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MODELLING PLANETARY ORBITS THROUGH GEODESICS ON A TORUS

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Abstract

In this work, we develop a formalism to determine the geodesic orbits at a point associated with the surface of a rotating planet which at the same time is forced to move on a toroid geometric structure. Such an approach is based on a Lagrangian method applied to the geodesics on the torus. In particular, we consider a circular as well as elliptic toroids. We find parameters relating the angular velocity with the internal angular momentum (spin) of the planet. We show that this relation leads to a non-Newtonian potential. Thus, through our Lagrangian formalism we determine the constants of the motion, including the orbital angular momentum and the energy.

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1. Introduction

Since the discovery of first exoplanet, it has been a growing interest in this new science [5]. This year, NASA's Kepler Mission announced 1284 new planets [14]. Recently [6], it was discovered the first exoplanet in the habitable zone for a star less massive than the Sun. Most of such exoplanets have been found in a distance less than 3000 light years from Earth.

Exoplanetary science has challenged the models of planetary systems formation [12]. Since the discovery of the first exoplanet, many questions have been arised about the formation of planetary systems. In the case of PEG51, a planet with the 60% of Jupiter mass, which is very close to his parent star, put in doubt the formation of this planet at its present position. As a consequence of this, a planet migration theory was proposed [11].

Moreover, the recent growing number of exoplanets with high eccentricities and quite close to their parent stars can put in doubt our current understanding of planetary system formation [13]. For those reasons, the development of better models can help us to increase our panorama in exoplanetary science.

More recently, it has been determined the angular velocity of rotating exoplanet [8]. With those data, a relation between internal angular momentum and mass (Regge trajectories) for various planets has been established. It is worth remarking that the rotation and translation periods in combination with the distance to its central star can affect the surface temperature and consequently the atmosphere and climate of a planet. The main motivation of this work emerged in the search of the understanding this combined rotation and translation movements.

One of our guides in our goal arises from models of geometric torus structures around astronomical objects, including accretion disks around collapsed systems [1] and black-holes [2]. In analogy of the initial condition of the formation of a planetary system, one may expect that such a accretion toroidal disks leads eventually to a planet to move along the geometry of a torus. Thus, the motion of a point on a surface of the evolving planet will describe geodesics determined by the geometry of the torus. In this context, it turns out convenient to develop a Lagrangian formalism associated with curved spaces. This will have the advantage to have a systematic method to obtain the constants of the motion knowing the symmetries of the corresponding Lagrangian.

Following, the above method we explore the possibility of applying not only the solar planetary system but also to exoplanets. In the case of our solar system, we show that our formalism may allows to find a relation between the periods of the planet around the Sun and the planet rotation. While in the case of exoplanets, our approach may help to explain why the planets rotate according the Regge trajectories. Specifically, in this work, we propose a model in which the geodesic path of a point on the surface of a planet is determined by geometric tools of the torus and a Lagrangian formalism.

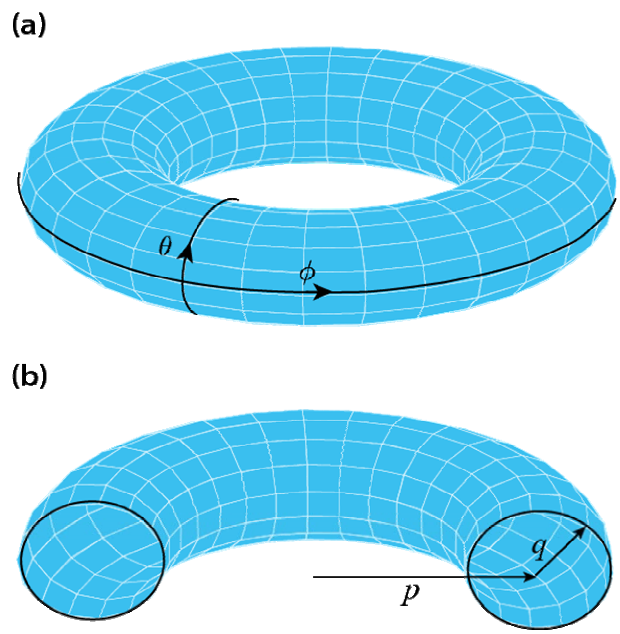

Figure 1. Circular torus. (a) The angle ϕ represents the translational motion and θ the spin angle rotation. (b) The radius of the torus is denoted by p and the radius of the tube by q.

