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WEIGHTED BOUNDEDNESS OF MULTILINEAR OPERATOR ASSOCIATED TO SINGULAR INTEGRAL OPERATOR SATISFYING A VARIANT OF HÖRMANDER'S CONDITION

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Abstract

In this paper, we establish the weighted sharp maximal function inequalities for the multilinear operator associated to the singular integral operator satisfying a variant of Hörmander's condition. As an application, we obtain the boundedness of the operator on weighted Lebesgue and Morrey spaces.

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1. Introduction

As the development of singular integral operators (see [10], [23], [24]), their commutators and multilinear operators have been well studied. In [5], [21], [22], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [2]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [14], [18], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [1], [13], the boundedness for the commutators generated by the singular integral operators and the weighted BMO and Lipschitz functions on $L^p(R^n)(1 < p < \infty)$ spaces are obtained (also see [12]). In [11], some singular integral operators satisfying a variant of Hörmander's condition are introduced, and the boundedness for the operators and their commutators are obtained (see [16], [25]). Motivated by these, in this paper, we will study the multilinear operator generated by the singular integral operator satisfying a variant of Hörmander's condition and the weighted Lipschitz and BMO functions.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well-known that (see [10], [23])

$$M^{\#}(f)(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For $\eta > 0$, let $M_\eta^{\#}(f)(x) = M^{\#}(|f|^\eta)^{1/\eta}(x)$ and $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < n$, $1 \le p < \infty$ and the non-negative weight function w, set

$$M_{\eta, p, w}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{w(Q)^{1-p\eta/n}} \int_Q |f(y)|^p w(y) dy \right)^{1/p}.$$

We write $M_{\eta, p, w}(f) = M_{p, w}(f)$ if $\eta = 0$.

The A_p weight is defined by (see [10]), for $1 < p < \infty$,

$$A_p = \left\{ 0 < w \in L^1_{\text{loc}}(R^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty \right\},$$

and

$$A_1 = \{ 0 < w \in L^p_{\text{loc}}(R^n) : M(w)(x) \le Cw(x),\ a.\ e. \}.$$

Given a non-negative weight function w. For $1 \le p < \infty$, the weighted Lebesgue space $L^p(R^n, w)$ is the space of functions f such that

$$\|f\|_{L^p(w)} = \left(\int_{R^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

For $0 < \beta < 1$ and the non-negative weight function w, the weighted Lipschitz space $Lip_\beta(w)$ is the space of functions b such that

$$\|b\|_{Lip_\beta(w)} = \sup_Q \frac{1}{w(Q)^{\beta/n}} \left(\frac{1}{w(Q)} \int_Q |b(y) - b_Q|^p w(x)^{1-p} dy \right)^{1/p} < \infty,$$

and the weighted BMO space $BMO(w)$ is the space of functions b such that

$$\|b\|_{BMO(w)} = \sup_Q \left(\frac{1}{w(Q)} \int_Q |b(y) - b_Q|^p w(x)^{1-p} dy \right)^{1/p} < \infty.$$

Remark. (1) It has been known that (see [9]), for $b \in Lip_\beta(w), w \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^k Q}| \leq Ck\|b\|_{Lip_\beta(w)} w(x) w(2^k Q)^{\beta/n}.$$

(2) It has been known that (see [1], [9]), for $b \in BMO(w)$, $w \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^k Q}| \leq Ck\|b\|_{BMO(w)} w(x).$$

(3) Let $b \in Lip_\beta(w)$ or $b \in BMO(w)$ and $w \in A_1$. By [8], we know that spaces $Lip_\beta(w)$ or $BMO(w)$ coincide and the norms $\|b\|_{Lip_\beta(w)}$ or $\|b\|_{BMO(w)}$ are equivalent with respect to different values $1 \leq p < \infty$.

Definition 1. Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \leq D\varphi(t) \quad \text{for } t \geq 0.$$

Let w be a non-negative weight function on R^n and f be a locally integrable function on R^n. Set, for $1 \leq p < \infty$,

$$\|f\|_{L^{p,\varphi}(w)} = \sup_{x \in R^n, d > 0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} |f(y)|^p w(y) dy \right)^{1/p},$$

where $Q(x, d) = \{y \in R^n : |x - y| < d\}$. The generalized weighted Morrey space is defined by

$$L^{p,\varphi}(R^n, w) = \{f \in L^1_{\text{loc}}(R^n) : \|f\|_{L^{p,\varphi}(w)} < \infty\}.$$

If $\varphi(d) = d^{\delta}$, $\delta > 0$, then $L^{p,\varphi}(R^n, w) = L^{p,\delta}(R^n, w)$, which is the classical Morrey spaces (see [19], [20]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n, w) = L^p(R^n, w)$, which is the weighted Lebesgue spaces (see [10]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [6], [7], [15], [17]).

Definition 2. Let $\Phi = \{\phi_1, \ldots, \phi_l\}$ be a finite family of bounded functions in R^n. For any locally integrable function *f*, the Φ sharp maximal function of *f* is defined by

$$M_{\Phi}^{\#}(f)(x) = \sup_{Q \ni x} \inf_{\{c_1, \ldots, c_l\}} \frac{1}{|Q|} \int_Q \left| f(y) - \sum_{i=1}^{l} c_i \phi_i(x_Q - y) \right| dy,$$

where the infimum is taken over all m-tuples $\{c_1, \ldots, c_l\}$ of complex numbers and x_Q is the center of Q. For $\eta > 0$, let

$$M_{\Phi,\eta}^{\#}(f)(x) = \sup_{Q \ni x} \inf_{\{c_1, \ldots, c_l\}} \left(\frac{1}{|Q|} \int_Q \left| f(y) - \sum_{i=1}^{l} c_j \phi_i(x_Q - y) \right|^{\eta} dy \right)^{1/\eta}.$$

Remark. We note that $M_{\Phi}^{\#} \approx f^{\#}$ if $l = 1$ and $\phi_1 = 1$.

Definition 3. Given a positive and locally integrable function f in R^n, we say that *f* satisfies the reverse Hölder's condition (write this as $f \in RH_{\infty}(R^n)$), if for any cube Q centered at the origin, we have

$$0 < \sup_{x \in Q} f(x) \leq C \frac{1}{|Q|} \int_Q f(y) dy.$$

In this paper, we will study some singular integral operators as following (see [11]).

