has all usual properties of the Brouwer topological degree (see [3]), where i denotes the inclusion map.

2.2. Fredholm operators

Now, we recall some basic notions of the theory of linear Fredholm operators (see, e.g., [8]). Let X and Y be Banach spaces.

Definition 6. A bounded linear operator $L : X \to Y$ is said to be a Fredholm operator of index zero, if

(1i) $\operatorname{Im} L$ is closed in Y;

(2i) $\operatorname{Ker} L$ and $\operatorname{Coker} L$ have finite dimensions and $\dim \operatorname{Ker} L = \dim \operatorname{Coker} L$.

Let $L : \operatorname{dom} L \subseteq X \to Y$ be a linear Fredholm operator of index zero. Then there exist projections $p_L : X \to X$ and $q_L : Y \to Y$ such that $\operatorname{Im} p_L = \operatorname{Ker} L$ and $\operatorname{Ker} q_L = \operatorname{Im} L$. If the operator

$$L_{p_L} : \operatorname{dom} L \cap \operatorname{Ker} p_L \to \operatorname{Im} L$$

is defined as the restriction of L on $\operatorname{dom} L \cap \operatorname{Ker} p_L$, then L_{p_L} is a linear isomorphism and so the linear operator $k_{p_L} : \operatorname{Im} L \to \operatorname{dom} L$, $k_{p_L} = L_{p_L}^{-1}$ is well-defined. Now, let $\operatorname{Coker} L = Y / \operatorname{Im} L$. Define a canonical projection operator $\pi_L : Y \to \operatorname{Coker} L$,

$$\pi_L(z) = z + \operatorname{Im} L,$$

and let $\ell_L : \operatorname{Coker} L \to \operatorname{Ker} L$ be a linear continuous isomorphism. Then, the equation

$$Lx = y, \quad y \in Y,$$

is equivalent to the following relation:

$$x = p_L x + (\ell_L \pi_L + k_L) y, \tag{2.1}$$

where $k_L : Y \to X$ is defined as

$$k_L = k_{p_L}(i - q_L).$$

2.3. Global bifurcation theorem for J^c-inclusions

Let X be a Banach space. Denote by $B_X(0, r)$ the ball of radius r centered at 0 in X. Consider the following one-parameter family of inclusions:

$$x \in \mathcal{F}(x, \mu), \tag{2.2}$$

where $\mathcal{F} : X \times \mathbb{R} \to K(X)$ is a completely u.s.c. J^c-multimap satisfying the following conditions:

$(\mathcal{F}1)$ $0 \in \mathcal{F}(0, \mu)$ for all $\mu \in \mathbb{R}$;

$(\mathcal{F}2)$ for each $\mu, 0 < |\mu - \mu_0| \le r_0$, there is $\delta_\mu > 0$ such that $x \notin \mathcal{F}(x, \mu)$ when $0 < \|x\| \le \delta_\mu$, where μ_0, r_0 are given numbers.

A point $(0, \mu_*)$ is said to be a bifurcation point of inclusion (2.2), if for every open subset $U \subset X \times \mathbb{R}$ with $(0, \mu_*) \in U$, there exists a point $(x, \mu) \in U$ such that $x \neq 0$ and $x \in \mathcal{F}(x, \mu)$.

From $(\mathcal{F}2)$, it follows that for each $\mu, 0 < |\mu - \mu_0| \le r_0$ the topological degree

$$\deg(i - \mathcal{F}(\cdot, \mu), B_X(0, \delta_\mu)),$$

is well defined. Then, the bifurcation index of the multimap $\mathcal{F}$ at $(0, \mu_0)$ may be defined as

$$Bi[\mathcal{F}; (0, \mu_0)] = \lim_{\mu \to \mu_0^+} \deg\big(i - \mathcal{F}(\cdot, \mu), B_X(0, \delta_\mu)\big)$$
$$- \lim_{\mu \to \mu_0^-} \deg\big(i - \mathcal{F}(\cdot, \mu), B_X(0, \delta_\mu)\big).$$

Let us denote by $\mathcal{S}$ the set of all non-trivial solutions to inclusion (2.2), i.e.,

$$\mathcal{S} = \{(x, \mu) \in X \times \mathbb{R};\ x \neq 0 \text{ and } x \in \mathcal{F}(x, \mu)\}.$$

The following assertion can be easily followed from the global bifurcation theorems presented in [7, 14].