The technical plan of this work is as follows. In Section 2, we introduce the geometry of a torus, the constants of the motion and compute the corresponding potential. Section 3 roughly describes the elliptic generalization of the torus. Finally, in Section 4, we explain our results and describe possible further research.

2. Circular Torus

Consider the coordinates transformation

$$\begin{aligned} x &= (p + q\cos\theta)\cos\phi, \\ y &= (p + q\cos\theta)\sin\phi, \\ z &= q\sin\theta. \end{aligned} \tag{1}$$

It can be shown that (1) describes a circular torus with p as its the radius and q the radius of the torus tube. Here, the variable θ is the angle inside the tube measured in the clockwise direction, and ϕ is the angle of the orbit (see Figure 1).

From the transformation (1), one obtains the Lagrangian

$$\mathcal{L} = m_0 \frac{1}{2}\left[q^2\dot{\theta}^2 + (p + q\cos\theta)^2\dot{\phi}^2\right], \tag{2}$$

where the potential associated to some external force is not considered. Instead, we shall determine a potential due to the geometry (1).

Observe that

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = m_0 q^2\dot{\theta}, \qquad \frac{\partial\mathcal{L}}{\partial\theta} = -m_0 q\sin\theta(p + q\cos\theta)\dot{\phi}^2, \tag{3}$$

and

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = m_0(p + \cos\theta)^2\dot{\phi}, \qquad \frac{\partial\mathcal{L}}{\partial\phi} = 0. \tag{4}$$

Hence, the Euler-Lagrange equations for θ and ϕ

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0, \tag{5}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = 0, \tag{6}$$

lead to

$$\frac{d}{dt}\left[m_0 q^2 \dot{\theta}\right] + m_0 q \sin\theta (p + q\cos\theta)\dot{\phi}^2 = 0, \tag{7}$$

and

$$\frac{d}{dt}\left[m_0 (p + \cos\theta)^2 \dot{\phi}\right] = 0, \tag{8}$$

respectively. Note that from (7), one obtains

$$q^2 \ddot{\theta} + q \sin\theta (p + q\cos\theta)\dot{\phi}^2 = 0. \tag{9}$$

Let us define

$$P_\phi \equiv m_0 (p + \cos\theta)^2 \dot{\phi}. \tag{10}$$

Observe that P_ϕ is the canonical momentum associated with ϕ. Moreover, from (8), one discovers that P_ϕ is a constant of the motion. It is not difficult to prove that P_ϕ is equivalent to the z-component of the angular momentum of the torus l_z. Thus, (10) can also be written as

$$l_z = m_0 (p + \cos\theta)^2 \dot{\phi}. \tag{11}$$

Solving (11) for $\dot{\phi}$ gives

$$\dot{\phi} = \frac{l_z}{m_0 (p + q\cos\theta)^2}. \tag{12}$$

On the other hand, substituting Equation (12) into (9) yields

$$q^2 \ddot{\theta} + \frac{q \sin\theta l_z^2}{m_0 (p + q\cos\theta)^3} = 0. \tag{13}$$

Multiplying this equation by $\dot{\theta}/2$ and simplifying, we find

$$\frac{d}{dt}\left[\frac{1}{2}m_0 q^2\dot{\theta}^2 + \frac{1}{2}\frac{l_z^2}{m_0(p+q\cos\theta)^2}\right] = 0. \tag{14}$$

This equation leads us to another constant of the motion which can be identified with the Hamiltonian and the total energy of the system E in the form

$$\frac{1}{2}m_0 q^2\dot{\theta}^2 + \frac{1}{2}\frac{l_z^2}{m_0(p+q\cos\theta)^2} = E. \tag{15}$$

Solving (15) for $\dot{\theta}$, we find the equation

$$\dot{\theta}^2 = \frac{1}{m_0 q^2}\left[2E - \frac{l_z^2}{m_0(p+q\cos\theta)^2}\right]. \tag{16}$$

Introducing the analogue parameter

$$h^2 = \frac{l_z^2}{2m_o E}, \tag{17}$$

the expression (16) can be rewrite as

$$\dot{\theta} = \frac{l_z}{qm_0 h}\frac{\sqrt{(p+q\cos\theta)^2 - h^2}}{(p+q\cos\theta)}. \tag{18}$$