Definition 4. Let $K \in L^2(R^n)$ and satisfy

$$\|K\|_{L^\infty} \leq C,$$

$$|K(x)| \leq C|x|^{-n},$$

there exist functions $B_1, \ldots, B_l \in L^1_{\text{loc}}(R^n - \{0\})$ and $\Phi = \{\phi_1, \ldots, \phi_l\} \subset L^\infty(R^n)$ such that $|\det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nl})$, and for a fixed $\delta > 0$ and any $|x| > 2|y| > 0$,

$$|K(x - y) - \sum_{i=1}^{l} B_i(x)\phi_i(y)| \leq C \frac{|y|^\delta}{|x - y|^{n+\delta}}.$$

For $f \in C_0^\infty$, we define the singular integral operator related to the kernel K by

$$T(f)(x) = \int_{R^n} K(x - y)f(y)dy.$$

Moreover, let m be the positive integer and b be the function on R^n. Set

$$R_{m+1}(b;\ x,\ y) = b(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha b(y)(x - y)^\alpha.$$

The multilinear operator related to the operator T is defined by

$$T^b(f)(x) = \int_{R^n} \frac{R_{m+1}(b;\ x,\ y)}{|x - y|^m} K(x - y)f(y)dy.$$

Note that the classical Calderón-Zygmund singular integral operator satisfies Definition 4 (see [10], [23]). Also note that the commutator $[b, T](f) = bT(f) - T(bf)$ is a particular operator of the multilinear operator T^b if $m = 0$. The multilinear operator T^b are the non-trivial generalizations of the commutator. It is well-known that commutators and multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [3], [4], [8]). The main purpose of this paper is to prove the sharp maximal inequalities for the multilinear operator T^b. As the application, we obtain the weighted L^p-norm inequality and Morrey space boundedness for the multilinear operator T^b.

2. Theorems and Lemmas

We shall prove the following theorems:

Theorem 1. *Let T be the singular integral operator as Definition* 4, $w \in A_1$, $0 < \eta < 1$, $1 < r < \infty$, *and* $D^\alpha b \in BMO(w)$ *for all* α *with* $|\alpha| = m$. *Then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$M^{\#}_{\Phi,\eta}(T^b(f))(\tilde{x}) \le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).$$

Theorem 2. *Let T be the singular integral operator as Definition* 4, $w \in A_1$, $0 < \eta < 1$, $1 < r < \infty$, $0 < \beta < 1$, *and* $D^\alpha b \in Lip_\beta(w)$ *for all* α *with* $|\alpha| = m$. *Then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$M^{\#}_{\Phi,\eta}(T^b(f))(\tilde{x}) \le C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,r,w}(f)(\tilde{x}).$$

Theorem 3. *Let T be the singular integral operator as Definition* 4, $w \in A_1$, $1 < p < \infty$, *and* $D^\alpha b \in BMO(w)$ *for all* α *with* $|\alpha| = m$. *Then* T^b *is bounded from* $L^p(R^n, w)$ *to* $L^p(R^n, w^{1-p})$.

Theorem 4. *Let T be the singular integral operator as Definition* 4, $w \in A_1$, $1 < p < \infty$, $0 < D < 2^n$, *and* $D^\alpha b \in BMO(w)$ *for all* α *with* $|\alpha| = m$. *Then* T^b *is bounded from* $L^{p,\varphi}(R^n, w)$ *to* $L^{p,\varphi}(R^n, w^{1-p})$.

Theorem 5. *Let T be the singular integral operator as Definition* 4, $w \in A_1$, $0 < \beta < 1$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$, *and* $D^\alpha b \in Lip_\beta(w)$ *for all* α *with* $|\alpha| = m$. *Then* T^b *is bounded from* $L^p(R^n, w)$ *to* $L^q(R^n, w^{1-q})$.

Theorem 6. *Let T be the singular integral operator as Definition* 4, $w \in A_1$, $0 < \beta < 1$, $0 < D < 2^n$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$, *and* $D^\alpha b \in Lip_\beta(w)$ *for all* α *with* $|\alpha| = m$. *Then* T^b *is bounded from* $L^{p,\varphi}(R^n, w)$ *to* $L^{q,\varphi}(R^n, w^{1-q})$.

To prove the theorems, we need the following lemmas:

Lemma 1 (See [10, p.485])**.** *Let* $0 < p < q < \infty$ *and for any function* $f \geq 0$. *We define that, for* $1/r = 1/p - 1/q$,

$$\|f\|_{WL^q} = \sup_{\lambda > 0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q}, \qquad N_{p,q}(f) = \sup_Q \|f\chi_Q\|_{L^p} / \|\chi_Q\|_{L^r},$$

where the sup *is taken for all measurable sets Q with* $0 < |Q| < \infty$. *Then*

$$\|f\|_{WL^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p} \|f\|_{WL^q}.$$

Lemma 2 (See [2])**.** *Let T be the singular integral operator as Definition* 4. *Then T is bounded on* $L^p(R^n, w)$ *for* $w \in A_p$ *with* $1 < p < \infty$, *and weak* (L^1, L^1) *bounded.*

Lemma 3 (See [11], [25])**.** *Let* $1 < p < \infty$, $0 < \eta < \infty$, $w \in A_\infty$ *and* $\Phi = \{\phi_1, \dots, \phi_l\} \subset L^\infty(R^n)$ *such that* $|\det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nl})$. *Then, for any smooth function f for which the left-hand side is finite,*

$$\int_{R^n} M_\eta(f)(x)^p w(x)dx \leq C \int_{R^n} M^{\#}_{\Phi,\eta}(f)(x)^p w(x)dx.$$

Lemma 4 (See [2], [9]). *Let* $0 \leq \eta < n, 1 \leq s < p < n/\eta, 1/q = 1/p - \eta/n$ *and* $w \in A_1$. *Then*

$$\|M_{\eta, s, w}(f)\|_{L^q(w)} \leq C\|f\|_{L^p(w)}.$$

Lemma 5 (See [9]). *Let* $0 \leq \eta < n, 0 < D < 2^n, 1 \leq s < p < n/\eta, 1/q = 1/p - \eta/n$ *and* $w \in A_1$. *Then*

$$\|M_{\eta, s, w}(f)\|_{L^{q,\varphi}(w)} \leq C\|f\|_{L^{p,\varphi}(w)}.$$

Lemma 6 (See [4]). *Let* b *be a function on* R^n *and* $D^\alpha A \in L^q(R^n)$ *for all* α *with* $|\alpha| = m$ *and any* $q > n$. *Then*

$$|R_m(b;\, x,\, y)| \leq C|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\widetilde{Q}(x,\, y)|} \int_{\widetilde{Q}(x,\, y)} |D^\alpha b(z)|^q \, dz \right)^{1/q},$$

where $\widetilde{Q}$ *is the cube centered at x and having side length* $5\sqrt{n}|x - y|$.