Theorem 1. *Under conditions* $(\mathcal{F}1)-(\mathcal{F}2)$, *assume that* $Bi[\mathcal{F}; (0, \mu_0)] \neq 0$. *Then, there exists a connected subset* $\mathcal{C} \subset \mathcal{S}$ *such that* $(0, \mu_0) \in \overline{\mathcal{C}}$ *and one of the following occurs*:

(a) $\mathcal{C}$ *is unbounded*;

(b) $(0, \mu_*) \in \overline{\mathcal{C}}$ *for some* $\mu_* \neq \mu_0$.

3. Main Result

We will use the same notation for scalar products and norms $\langle z', z'' \rangle$ and $|z|$ in all spaces $\mathbb{R}^n$ and $\mathbb{R}^{m_i}$. Denote by the symbols $C(I, \mathbb{R}^n)$ $[L_p(I, \mathbb{R}^n), (p \geq 1)]$ the space of continuous [resp., p-summable] functions $x : I \to \mathbb{R}^n$ with usual norms

$$\|x\|_C = \max_{t \in I} |x(t)| \text{ and } \|f\|_p = \left(\int_0^T |f(s)|^p \, ds \right)^{\frac{1}{p}}.$$

A ball [sphere] of radius r centered at 0 in $C(I, \mathbb{R}^n)$ is denoted by $B_C(0, r)$ [resp., $\partial B_C(0, r)$]. Consider the space of all absolutely continuous functions $x : I \to \mathbb{R}^n$, whose derivatives belong to $L_2(I, \mathbb{R}^n)$. It is known (see, e.g., [1]) that this space can be identified with the Sobolev space $W^{1,2}(I, \mathbb{R}^n)$ endowed with the norm

$$\|x\|_W = \left(\|x\|_2^2 + \|x'\|_2^2 \right)^{1/2}.$$

Notice that (see, e.g., [6]) the embedding $W^{1,2}(I, \mathbb{R}^n) \hookrightarrow C(I, \mathbb{R}^n)$ is compact. By the symbol $W_T^{1,2}(I, \mathbb{R}^n)$, we will denote the subspace of all functions $x \in W^{1,2}(I, \mathbb{R}^n)$ such that $x(0) = x(T)$.

We will consider system (1.1) with the following assumptions:

(f) there exists $0 < c < a$ such that

$$|f(t, z, w_1, \cdots, w_k, \mu)| \leq c|z| \left(|\mu| + |w_1| + \cdots + |w_k| \right),$$

for all $(z, w_1, \cdots, w_k, \mu)$ and a.e. $t \in I$.

For each $1 \leq \tau \leq k$, the multimap G_τ satisfies conditions:

(G1) for every $(z, w, \mu) \in \mathbb{R}^n \times \mathbb{R}^{m_\tau} \times \mathbb{R}$, the multifunction $G_\tau(\cdot, z, w, \mu) : I \to Kv(\mathbb{R}^{m_\tau})$ has a measurable selection;

(G2) for a.e. $t \in I$, the multimap $G_\tau(t, \cdot, \cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^{m_\tau} \times \mathbb{R} \to Kv(\mathbb{R}^{m_\tau})$ is u.s.c.;

(G3) the multimap G_τ is uniformly continuous with respect to the second and fourth arguments in the following sense: For every $\varepsilon > 0$, there is $\delta > 0$ such that

$$G_\tau(t, \bar{z}, w, \bar{\mu}) \subset O_\varepsilon(G_\tau(t, z, w, \mu)), \quad \forall (t, w) \in I \times \mathbb{R}^{m_\tau},$$

provided $\max\{|\bar{z} - z|, |\bar{\mu} - \mu|\} < \delta$;