Thus, from Equations (12) and (18), the ratio between $\dot{\theta}$ and $\dot{\phi}$ becomes

$$\frac{\dot{\theta}}{\dot{\phi}} = \frac{1}{qh}(p+q\cos\theta)\sqrt{(p+q\cos\theta)^2 - h^2}. \tag{19}$$

Observe that Equation (19) is a continue function of h and p. The values of h and p lead to different paths on the toroid. It is noticeable that the Equations (18) and (19) are similar to the geodesic equations derived by [2]. However, the advantage of our derivation is that (19) can be written in terms of the fundamentals constants l_z and E. This shall allow us a deeper understanding of the physics meaning provided by the dynamics of the system. In particular, observe from (17) that h^2 is negative if the total energy is negative. In fact, the parameter h^2 will play an important role in the determination of the different geodesics on the torus as we shall show in detail in the next section.

2.1. Geodesic on the surface of the torus

In this subsection, we shall analyze the possible trajectories of a particle on the torus. In particular, we shall focus our attention on the energy of the system E giving in (15).

Let us first consider the case $l_z = 0.$ From Equation (15), one obtains

$$E = \frac{1}{2} m_0 q^2 \dot{\theta}^2. \tag{20}$$

Note that according to (17), this case corresponds to $h = 0.$

If $l_z \neq 0,$ then we can assume that $\dot{\theta} = 0$ in Equation (15) and E becomes

$$E = \frac{l_z^2}{2(p + q\cos\theta_0)^2}, \tag{21}$$

where we set $\theta = \theta_0 = \text{const.}$ For $\theta_0 = 0,$ we get

$$E_0 = \frac{l_z^2}{2(p + q)^2}. \tag{22}$$

In the other words, from (17), we find $h_0 = (p + q)$ which corresponds to the outer equator of the torus. For $\theta_0 = \pi,$ we get

$$E_\pi = \frac{l_z^2}{2(p - q)^2}, \tag{23}$$

given $h_\pi = (p - q)$ for the inner equator.

In the case that $l_z \neq 0$ and $\dot{\theta} \neq 0$, the bounded orbits must satisfy that $E < 0$. In this case, from (18), one needs to impose the condition

$$0 \leq (p + q \cos \theta)^2 - h^2. \tag{24}$$

Table 1. Paths on the toroid

h	Geodesic
0	Meridians
$p - q$	The inner equator
$0 < h \leq (p - q)$	Unbounded orbits
$p + q$	The outer equator
$(p - q) \leq \mathcal{R} \leq (p + q)$	Bounded geodesics $(h^2 < 0)$

This can be rewritten as

$$h^2 \leq \mathcal{H}^2, \tag{25}$$

with

$$\mathcal{H}^2 = (p + q \cos \theta)^2. \tag{26}$$

Observe that

$$(p - q) \leq \mathcal{H} \leq (p + q). \tag{27}$$

Thus, we have discovered that bounded orbits correspond to the energy between E_0 and E_π according to Equations (22) and (23).

On the other hand, for $0 < E$, the non allowed orbits are determined when

$$(p + q \cos \theta)^2 - h^2 < 0. \tag{28}$$

Considering (27), from (28), we arrive to the relation

$$0 < h \leq (p - q). \tag{29}$$

As we mentioned, the parameter h determines the type of geodesic that travels around the torus. We observe h is related to the angular momentum l_z and the total energy of the system E. Table 1 shows some geodesics for different values of h.

2.2. The potential associated with the torus

The potential associated with the geometry can be obtained through the relations (1) (see [3]). In fact, from (1), one finds

$$r^4 + 2z^2r^2 - 2(p^2 + q^2)r^2 + [z^4 + 2(p^2 - q^2)z^2 + (p^2 - q^2)^2] = 0, \tag{30}$$

where we used polar coordinates for x and y. Our main goal here is to start with (30) and subsequently derive the analogue of the equations of motion.