Lemma 7. *Let* $1 < p < \infty, 0 < \eta < \infty, w \in A_1, 0 < D < 2^n$, *and* $\Phi = \{\phi_1, \ldots, \phi_l\} \subset L^\infty(R^n)$ *such that* $|\det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nl})$. *Then, for any smooth function f for which the left-hand side is finite*

$$\|M_\eta(f)\|_{L^{p,\varphi}(w)} \leq C\|M^{\#}_{\Phi,\eta}(f)\|_{L^{p,\varphi}(w)}.$$

Proof. For any cube $Q = Q(x_0, d)$ in R^n, we know $M(w\chi_Q) \in A_1$ for any cube $Q = Q(x, d)$ by [10]. By Lemma 3, we have, for $f \in L^{p,\varphi}(R^n, w)$,

$$\int_Q |M_\eta(f)(y)|^p w(y)dy = \int_{R^n} |M_\eta(f)(y)|^p w(y)\chi_Q(y)dy$$

$$\leq \int_{R^n} |M_\eta(f)(y)|^p M(w\chi_Q)(y)dy \leq C\int_{R^n} |M^{\#}_{\Phi,\eta}(f)(y)|^p M(w\chi_Q)(y)dy$$

$$= C\left(\int_Q |M^{\#}_{\Phi,\eta}(f)(y)|^p M(w\chi_Q)(y)dy + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} |M^{\#}_{\Phi,\eta}(f)(y)|^p M(w\chi_Q)(y)dy \right)$$

$$\leq C\left(\int_Q |M^{\#}_{\Phi,\eta}(f)(y)|^p w(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|M^{\#}_{\Phi,\eta}(f)(y)|^p \frac{w(Q)}{|2^{k+1}Q|}dy\right)$$

$$\leq C\left(\int_Q |M^{\#}_{\Phi,\eta}(f)(y)|^p w(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}Q}|M^{\#}_{\Phi,\eta}(f)(y)|^p \frac{M(w)(y)}{2^{n(k+1)}}dy\right)$$

$$\leq C\left(\int_Q |M^{\#}_{\Phi,\eta}(f)(y)|^p w(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}Q}|M^{\#}_{\Phi,\eta}(f)(y)|^p \frac{w(y)}{2^{nk}}dy\right)$$

$$\leq C\|M^{\#}_{\Phi,\eta}(f)\|^p_{L^{p,\varphi}(w)}\sum_{k=0}^{\infty}2^{-nk}\varphi(2^{k+1}d)$$

$$\leq C\|M^{\#}_{\Phi,\eta}(f)\|^p_{L^{p,\varphi}(w)}\sum_{k=0}^{\infty}(2^{-n}D)^k\varphi(d)$$

$$\leq C\|M^{\#}_{\Phi,\eta}(f)\|^p_{L^{p,\varphi}(w)}\varphi(d),$$

thus,

$$\left(\frac{1}{\varphi(d)}\int_Q M_\eta(f)(x)^p w(x)dx\right)^{1/p} \leq C\left(\frac{1}{\varphi(d)}\int_Q M^{\#}_{\Phi,\eta}(f)(x)^p w(x)dx\right)^{1/p},$$

and

$$\|M_\eta(f)\|_{L^{p,\varphi}(w)} \leq C\|M^{\#}_{\Phi,\eta}(f)\|_{L^{p,\varphi}(w)}.$$

This finishes the proof.

3. Proofs of Theorems

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - C_0\right|^\eta dx\right)^{1/\eta} \leq C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}w(\tilde{x})M_{r,w}(f)(\tilde{x}),$$

where Q is any a cube centered at x_0, $C_0 = \sum_{j=1}^{l} c_j \phi_j(x_0 - x)$ and $c_j = \int_{R^n} \frac{K(x_0, y)}{|x_0 - y|^m} B_j(x_0 - y) f_2(y) dy$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}(x) = b(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha b)_{\tilde{Q}} x^\alpha$, then $R_m(b; x, y) = R_m(\tilde{b}; x, y)$ and $D^\alpha \tilde{b} = D^\alpha b - (D^\alpha b)_{\tilde{Q}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$T^b(f)(x) = \int_{R^n} \frac{R_m(\tilde{b}; x, y)}{|x - y|^m} K(x - y) f_1(y) dy$$

$$- \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{(x - y)^\alpha D^\alpha \tilde{b}(y)}{|x - y|^m} K(x - y) f_1(y) dy$$

$$+ \int_{R^n} \frac{R_{m+1}(\tilde{b}; x, y)}{|x - y|^m} K(x - y) f_2(y) dy$$

$$= T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x - \cdot|^m} f_1\right) - T\left(\sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x - \cdot)^\alpha D^\alpha \tilde{b}}{|x - \cdot|^m} f_1\right) + T^{\tilde{b}}(f_2)(x),$$

then

$$\left(\frac{1}{|Q|} \int_Q \left|T^b(f)(x) - C_0\right|^\eta dx\right)^{1/\eta}$$

$$\leq C\left(\frac{1}{|Q|} \int_Q \left|T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x - \cdot|^m} f_1\right)\right|^\eta dx\right)^{1/\eta} + C\left(\frac{1}{|Q|} \int_Q \left|T\left(\sum_{|\alpha|=m} \frac{(x - \cdot)^\alpha D^\alpha \tilde{b}}{|x - \cdot|^m} f_1\right)\right|^\eta dx\right)^{1/\eta}$$

$$+ C\left(\frac{1}{|Q|} \int_Q \left|T^{\tilde{b}}(f_2)(x) - C_0\right|^\eta dx\right)^{1/\eta}$$

$$= I_1 + I_2 + I_3.$$

For I_1, noting that $w \in A_1$, w satisfies the reverse of Hölder's inequality:

$$\left(\frac{1}{|Q|}\int_Q w(x)^{p_0}\,dx\right)^{1/p_0} \le \frac{C}{|Q|}\int_Q w(x)dx,$$

for all cube Q and some $1 < p_0 < \infty$ (see [10]). We take $q = rp_0 / (r + p_0 - 1)$ in Lemma 6 and have $1 < q < r$ and $p_0 = q(r-1)/(r-q)$, then by the Lemma 6 and Hölder's inequality, we gain

$$|R_m(\tilde{b};\, x,\, y)| \le C|x-y|^m \sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}(x,\, y)|}\int_{\tilde{Q}(x,y)}|D^\alpha\tilde{b}(z)|^q\,dz\right)^{1/q}$$