(G4) there are $d_\tau > 0$ such that $\sum_{\tau=1}^{k} d_\tau e^{Td_\tau} < \dfrac{a - c}{cT}$ and

$$|G_\tau(t, z, w, \mu)| \leq d_\tau(|\mu| + |z| + |w|),$$

for all $(z, w, \mu) \in \mathbb{R}^n \times \mathbb{R}^{m_\tau} \times \mathbb{R}$ and a.e. $t \in I$.

Now, for each $(x, \mu) \in C(I, \mathbb{R}^n) \times \mathbb{R}$ define the multimaps

$$G_\tau^{(x,\mu)} : I \times \mathbb{R}^{m_\tau} \to Kv(\mathbb{R}^{m_\tau}),\ G_\tau^{(x,\mu)}(t, w) = G_\tau(t, x(t), w, \mu), \quad 1 \leq \tau \leq k.$$

By virtue of Theorem 1.3.5 [13] for each $w \in \mathbb{R}^{m_\tau}$, the multifunction $G_\tau^{(x,\mu)}(\cdot, w)$ has a measurable selection. Further, from conditions (G2) and (G3), it follows that for a.e. $t \in I$ the multimap $G_\tau^{(x,\mu)}(t, w)$ depends upper semicontinuously on (x, w, μ). We have the following assertion (see, e.g., [9, 13]): For each $(x, \mu) \in C(I, \mathbb{R}^n) \times \mathbb{R}$, the set $\prod_\tau^{(x,\mu)}$ of solutions of the following problem:

$$\begin{cases} u_\tau'(t) \in G_\tau(t, x(t), u_\tau(t), \mu), \text{ for a.e. } t \in I, \\ u_\tau(0) = 0, \end{cases}$$

is an R_δ-set in $C(I, \mathbb{R}^{m_\tau})$ and the multimap $\prod_\tau : C(I, \mathbb{R}^n) \times \mathbb{R} \to K\left(C(I, \mathbb{R}^{m_\tau})\right)$,

$$\prod_\tau(x, \mu) = \prod_\tau^{(x, \mu)}$$

is upper semicontinuous.

By a solution to problem (1.1), we mean a pair $(x, \mu) \in W_T^{1,2}(I, \mathbb{R}^n) \times \mathbb{R}$ such that there are $u_\tau \in \prod_\tau(x, \mu)$, $1 \le \tau \le k$, and

$$x'(t) = a\mu x(t) + f(t, x(t), u_1(t), \cdots, u_k(t), \mu), \text{ for a.e. } t \in I.$$

From (f), it follows that $(0, \mu)$ is a solution of (1.1) for every $\mu \in \mathbb{R}$. These solutions are called trivial. Let us denote by $\mathcal{S}$ the set of all non-trivial solutions of (1.1).

In what follows, we need the following statement:

Lemma 1 (Gronwall's lemma, see, e.g., [12])**.** *Let $u, v : [a, b] \to \mathbb{R}$ be continuous nonnegative functions and $C \ge 0$ be a constant and*

$$v(t) \le C + \int_a^t u(s)v(s)ds, \quad a \le t \le b.$$

Then

$$v(t) \le Ce^{\int_a^t u(s)ds}, \quad a \le t \le b.$$

Our main result is the following statement.