For this purpose, let us first write (30) as

$$r^2 + \frac{[z^4 + 2(p^2 - q^2)z^2 + (p^2 - q^2)^2]}{r^2} + 2z^2 - 2(p^2 + q^2) = 0. \tag{31}$$

By taking the derivative (with respect to the time) of this expression, one obtains

$$\dot{r} = -\frac{1}{r}\frac{[z^3 + (p^2 - q^2)z + zr^2]}{[r^2 + z^2 - (p^2 + q^2)]}\dot{z}. \tag{32}$$

From (1), we have

$$\dot{z} = q\cos\theta\,\dot{\theta}. \tag{33}$$

Since from (1), we get

$$r = \sqrt{x^2 + y^2} = p + q\cos\theta, \tag{34}$$

then z and $\dot{z}$ can be written in terms of r as

$$z = \sqrt{q^2 - (r - p)^2} \quad \text{and} \quad \dot{z} = \frac{l_z^2}{m_o^2 hqr}(r - p)\sqrt{r^2 - h^2}, \tag{35}$$

where we had used the relation (18) in $\dot{z}$. Then substituting Equation (35) into (32), one obtains

$$\dot{r} = \frac{pl_z^2}{m_0^2 hq} \frac{(r-p)\sqrt{r^2-h^2}\sqrt{q^2-(r-p)^2}}{r^2}. \tag{36}$$

Deriving (36) once again and using (36) itself yields

$$\ddot{r} = \dot{r}^2\left[-\frac{2}{r} + \frac{1}{r-p} + \frac{r}{r^2-h^2} - \frac{r-p}{q^2-(r-p)^2}\right]. \tag{37}$$

Adding and subtracting $l_z^2 / m_o^2 r^3$ in the last equation and substituting Equation (36), we can rewrite the Equation (37) as

$$\ddot{r} = \frac{l_z^2}{m_0 r^3} - \frac{\partial V}{\partial r}, \tag{38}$$

where

$$V = -2\left[\dot{r}^2 + \frac{l_z^2}{2m_0}\frac{1}{r^2}\right]. \tag{39}$$

Thus, using (36), we discovered that V is given by

$$V = -\frac{l_z^2}{2m_0}\left[\frac{4p^2 l_z^2}{m_0^3 h^2 q^2}\frac{(r-p)^2(r^2-h^2)(q^2-(r-p)^2)}{r^4} - \frac{1}{r^2}\right]. \tag{40}$$

This is the potential associated with the geometric structure of the circular torus.

3. Elliptic Toroid

A more realistic approach, it is to consider the orbit of a planet in terms of the geometry of an elliptic toroid, namely,

$$x = (d_1 + \cos\theta)\cos\phi,$$

$$y = (d_2 + \cos\theta)\sin\phi,$$

$$z = \sin\theta,$$

where d_1 and d_2 are the ratio of the radius of the axis of the tube of the torus (p_1 (semi-major axis) and p_2 (semi-minor axis)) and the radius of the tube of the torus q.

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}[\dot{x}^2 + \dot{y}^2 + \dot{z}^2]. \tag{41}$$

From Equations (40) and (41), we get

$$\mathcal{L} = \frac{1}{2}\Big[\dot{\theta}^2 + [(d_1 + \cos\theta)^2 \sin^2\phi + (d_2 + \cos\theta)^2 \cos^2\phi]\dot{\phi}^2$$
$$+ 2(d_1 - d_2)\sin\theta\sin\phi\cos\phi\dot{\theta}\dot{\phi}\Big]. \tag{42}$$

Instead of following the Euler-Lagrange equation of motion

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}_i} - \frac{\partial\mathcal{L}}{\partial x_i} = 0, \tag{43}$$

where $x_1 = \theta$ and $x_2 = \phi$, we shall start from the Hamiltonian

$$\mathcal{H} = \dot{\theta}P_\theta + \dot{\phi}P_\phi + \mathcal{L}. \tag{44}$$

Here, P_θ and P_ϕ are the canonical momentum associated with the coordinates θ and ϕ, respectively. Specifically, we have

$$\frac{\partial\mathcal{L}}{\partial\dot{x}_i} = P_{x_i}. \tag{45}$$

From (42) and (45), we get

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \dot{\theta} + (d_1 - d_2)\sin\theta\sin\phi\cos\phi\dot{\phi}. \tag{46}$$

Thus, the canonical momentum P_θ can be written as

$$P_\theta = \dot{\theta} + (d_1 - d_2)\sin\theta\sin\phi\cos\phi\dot{\phi}. \tag{47}$$

While, the canonical moment P_ϕ becomes

$$\begin{aligned} P_\phi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\ &= [(d_1 + \cos\theta)^2 \sin^2\phi + (d_2 + \cos\theta)^2 \cos^2\phi]\dot{\phi} \\ &\quad + (d_1 - d_2)\sin\theta \sin\phi \cos\phi\dot{\theta}. \end{aligned} \tag{48}$$