$$\le C|x-y|^m \sum_{|\alpha|=m}|\tilde{Q}|^{-1/q}\left(\int_{\tilde{Q}(x,y)}|D^\alpha\tilde{b}(z)|^q\,w(z)^{q(1-r)/r}\,w(z)^{q(r-1)/r}\,dz\right)^{1/q}$$

$$\le C|x-y|^m \sum_{|\alpha|=m}|\tilde{Q}|^{-1/q}\left(\int_{\tilde{Q}(x,y)}|D^\alpha\tilde{b}(z)|^r\,w(z)^{1-r}\,dz\right)^{1/r}\left(\int_{\tilde{Q}(x,y)}w(z)^{q(r-1)/(r-q)}dz\right)^{(r-q)/rq}$$

$$\le C|x-y|^m \sum_{|\alpha|=m}|\tilde{Q}|^{-1/q}\|D^\alpha b\|_{BMO(w)}w(\tilde{Q})^{1/r}|\tilde{Q}|^{(r-q)/rq}$$

$$\times\left(\frac{1}{|\tilde{Q}(x,\, y)|}\int_{\tilde{Q}(x,y)}w(z)^{p_0}\,dz\right)^{(r-q)/rq}$$

$$\le C|x-y|^m \sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}|\tilde{Q}|^{-1/q}\,w(\tilde{Q})^{1/r}|\tilde{Q}|^{1/q-1/r}$$

$$\times\left(\frac{1}{|\tilde{Q}(x,\, y)|}\int_{\tilde{Q}(x,y)}w(z)dz\right)^{(r-1)/r}$$

$$\le C|x-y|^m \sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}|\tilde{Q}|^{-1/q}\,w(\tilde{Q})^{1/r}|\tilde{Q}|^{1/q-1/r}\,w(\tilde{Q})^{1-1/r}|\tilde{Q}|^{1/r-1}$$

$$\le C|x-y|^m \sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}\,\frac{w(\tilde{Q})}{|\tilde{Q}|}$$

$$\leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}),$$

thus, by the L^s-boundedness of T (see Lemma 2) for $1 < s < r$ and $w \in A_1 \subseteq A_{r/s}$, we obtain

$$I_1 \leq \frac{C}{|Q|} \int_Q \left| T\left(\frac{R_m(\tilde{b};\, x,\, \cdot\,)}{|x - \cdot|^m} f_1 \right) \right| dx$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \left(\frac{1}{|Q|} \int_{R^n} |T(f_1)(x)|^s dx \right)^{1/s}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) |Q|^{-1/s} \left(\int_{R^n} |f_1(x)|^s dx \right)^{1/s}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) |Q|^{-1/s} \left(\int_{\tilde{Q}} |f(x)|^s w(x)^{s/r} w(x)^{-s/r} dx \right)^{1/s}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) |Q|^{-1/s} \left(\int_{\tilde{Q}} |f(x)|^r w(x) dx \right)^{1/r}$$

$$\times \left(\int_{\tilde{Q}} w(x)^{-s/(r-s)} dx \right)^{(r-s)/rs}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) |Q|^{-1/s} w(\tilde{Q})^{1/r} \left(\frac{1}{w(\tilde{Q})} \int_{\tilde{Q}} |f(x)|^r w(x) dx \right)^{1/r}$$

$$\times \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x)^{-s/(r-s)} dx \right)^{(r-s)/rs} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x) dx \right)^{1/r} |\tilde{Q}|^{1/s} w(\tilde{Q})^{-1/r}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).$$

For I_2, by the weak (L^1, L^1) boundedness of T (see Lemma 2) and Kolmogoro's inequality (see Lemma 1), we obtain

$$I_2 \leq C \sum_{|\alpha|=m} \left(\frac{1}{|Q|} \int_Q |T(D^\alpha \tilde{b} f_1)(x)|^\eta dx \right)^{1/\eta}$$

$$\leq C \sum_{|\alpha|=m} \frac{|Q|^{1/\eta-1}}{|Q|^{1/\eta}} \frac{\|T(D^\alpha \tilde{b} f_1)\chi_Q\|_{L^\eta}}{\|\chi_Q\|_{L^{\eta/(1-\eta)}}}$$

$$\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \|T(D^\alpha \tilde{b} f_1)\|_{WL^1}$$

$$\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \int_{R^n} |D^\alpha \tilde{b}(x) f_1(x)| dx$$

$$\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \int_{\tilde{Q}} |D^\alpha b(x) - (D^\alpha b)_{\tilde{Q}}| w(x)^{-1/r} |f(x)| w(x)^{1/r} dx$$

$$\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \left(\int_{\tilde{Q}} |(D^\alpha b(x) - (D^\alpha b)_{\tilde{Q}})|^{r'} w(x)^{1-r'} dx \right)^{1/r'}$$

$$\times \left(\int_{\tilde{Q}} |f(x)|^r w(x) dx \right)^{1/r}$$

$$\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \|D^\alpha b\|_{BMO(w)} w(\tilde{Q})^{1/r'} w(\tilde{Q})^{1/r} \left(\frac{1}{w(\tilde{Q})} \int_{\tilde{Q}} |f(x)|^r w(x) dx \right)^{1/r}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \frac{w(\tilde{Q})}{|\tilde{Q}|} M_{r,w}(f)(\tilde{x})$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).$$

For I_3, we write

$$|T^{\tilde{b}}(f_2)(x) - C_0| \le \int_{R^n} \left| \frac{R_m(\tilde{b};\, x,\, y)}{|x-y|^m} - \frac{R_m(\tilde{b};\, x_0,\, y)}{|x_0-y|^m} \right| |K(x-y)|\,|f_2(y)| dy$$

$$+ \int_{R^n} \frac{|R_{m+1}(\tilde{b};\, x_0,\, y)|}{|x_0-y|^m} \left|K(x-y) - \sum_{j=1}^{l} B_j(x_0-y)\phi_j(x_0-x)\right| |f_2(y)|\,dy$$

$$+ C \sum_{|\alpha|=m} \int_{R^n} \left| \frac{(x-y)^\alpha}{|x-y|^m} - \frac{(x_0-y)^\alpha}{|x_0-y|^m} \right| |K(x-y)|\,|D^\alpha \tilde{b}(y)|\,|f_2(y)|\,dy$$

$$= I_3^{(1)}(x) + I_3^{(2)}(x) + I_3^{(3)}(x).$$

By the formula (see [4]):

$$R_m(\tilde{b};\, x,\, y) - R_m(\tilde{b};\, x_0,\, y) = \sum_{|\gamma|<m} \frac{1}{\gamma!} R_{m-|\gamma|}(D^\gamma \tilde{b};\, x,\, x_0)(x-y)^\gamma,$$