Theorem 2. *Let conditions* (f) *and* (G1)-(G4) *hold. Then, there is an unbounded connected subset $\mathcal{C} \subset \mathcal{S}$ such that $(0, 0) \in \overline{\mathcal{C}}$.*

Proof. For every $(x, \mu) \in C(I, \mathbb{R}^n) \times \mathbb{R}$, define the following multimaps:

$$\widetilde{\prod}_1 : C(I, \mathbb{R}^n) \times \mathbb{R} \to K(C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^{m_1}) \times \mathbb{R}),$$

$$\widetilde{\prod}_1(x, \mu) = \{x\} \times \prod_1(x, \mu) \times \{\mu\},$$

$$\widetilde{\prod}_2 : C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^{m_1}) \times \mathbb{R} \to K(C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^{m_1}) \times C(I, \mathbb{R}^{m_2}) \times \mathbb{R}),$$

$$\widetilde{\prod}_2(x, u, \mu) = \{x\} \times \{u\} \times \prod_2(x, \mu) \times \{\mu\},$$

and so on

$$\widetilde{\prod}_k : C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^{m_1}) \times \cdots \times C(I, \mathbb{R}^{m_{k-1}}) \times \mathbb{R}$$

$$\to K(C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^{m_1}) \times \cdots \times C(I, \mathbb{R}^{m_k}) \times \mathbb{R}),$$

$$\widetilde{\prod}_k(x, u_1, \cdots, u_{k-1}, \mu) = \{x\} \times \{u_1\} \times \cdots \times \{u_{k-1}\} \times \prod_k(x, \mu) \times \{\mu\},$$

and a map $\widetilde{f} : C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^{m_1}) \times \cdots \times C(I, \mathbb{R}^{m_k}) \times \mathbb{R} \to L_2(I, \mathbb{R}^n)$,

$$\widetilde{f}(x, u_1, \cdots, u_k, \mu)(t) = a\mu x(t) + f(t, x(t), u_1(t), \cdots, u_k(t), \mu), \quad t \in I.$$

It is clear that for every $i, 1 \leq i \leq k$, $\widetilde{\prod}_i$ is a J-multimap and $\widetilde{f}$ is a continuous map. Set $Q : C(I, \mathbb{R}^n) \times \mathbb{R} \to K(L_2(I, \mathbb{R}^n))$,

$$Q(x, \mu) = \widetilde{f} \circ \widetilde{\prod}_k \circ \cdots \circ \widetilde{\prod}_1(x, \mu).$$

Then problem (1.1) can be substituted by the following operator-inclusion:

$$Ax \in Q(x, \mu), \tag{3.1}$$

where $A : W_T^{1,2}(I, \mathbb{R}^n) \to L_2(I, \mathbb{R}^n)$, $Ax = x'$.

It is easy to see that A is a linear Fredholm operator of index zero and

$$\operatorname{Ker} A \cong \mathbb{R}^n \cong \operatorname{Coker} A.$$

The projection

$$\pi_A : L_2(I, \mathbb{R}^n) \to \mathbb{R}^n,$$

is defined as

$$\pi_A(g) = \frac{1}{T}\int_0^T g(s)ds,$$

and the homeomorphism $\ell_A : \mathbb{R}^n \to \mathbb{R}^n$ is an identity operator. The space $L_2(I, \mathbb{R}^n)$ can be represented as

$$L_2(I, \mathbb{R}^n) = \mathcal{L}_0 \oplus \mathcal{L}_1,$$

where $\mathcal{L}_0 = \operatorname{Coker} A$ and $\mathcal{L}_1 = \operatorname{Im} A$.

The decomposition of an element $g \in L_2(I, \mathbb{R}^n)$ is denoted by

$$g = g_{(0)} + g_{(1)}, \quad g_{(0)} \in \mathcal{L}_0, \quad g_{(1)} \in \mathcal{L}_1.$$

By virtue of (2.1) inclusion (3.1) is equivalent to the following inclusion:

$$x \in H(x, \mu), \tag{3.2}$$

where $H : C(I, \mathbb{R}^n) \times \mathbb{R} \to K(C(I, \mathbb{R}^n))$,

$$H(x, \mu) = p_A x + (\pi_A + k_A) \circ Q(x, \mu).$$

It is clear that H is a J^c-multimap. Let us show that H is completely u.s.c.. In fact, let $\Omega \subset C(I, \mathbb{R}^n) \times \mathbb{R}$ be a bounded subset and $(x, \mu) \in \Omega$. Taking arbitrarily $\gamma \in Q(\Omega)$, then there exist $u_\tau \in \prod_\tau(x, \mu)$, $1 \le \tau \le k$, such that

$$\gamma(t) = \mu a x(t) + f(t, x(t), u_1(t), \cdots, u_k(t), \mu), \text{ for a.e. } t \in I.$$