From Equations (47) and (48), we find that the angular velocities $\dot{\theta}$ and $\dot{\phi}$ can be written as

$$\dot{\theta} = \frac{1}{g}\left[fP_\theta - hP_\phi\right], \tag{49}$$

and

$$\dot{\phi} = \frac{1}{g}\left[P_\phi - hP_\theta\right], \tag{50}$$

where

$$f = [(d_1 + \cos\theta)^2 \sin^2\phi + (d_2 + \cos\theta)^2 \cos^2\phi], \tag{51}$$

$$h = (d_1 - d_2)\sin\theta \sin\phi \cos\phi. \tag{52}$$

Thus, we discovered that the Lagrangian $\mathcal{L}$ can be rewritten in the form

$$\mathcal{L} = \frac{1}{2}\left[\ddot{\theta} + 2h\dot{\theta}\dot{\phi} + f\ddot{\phi}\right]. \tag{53}$$

Noticed that the determinant g of the metric is

$$g = f - h^2. \tag{54}$$

Considering the Lagrangian (53), the angular velocities (49) and (50), and the canonical momenta (47) and (48) one finds that the Hamiltonian (45) becomes

$$\mathcal{H} = \frac{1}{2g}\left[fP_\theta^2 - 2hfP_\theta P_\phi + P_\phi^2\right]. \tag{55}$$

If we assume that the Hamiltonian represents the total energy of the system E, then (55) leads

$$E = \frac{1}{2}\frac{\left[fP_\theta^2 - 2hfP_\theta P_\phi + P_\phi^2\right]}{f - h^2}. \tag{56}$$

Solving (56) for f, we obtain

$$f = \frac{P_\phi^2 + 2Eh^2}{2E - P_\theta^2 - 2hP_\theta P_\phi}. \tag{57}$$

Substituting Equations (52), (53), and (54) into the last equation and solving to f, we find a relation between the angles θ and ϕ in terms of the canonical moments, P_θ and P_ϕ, and the total energy of the system E

$$\begin{aligned}&[(d_1 + \cos\theta)^2 \sin^2\phi + (d_2 + \cos\theta)^2 \cos^2\phi] \\ &\quad = \frac{P_\phi^2 - 2E(d_1 - d_2)\sin\theta\sin\phi\cos\phi}{2E + 2(d_1 - d_2)\sin\theta\sin\phi\cos\phi P_\theta P_\phi - P_\theta^2}.\end{aligned} \tag{58}$$

The relation (58) shows us that it is not possible to express the last equation in terms of the constant of motion related to the canonical momentum P_θ and P_ϕ as we did for the circular toroid. In fact, we can see that this relation is consistent with the circular analysis due to the fact that when $d_1 = d_2$ (58) can be reduced to (14).

4. Final Remarks

We have developed a toroidal model that determines the rotation and translation movements of a planet around a source object. Following the geodesic path of a particle attached to the rotating planet adapted to the torus geometry, we show that the total energy and the z-component of the angular momentum, which are constants of motion, play a key role in the orbit described by a given planet. We have also developed the model in the case of elliptic toroid which is appropriated for planets with large eccentric orbits. It is worth mentioning that the elliptic case leads to a complicated formulae which can be solved through numerical approximations for further research.

One of the main results of our formalism is the link between the translation and rotation motions of a planet. In fact, by introducing a parameter h with length units, which is a function of the constants of the motion (energy and angular momentum) of the system, we find that for positive energies, $h^2 > 0$ the orbit described on the torus do not corresponds to planetary orbits. On the other hand, for negative energies one gets that $h^2 < 0$ and, therefore, this parameter can not be identified with a physical length. Nevertheless, in this case, we obtain the corresponding planetary orbits.

A number of interesting possible application of our toroidal model may emerge. First, one may try to determine the spin period of the exoplanet which recently has shown increasing attention [8, 9]. Second, the potential found (42) can be used in the context of the gravitational Poisson equation to obtain the density associated to a toroidal geometry. Moreover, it seems attractive to extend our toroidal model for the description of other astrophysical phenomena, such as the path of binary stars through the galaxy, globular clusters, epicyclic movements of the stars through the spiral galaxies and ring galaxies. Finally, our theory could be useful in the description of the toroidal dust surrounding super massive black-holes [10].

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