and Lemma 8, we have, similar to the proof of I_1,

$$|R_m(\tilde{b};\, x,\, y) - R_m(\tilde{b};\, x_0,\, y) \le C \sum_{|\gamma|<m} \sum_{|\alpha|=m} |x-x_0|^{m-|\gamma|} |x-y|^{|\gamma|}$$

$$\times \|D^\alpha b\|_{BMO(w)} w(\tilde{x}).$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, thus, by $w \in A_1 \subseteq A_r$,

$$I_3^{(1)}(x) \le \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} |R_m(\tilde{b};\, x,\, y) - R_m(\tilde{b};\, x_0,\, y)| \frac{|K(x-y)|}{|x-y|^m} |f(y)| dy$$

$$+ \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} \left| \frac{1}{|x-y|^m} - \frac{1}{|x_0-y|^m} \right| |R_m(\tilde{b};\, x_0,\, y)|\,|K(x-y)|\,|f(y)| dy$$

$$\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} \frac{|x-x_0|}{|x_0-y|^{n+1}} |f(y)|\,dy$$

$$\leq C\sum_{|\alpha|=m}\|D^{\alpha}b\|_{BMO(w)}w(\widetilde{x})\sum_{k=1}^{\infty}\frac{d}{(2^k d)^{n+1}}\int_{2^k\widetilde{Q}}|f(y)|w(y)^{1/r}w(y)^{-1/r}dy$$

$$\leq C\sum_{|\alpha|=m}\|D^{\alpha}b\|_{BMO(w)}w(\widetilde{x})\sum_{k=1}^{\infty}\frac{d}{(2^k d)^{n+1}}\left(\int_{2^k\widetilde{Q}}|f(y)|^r w(y)dy\right)^{1/r}$$

$$\times\left(\int_{2^k\widetilde{Q}}w(y)^{-1/(r-1)}dy\right)^{(r-1)/r}$$

$$\leq C\sum_{|\alpha|=m}\|D^{\alpha}b\|_{BMO(w)}w(\widetilde{x})\sum_{k=1}^{\infty}\frac{d}{(2^k d)^{n+1}}\,w(2^k\widetilde{Q})^{1/r}$$

$$\times\left(\frac{1}{w(2^k\widetilde{Q})}\int_{2^k\widetilde{Q}}|f(y)|^r w(y)dx\right)^{1/r}$$

$$\times\left(\frac{1}{|2^k\widetilde{Q}|}\int_{2^k\widetilde{Q}}w(y)^{-1/(r-1)}dy\right)^{(r-1)/r}\left(\frac{1}{|2^k\widetilde{Q}|}\int_{2^k\widetilde{Q}}w(y)dy\right)^{1/r}$$

$$\times\,|2^k\widetilde{Q}|w(2^k\widetilde{Q})^{-1/r}$$

$$\leq C\sum_{|\alpha|=m}\|D^{\alpha}b\|_{BMO(w)}w(\widetilde{x})M_{r,w}(f)(\widetilde{x})\sum_{k=1}^{\infty}2^{-k}$$

$$\leq C\sum_{|\alpha|=m}\|D^{\alpha}b\|_{BMO(w)}w(\widetilde{x})M_{r,w}(f)(\widetilde{x}).$$

For $I_3^{(2)}(x)$ by the conditions on K, we get

$$I_3^{(2)}(x)\leq C\sum_{k=0}^{\infty}\int_{2^{k+1}\widetilde{Q}\setminus 2^k\widetilde{Q}}\frac{|R_m(\widetilde{b};\,x_0,\,y)|}{|x_0-y|^m}$$

$$\left|K(x-y)-\sum_{j=1}^{l}B_j(x_0-y)\phi_j(x_0-x)\right||f(y)|dy$$

$$+ C\sum_{|\alpha|=m}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|D^\alpha\tilde{b}(y)||(x_0-y)^\alpha|}{|x_0-y|^m}$$

$$|K(x-y)-\sum_{j=1}^{l}B_j(x_0-y)\phi_j(x_0-x)||f(y)|dy$$

$$\leq C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}w(\tilde{x})\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|^\delta}{|x_0-y|^{n+\delta}}|f(y)|dy$$

$$+ C\sum_{|\alpha|=m}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|D^\alpha b(y)-(D^\alpha b)_{2^{k+1}\tilde{Q}}|\frac{|x-x_0|^\delta}{|x_0-y|^{n+\delta}}|f(y)|dy$$

$$+ C\sum_{|\alpha|=m}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|(D^\alpha b)_{2^{k+1}\tilde{Q}}-(D^\alpha b)_{\tilde{Q}}|\frac{|x-x_0|^\delta}{|x_0-y|^{n+\delta}}|f(y)|dy$$

$$\leq C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}\frac{d^\delta}{(2^k d)^{n+\delta}}\left(\int_{2^k\tilde{Q}}|D^\alpha b(y)-(D^\alpha b)_{2^k\tilde{Q}}|^{r'}w(y)^{1-r'}dy\right)^{1/r'}$$

$$\times\left(\int_{2^k\tilde{Q}}|f(y)|^r w(y)dx\right)^{1/r}$$

$$+ C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}w(\tilde{x})\sum_{k=1}^{\infty}k\,\frac{d^\delta}{(2^k d)^{n+\delta}}\left(\int_{2^k\tilde{Q}}|f(y)|^r w(y)dx\right)^{1/r}$$

$$\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)^{-1/(r-1)}dy\right)^{(r-1)/r}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)dy\right)^{1/r}$$

$$\times|2^k\tilde{Q}|w(2^k\tilde{Q})^{-1/r}$$

$$\leq C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}w(\tilde{x})\sum_{k=1}^{\infty}k2^{-k\delta}\left(\frac{1}{w(2^k\tilde{Q})}\int_{2^k\tilde{Q}}|f(y)|^r w(y)dx\right)^{1/r}$$

$$\leq C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(w)}w(\tilde{x})M_{r,w}(f)(\tilde{x}).$$