From (f), it follows that

$$|\gamma(t)| \le |x(t)|\Big((a + c)|\mu| + c|u_1(t)| + \cdots + c|u_k(t)|\Big), \text{ for a.e. } t \in I. \tag{3.3}$$

From the fact that $u_i \in \prod_\tau(x, \mu)$, it follows that there are $g_\tau \in L_1(I, \mathbb{R}^{m_\tau})$, $1 \le \tau \le k$, such that

$$g_\tau(t) \in G_\tau(t, x(t), u_\tau(t), \mu), \text{ for a.e. } t \in I,$$

and

$$u_\tau(t) = \int_0^t g_\tau(s)ds.$$

By virtue of (G4), we have that

$$|u_\tau(t)| \le \int_0^t |g_\tau(s)| ds \le \int_0^t d_\tau(|\mu| + |x(s)| + |u_\tau(s)|)ds$$

$$\le Td_\tau|\mu| + d_\tau\sqrt{T}\|x\|_2 + \int_0^t d_\tau|u_\tau(s)|ds.$$

From Lemma 1, it follows that

$$|u_\tau(t)| \le \left(Td_\tau|\mu| + d_\tau\sqrt{T}\|x\|_2\right)e^{Td_\tau}, \text{ for all } t \in I. \tag{3.4}$$

Since (3.3) and (3.4), there exists $M_\Omega > 0$ such that $|\gamma(t)| < M_\Omega$, for a.e. $t \in I$, i.e., the set $Q(\Omega)$ is bounded in $L_2(I, \mathbb{R}^n)$. Notice that the operator

$$(\pi_A + k_A) : L_2(I, \mathbb{R}^n) \to W_T^{1,2}(I, \mathbb{R}^n),$$

is continuous and the map p_A takes values in $\mathbb{R}^n$. Then, the set $H(\Omega)$ is bounded in $W_T^{1,2}(I, \mathbb{R}^n)$, and hence, it is a relative compact set in $C(I, \mathbb{R}^n)$. So, H is a completely u.s.c. J^c-multimap.

For each $\mu \neq 0$, let us show that there exists $\delta_\mu > 0$ such that inclusion (3.2) has no non-trivial solution on $B_C(0, \delta_\mu) \times \{\mu\}$.

In fact, to contrary assume that $(x, \mu) \in B_C(0, \delta_\mu) \times \{\mu\}$ is a non-trivial solution of (3.2). Then there are $u_\tau \in \prod_\tau(x, \mu), 1 \le \tau \le k$, such that

$$x'(t) = a\mu x(t) + f(t, x(t), u_1(t), \cdots, u_k(t), \mu), \text{ for a.e. } t \in I. \tag{3.5}$$

Therefore,

$$\int_0^T \left\langle \mu x(t), a\mu x(t) + f(t, x(t), u_1(t), \cdots, u_k(t), \mu) \right\rangle dt = \int_0^T \langle \mu x(t), x'(t)\rangle dt = 0.$$

On the other hand,

$$\int_0^T \left\langle \mu x(t), a\mu x(t) + f(t, x(t), u_1(t), \cdots, u_k(t), \mu) \right\rangle dt$$

$$\geq a\mu^2 \|x\|_2^2 - |\mu| \int_0^T |x(t)|\,|f(t, x(t), u_1(t), \cdots, u_k(t), \mu)| dt$$

$$\geq a\mu^2 \|x\|_2^2 - c|\mu| \int_0^T x^2(t)\,(|\mu| + |u_1(t)| + \cdots + |u_k(t)|)\, dt$$