Similarly, we have

$$I_3^{(3)}(x) \leq C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\widetilde{Q}\setminus 2^k\widetilde{Q}} \frac{|x-x_0|^{\delta}}{|x_0-y|^{n+1}} |f(y)|\,|D^{\alpha}\widetilde{b}(y)|dy$$

$$\leq C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \frac{d}{(2^k d)^{n+1}} \int_{2^k\widetilde{Q}} |D^{\alpha}b(y) - (D^{\alpha}b)_{2^k\widetilde{Q}}|w(y)^{-1/r}|f(y)|w(y)^{1/r}dy$$

$$+ C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \frac{d}{(2^k d)^{n+1}} \int_{2^k\widetilde{Q}} |(D^{\alpha}b)_{2^k\widetilde{Q}} - (D^{\alpha}b)_{\widetilde{Q}}|\,|f(y)|w(y)^{1/r}w(y)^{-1/r}dy$$

$$\leq C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \frac{d}{(2^k d)^{n+1}} \left(\int_{2^k\widetilde{Q}} |(D^{\alpha}b(y) - (D^{\alpha}b)_{2^k\widetilde{Q}})^{r'}|w(y)^{1-r'}dy\right)^{1/r'}$$

$$\times \left(\int_{2^k\widetilde{Q}} |f(y)|^r w(y)dy\right)^{1/r}$$

$$+ C \sum_{|\alpha|=m} \|D^{\alpha}b\|_{BMO(w)} w(\widetilde{x}) \sum_{k=1}^{\infty} k \frac{d}{(2^k d)^{n+1}} \left(\int_{2^k\widetilde{Q}} |f(y)|^r w(y)dx\right)^{1/r}$$

$$\times \left(\frac{1}{|2^k\widetilde{Q}|}\int_{2^k\widetilde{Q}} w(y)^{-1/(r-1)}dy\right)^{(r-1)/r} \left(\frac{1}{|2^k\widetilde{Q}|}\int_{2^k\widetilde{Q}} w(y)dy\right)^{1/r}$$

$$\times |2^k\widetilde{Q}|w(2^k\widetilde{Q})^{-1/r}$$

$$\leq C \sum_{|\alpha|=m} \|D^{\alpha}b\|_{BMO(w)} \sum_{k=1}^{\infty} 2^{-k} \frac{w(2^k\widetilde{Q})}{|2^k\widetilde{Q}|} \left(\frac{1}{w(2^k\widetilde{Q})}\int_{2^k\widetilde{Q}} |f(y)|^r w(y)dx\right)^{1/r}$$

$$+ C \sum_{|\alpha|=m} \|D^{\alpha}b\|_{BMO(w)} w(\widetilde{x}) \sum_{k=1}^{\infty} k2^{-k} \left(\frac{1}{w(2^k\widetilde{Q})}\int_{2^k\widetilde{Q}} |f(y)|^r w(y)dx\right)^{1/r}$$

$$\leq C \sum_{|\alpha|=m} \|D^{\alpha}b\|_{BMO(w)} w(\widetilde{x}) M_{r,w}(f)(\widetilde{x}).$$

Thus,

$$I_3 \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).$$

These complete the proof of Theorem 1.

Proof of Theorem 2. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - C_0\right|^\eta dx\right)^{1/\eta} \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,s,w}(f)(\tilde{x}),$$

where Q is any a cube centered at x_0, $C_0 = \sum_{j=1}^m c_j \phi_j(x_0 - x)$ and $c_j = \int_{R^n} \frac{K(x_0, y)}{|x_0 - y|^m} B_j(x_0 - y) f_2(y) dy$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similar to the proof of Theorem 1, we have, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - C_0\right|^\eta dx\right)^{1/\eta}$$

$$\leq C\left(\frac{1}{|Q|}\int_Q \left|T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x - \cdot|^m} f_1\right)\right|^\eta dx\right)^{1/\eta}$$

$$+ C\left(\frac{1}{|Q|}\int_Q \left|T\left(\sum_{|\alpha|=m} \frac{(x - \cdot)^\alpha D^\alpha \tilde{b}}{|x - \cdot|^m} f_1\right)\right|^\eta dx\right)^{1/\eta}$$

$$+ C\left(\frac{1}{|Q|}\int_Q \left|T^{\tilde{b}}(f_2)(x) - C_0\right|^\eta dx\right)^{1/\eta}$$

$$= J_1 + J_2 + J_3.$$

For J_1 and J_2, by using the same argument as in the proof of Theorem 1, we get

$$|R_m(\widetilde{b};\, x,\, y)| \leq C|x-y|^m \sum_{|\alpha|=m} |\widetilde{Q}|^{-1/q}\left(\int_{\widetilde{Q}(x,\,y)} |D^\alpha \widetilde{b}(z)|^q w(z)^{q(1-r)/r} w(z)^{q(r-1)/r} dz\right)^{1/q}$$

$$\leq C|x-y|^m \sum_{|\alpha|=m} |\widetilde{Q}|^{-1/q}\left(\int_{\widetilde{Q}(x,\,y)} |D^\alpha \widetilde{b}(z)|^r w(z)^{1-r} dz\right)^{1/r}$$

$$\times \left(\int_{\widetilde{Q}(x,\,y)} w(z)^{q(r-1)/(r-q)} dz\right)^{(r-q)/rq}$$

$$\leq C|x-y|^m \sum_{|\alpha|=m} |\widetilde{Q}|^{-1/q} \|D^\alpha b\|_{Lip_\beta(w)} w(\widetilde{Q})^{\beta/n+1/r} |\widetilde{Q}|^{(r-q)/rq}$$

$$\times \left(\frac{1}{|\widetilde{Q}(x,\,y)|}\int_{\widetilde{Q}(x,\,y)} w(z)^{p_0} dz\right)^{(r-q)/rq}$$

$$\leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} |\widetilde{Q}|^{-1/q} w(\widetilde{Q})^{\beta/n+1/r} |\widetilde{Q}|^{1/q-1/r}$$

$$\times \left(\frac{1}{|\widetilde{Q}(x,\,y)|}\int_{\widetilde{Q}(x,\,y)} w(z) dz\right)^{(r-1)/r}$$

$$\leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} |\widetilde{Q}|^{-1/q} w(\widetilde{Q})^{\beta/n+1/r} |\widetilde{Q}|^{1/q-1/r}$$

$$\times\, w(\widetilde{Q})^{1-1/r} |\widetilde{Q}|^{1/r-1}$$

$$\leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\widetilde{Q})^{\beta/n} w(\widetilde{x}),$$

thus

$$J_1 \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\widetilde{Q})^{\beta/n} w(\widetilde{x}) |Q|^{-1/s} \left(\int_{R^n} |f_1(x)|^s dx \right)^{1/s}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\widetilde{Q})^{\beta/n} w(\widetilde{x}) |Q|^{-1/s} \left(\int_{\widetilde{Q}} |f(x)|^r w(x) dx \right)^{1/r}$$

$$\times \left(\int_{\widetilde{Q}} w(x)^{-s/(r-s)} dx \right)^{(r-s)/rs}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\widetilde{x}) |\widetilde{Q}|^{-1/s} w(\widetilde{Q})^{1/r}$$

$$\times \left(\frac{1}{w(\widetilde{Q})^{1-r\beta/n}} \int_{\widetilde{Q}} |f(x)|^r w(x) dx \right)^{1/r}$$

$$\times \left(\frac{1}{|\widetilde{Q}|} \int_{\widetilde{Q}} w(x)^{-s/(r-s)} dx \right)^{(r-s)/rs} \left(\frac{1}{|\widetilde{Q}|} \int_{\widetilde{Q}} w(x) dx \right)^{1/r} |\widetilde{Q}|^{1/s} w|\widetilde{Q}|^{-1/r}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} w(\widetilde{x}) M_{\beta, r, w}(f)(\widetilde{x}),$$

$$J_2 \leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \int_{\widetilde{Q}} |D^\alpha b(x) - (D^\alpha b)_{\widetilde{Q}}| \, |w(x)^{-1/r}| \, |f(x)| w(x)^{1/r} dx$$

$$\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \left(\int_{\widetilde{Q}} |(D^\alpha b(x) - (D^\alpha b)_{\widetilde{Q}}|^{r'} w(x)^{1-r'} dx \right)^{1/r'}$$

$$\times \left(\int_{\widetilde{Q}} |f(x)|^r w(x) dx \right)^{1/r}$$

$$\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \|D^\alpha b\|_{Lip_\beta(w)} w(\widetilde{Q})^{\beta/n + 1/r'} w(\widetilde{Q})^{1/r - \beta/n}$$

$$\times \left(\frac{1}{w(\widetilde{Q})^{1-r\beta/n}} \int_{\widetilde{Q}} |f(x)|^r w(x) dx \right)^{1/r}$$

$$\leq C \sum_{|\alpha|=m} \|D^{\alpha} b\|_{Lip_{\beta}(w)} \frac{w(\widetilde{Q})}{|\widetilde{Q}|} M_{\beta, r, w}(f)(\widetilde{x})$$

$$\leq C \sum_{|\alpha|=m} \|D^{\alpha} b\|_{Lip_{\beta}(w)} w(\widetilde{x}) M_{\beta, r, w}(f)(\widetilde{x}).$$

For J_3, we have

$$|R_m(\widetilde{b};\, x,\, y) - R_m(\widetilde{b};\, x_0,\, y)|$$

$$\leq C \sum_{|\gamma|<m} \sum_{|\alpha|=m} |x - x_0|^{m-|\gamma|} |x - y|^{|\gamma|} \|D^{\alpha} b\|_{Lip_{\beta}(w)} w(\widetilde{x}) w(2^k \widetilde{Q})^{\beta/n},$$

thus

$$|T^{\widetilde{b}}(f_2)(x) - C_0|$$

$$\leq \sum_{k=0}^{\infty} \int_{2^{k+1}\widetilde{Q} \setminus 2^k \widetilde{Q}} |R_m(\widetilde{b};\, x,\, y) - R_m(\widetilde{b};\, x_0,\, y)| \frac{|K(x-y)|}{|x-y|^m} |f(y)| dy$$

$$+ \sum_{k=0}^{\infty} \int_{2^{k+1}\widetilde{Q} \setminus 2^k \widetilde{Q}} \left| \frac{1}{|x-y|^m} - \frac{1}{|x_0 - y|^m} \right| |R_m(\widetilde{b};\, x_0,\, y)| |K(x-y)| |f(y)| dy$$

$$+ C \sum_{k=0}^{\infty} \int_{2^{k+1}\widetilde{Q} \setminus 2^k \widetilde{Q}} \frac{|R_m(\widetilde{b}; x_0, y)|}{|x_0 - y|^m} \left| K(x-y) - \sum_{j=1}^{l} B_j(x_0 - y) \phi_j(x_0 - x) \right| |f(y)| dy$$

$$+ C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\widetilde{Q} \setminus 2^k \widetilde{Q}} \frac{|D^{\alpha} \widetilde{b}(y)| |(x_0 - y)^{\alpha}|}{|x_0 - y|^m} \left| K(x-y) \right.$$

$$\left. - \sum_{j=1}^{l} B_j(x_0 - y) \phi_j(x_0 - x) \right| |f(y)| dy$$

$$+ C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\widetilde{Q} \setminus 2^k \widetilde{Q}} \left| \frac{(x-y)^{\alpha}}{|x-y|^m} - \frac{(x_0 - y)^{\alpha}}{|x_0 - y|^m} \right| |K(x-y)| |D^{\alpha} \widetilde{b}(y)| |f(y)| dy$$

$$\leq C\sum_{|\alpha|=m}\|D^{\alpha}b\|_{Lip_{\beta}(w)}w(\widetilde{x})\sum_{k=0}^{\infty}w(2^{k+1}\widetilde{Q})^{\beta/n}$$

$$\times\int_{2^{k+1}\widetilde{Q}\setminus 2^{k}\widetilde{Q}}\left(\frac{|x-x_0|}{|x_0-y|^{n+1}}+\frac{|x-x_0|^{\delta}}{|x_0-y|^{n+\delta}}\right)|f(y)|\,dy$$

$$+C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}\left(\frac{|x-x_0|}{|x_0-y|^{n+1}}+\frac{|x-x_0|^{\delta}}{|x_0-y|^{n+\delta}}\right)$$

$$\times\int_{2^{k}\widetilde{Q}}|D^{\alpha}b(y)-(D^{\alpha}b)_{2^{k}\widetilde{Q}}|w(y)^{-1/r}|f(y)|w(y)^{1/r}\,dy$$

$$+C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}\left(\frac{|x-x_0|}{|x_0-y|^{n+1}}+\frac{|x-x_0|^{\delta}}{|x_0-y|^{n+\delta}}\right)$$

$$\times\int_{2^{k}\widetilde{Q}}|(D^{\alpha}b)_{2^{k}\widetilde{Q}}-(D^{\alpha}b)_{\widetilde{Q}}||f(y)|w(y)^{-1/r}w(y)^{1/r}\,dy$$

$$\leq C\sum_{|\alpha|=m}\|D^{\alpha}b\|_{Lip_{\beta}(w)}w(\widetilde{x})\sum_{k=0}^{\infty}\left(\frac{d}{(2^{k}d)^{n+1}}+\frac{d^{\delta}}{(2^{k}d)^{n+\delta}}\right)w(2^{k}\widetilde{Q})^{\beta/n}$$

$$\times\left(\int_{2^{k}\widetilde{Q}}|f(y)|^{r}w(y)dx\right)^{1/r}\left(\frac{1}{|2^{k}\widetilde{Q}|}\int_{2^{k}\widetilde{Q}}w(y)^{-1/(r-1)}dy\right)^{(r-1)/r}$$