$$\geq (a - c)\mu^2 \|x\|_2^2 - c|\mu| \int_0^T x^2(t) \left(\sum_{\tau=1}^{k} \left(Td_\tau |\mu| + d_\tau \sqrt{T} \|x\|_2\right) e^{Td_\tau} \right) dt$$

$$= \left(a - c - cT \sum_1^k d_\tau e^{Td_\tau}\right) \mu^2 \|x\|_2^2 - c|\mu| \sqrt{T} \sum_1^k d_\tau e^{Td_\tau} \|x\|_2^3 > 0, \tag{3.6}$$

provided

$$0 < \|x\|_2 < \frac{\left(a - c - cT \sum_1^k d_\tau e^{Td_\tau}\right) |\mu|}{c\sqrt{T} \sum_1^k d_\tau e^{Td_\tau}}. \tag{3.7}$$

Therefore, inclusion (3.2) has no solution (x, μ) that satisfies (3.7).

Thus, for sufficiently small δ_μ, we obtain a contradiction.

Now, for a given $\mu \neq 0$, we will evaluate the topological degree

$$\deg(i - H(\cdot, \mu), B_C(0, \delta_\mu)).$$

Toward this goal, let us consider the multimap

$$\Sigma_\mu : B_C(0, \delta_\mu) \times [0, 1] \to K(C(I, \mathbb{R}^n)),$$

$$\Sigma_\mu(x, \lambda) = p_A x + (\pi_A + k_A) \circ \varphi(Q(x, \mu), \lambda),$$

where $\varphi : L_2(I, \mathbb{R}^n) \times [0, 1] \to L_2(I, \mathbb{R}^n)$,

$$\varphi(g, \lambda) = g_{(0)} + \lambda g_{(1)}, \quad g_{(0)} \in \mathcal{L}_0, \quad g_{(1)} \in \mathcal{L}_1.$$

It is clear that Σ_μ is a compact J^c-multimap. Assume that there is $(x_*, \lambda_*) \in \partial B_C(0, \delta_\mu) \times [0, 1]$ such that $x_* \in \Sigma_\mu(x_*, \lambda_*)$. Then there are $\gamma^* \in L_2(I, \mathbb{R}^n)$ and $u_\tau^* \in \prod_\tau(x_*, \mu)$, $1 \leq \tau \leq k$, such that

$$\gamma^*(t) = a\mu x_*(t) + f(t, x_*(t), u_1^*(t), \cdots, u_k^*(t), \mu), \text{ for a.e. } t \in I,$$

and

$$\begin{cases} x_*' = \lambda_* \gamma_{(1)}^*, \\ 0 = \gamma_{(0)}^*, \end{cases}$$

where $\gamma_{(0)}^* + \gamma_{(1)}^* = \gamma^*$, $\gamma_{(0)}^* \in \mathcal{L}_0$, $\gamma_{(1)}^* \in \mathcal{L}_1$.

If $\lambda_* \neq 0$, then

$$\int_0^T \langle \mu x_*(t), \gamma^*(t) \rangle dt = \frac{1}{\lambda_*} \int_0^T \langle \mu x_*(t), x_*'(t) \rangle dt = 0.$$

On the other hand, from $\|x_*\|_2 \leq \sqrt{T}\|x_*\|_C = \sqrt{T}\delta_\mu$, it follows that x_* satisfies relation (3.7) for sufficiently small δ_μ. Therefore,

$$\int_0^T \langle \mu x_*(t), \gamma^*(t) \rangle dt > 0,$$

giving a contradiction.

If $\lambda_* = 0$, then $x_* \in \operatorname{Ker} A$, i.e., $x_*(t) = w \in \mathbb{R}^n$ for all $t \in I$. Since the fact that w satisfies relation (3.7), we have

$$\int_0^T \langle \mu w,\ \gamma(t) \rangle dt > 0,$$

for all $u_\tau \in \prod_\tau(w, \mu)$, $1 \le \tau \le k$, where

$$\gamma(t) = a\mu w + f(t, w, u_1(t), \cdots, u_k(t), \mu) \in Q(w, \mu), \text{ for a.e. } t \in I.$$

Notice that

$$\int_0^T \langle \mu w,\ \gamma(t) \rangle dt = T\langle \mu w,\ \pi_A \gamma \rangle.$$

Consequently,

$$\langle \mu w,\ \pi_A \gamma \rangle > 0, \text{ for all } \gamma \in Q(w, \mu). \tag{3.8}$$

In particular,

$$0 < \langle \mu w,\ \pi_A \gamma^* \rangle = \langle \mu w,\ \pi_A \gamma^*_{(0)} \rangle = 0,$$

that is a contradiction.

So, Σ_μ is a homotopy connecting the multimaps $\Sigma_\mu(\cdot, 1) = H(\cdot, \mu)$ and

$$\Sigma_\mu(\cdot, 0) = p_A + \pi_A Q(\cdot, \mu).$$

By virtue of the invariant property of the topological degree, we have that

$$\deg(i - H(\cdot, \mu),\ B_C(0,\ \delta_\mu)) = \deg(i - p_A - \pi_A Q(\cdot, \mu),\ B_C(0,\ \delta_\mu)).$$

Notice that the multimap $p_A + \pi_A Q(\cdot, \mu)$ takes values in $\mathbb{R}^n$, and hence

$$\deg(i - p_A - \pi_A Q(\cdot, \mu),\ B_C(0,\ \delta_\mu)) = \deg(i - p_A - \pi_A Q(\cdot, \mu),\ B_{\mathbb{R}^n}(0,\ \delta_\mu)).$$

In the space $\mathbb{R}^n$, the vector field $i - p_A - \pi_A Q(\cdot, \mu)$ has the form

$$i - p_A - \pi_A Q(\cdot, \mu) = -\pi_A Q(\cdot, \mu).$$

From (3.8), it follows that $\pi_A Q(\cdot, \mu)$ and μi are homotopic on $\partial B_{\mathbb{R}^n}(0,\ \delta_\mu)$. So, we obtain

$$\deg(-\pi_A Q(\cdot, \mu),\ B_{\mathbb{R}^n}(0,\ \delta_\mu)) = \deg(-\mu i,\ B_{\mathbb{R}^n}(0,\ \delta_\mu)) = -\operatorname{sign}(\mu).$$

Thus, the bifurcation index $Bi[H;\ (0,\ 0)] = -2.$ From (3.6)-(3.7), it follows that $(0,\ 0)$ is the unique bifurcation point of system (1.1). To complete the proof, we need only to apply Theorem 1 with a remark that the case (b) of Theorem 1 could not appear. □

Example 1. Consider the following feedback control system:

$$\begin{cases} x'(t) = a\mu x(t) + x(t)(\mu + u_1(t) + \cdots + u_k(t)), \text{ for a.e. } t \in [0, 1], \\ u'_\tau(t) \in [\mu + x(t),\ \mu + x(t) + |u_\tau(t)|], \text{ for a.e. } t \in [0, 1], \quad 1 \le \tau \le k, \\ x(0) = x(1),\ u_\tau(0) = 0, \end{cases} \tag{3.9}$$

where $a > 1 + ke.$

Here, $f : [0, 1] \times \mathbb{R}^{k+2} \to \mathbb{R},$

$$f(t, z, w_1, \cdots, w_k, \mu) = z(\mu + w_1 + \cdots + w_k), (z, w_1, \cdots, w_k, \mu) \in \mathbb{R}^{k+2},$$

and $G_\tau : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to Kv(\mathbb{R}),$

$$G_\tau(t, z, w, \mu) = [\mu + z,\ \mu + z + |w|], \quad 1 \le \tau \le k.$$

It is easy to verify that the map f satisfies condition (f) and the multimaps G_τ satisfy conditions (G1)-(G4). Therefore, applying Theorem 2, we obtain the global structure of the solution set of problem (3.9).

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