$$\times\left(\frac{1}{|2^{k}\widetilde{Q}|}\int_{2^{k}\widetilde{Q}}w(y)dy\right)^{1/r}|2^{k}\widetilde{Q}|w(2^{k}\widetilde{Q})^{-1/r}$$

$$+C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}\left(\frac{d}{(2^{k}d)^{n+1}}+\frac{d^{\delta}}{(2^{k}d)^{n+\delta}}\right)$$

$$\times\left(\int_{2^{k}\widetilde{Q}}|(D^{\alpha}b(y)-(D^{\alpha}b)_{2^{k}\widetilde{Q}})|^{r'}w(y)^{1-r'}\,dy\right)^{1/r'}$$

$$\times\left(\int_{2^{k}\widetilde{Q}}|f(y)|^{r}w(y)dy\right)^{1/r}$$

$$+ C\sum_{|\alpha|=m}\|D^{\alpha}b\|_{Lip_{\beta}(w)}w(\widetilde{x})\sum_{k=1}^{\infty}kw(2^k\widetilde{Q})^{\beta/n}\left(\frac{d}{(2^k d)^{n+1}}+\frac{d^{\delta}}{(2^k d)^{n+\delta}}\right)$$

$$\times\left(\int_{2^k\widetilde{Q}}|f(y)|^r w(y)dx\right)^{1/r}\left(\frac{1}{|2^k\widetilde{Q}|}\int_{2^k\widetilde{Q}}w(y)^{-1/(r-1)}dy\right)^{(r-1)/r}$$

$$\times\left(\frac{1}{|2^k\widetilde{Q}|}\int_{2^k\widetilde{Q}}w(y)dy\right)^{1/r}|2^k\widetilde{Q}|w(2^k\widetilde{Q})^{-1/r}$$

$$\leq C\sum_{|\alpha|=m}\|D^{\alpha}b\|_{Lip_{\beta}(w)}w(\widetilde{x})\sum_{k=1}^{\infty}k(2^{-k}+2^{-k\delta})$$

$$\times\left(\frac{1}{w(2^k\widetilde{Q})^{1-r\beta/n}}\int_{2^k\widetilde{Q}}|f(y)|^r w(y)dx\right)^{1/r}$$

$$+ C\sum_{|\alpha|=m}\|D^{\alpha}b\|_{Lip_{\beta}(w)}\sum_{k=1}^{\infty}(2^{-k}+2^{-k\delta})\frac{w(2^k\widetilde{Q})}{|2^k\widetilde{Q}|}$$

$$\times\left(\frac{1}{w(2^k\widetilde{Q})^{1-r\beta/n}}\int_{2^k\widetilde{Q}}|f(y)|^r w(y)dx\right)^{1/r}$$

$$\leq C\sum_{|\alpha|=m}\|D^{\alpha}b\|_{Lip_{\beta}(w)}w(\widetilde{x})M_{\beta,r,w}(f)(\widetilde{x}).$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Choose $1 < r < p$ in Theorem 1 and notice $w^{1-p} \in A_1$, then we have, by Lemmas 3 and 4,

$$\|T^b(f)\|_{L^p(w^{1-p})} \leq \|M_{\eta}(T^b(f))\|_{L^p(w^{1-p})} \leq C\|M^{\#}_{\Phi,\eta}(T^b(f))\|_{L^p(w^{1-p})}$$

$$\leq C\sum_{|\alpha|=m}\|D^{\alpha}b\|_{BMO(w)}\|wM_{r,w}(f)\|_{L^p(w^{1-p})}$$

$$= C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \|M_{r,w}(f)\|_{L^p(w)}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \|f\|_{L^p(w)}.$$

This completes the proof of Theorem 3.

Proof of Theorem 4. Choose $1 < r < p$ in Theorem 1 and notice $w^{1-p} \in A_1$, then we have, by Lemmas 5 and 7,

$$\|T^b(f)\|_{L^{p,\varphi}(w^{1-p})} \leq \|M_\eta(T^b(f))\|_{L^{p,\varphi}(w^{1-p})} \leq C\|M^{\#}_{\Phi,\eta}(T^b(f))\|_{L^{p,\varphi}(w^{1-p})}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \|wM_{r,w}(f)\|_{L^{p,\varphi}(w^{1-p})}$$

$$= C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \|M_{r,w}(f)\|_{L^{p,\varphi}(w)}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \|f\|_{L^{p,\varphi}(w)}.$$

This completes the proof of Theorem 4.

Proof of Theorem 5. Choose $1 < r < p$ in Theorem 2 and notice $w^{1-q} \in A_1$, then we have, by Lemmas 3 and 4,

$$\|T^b(f)\|_{L^q(w^{1-q})} \leq \|M_\eta(T^b(f))\|_{L^q(w^{1-q})} \leq C\|M^{\#}_{\Phi,\eta}(T^b(f))\|_{L^q(w^{1-q})}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|wM_{\beta,r,w}(f)\|_{L^q(w^{1-q})}$$

$$= C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|M_{\beta,r,w}(f)\|_{L^q(w)}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|f\|_{L^p(w)}.$$

This completes the proof of Theorem 5.

Proof of Theorem 6. Choose $1 < r < p$ in Theorem 2 and notice $w^{1-q} \in A_1$, then we have, by Lemmas 5 and 7,

$$\|T^b(f)\|_{L^{q,\varphi}(w^{1-q})} \leq \|M_\eta(T^b(f))\|_{L^{q,\varphi}(w^{1-q})} \leq C\|M^{\#}_{\Phi,\eta}(T^b(f))\|_{L^{q,\varphi}(w^{1-q})}$$

$$\leq C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)}\|wM_{\beta,r,w}(f)\|_{L^{q,\varphi}(w^{1-q})}$$

$$= C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)}\|M_{\beta,r,w}(f)\|_{L^{q,\varphi}(w)}$$

$$\leq C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)}\|f\|_{L^{p,\varphi}(w)}.$$

This completes the proof of Theorem 6.

References

[1] S. Bloom, A commutator theorem and weighted BMO, Trans. Amer. Math. Soc. 292 (1985), 103-122.

[2] S. Chanillo, A note on commutators, Indiana Univ. Math. J. 31 (1982), 7-16.

[3] J. Cohen and J. Gosselin, On multilinear singular integral operators on R^n, Studia Math. 72 (1982), 199-223.

[4] J. Cohen and J. Gosselin, A BMO estimate for multilinear singular integral operators, Illinois J. Math. 30 (1986), 445-465.

[5] R. R. Coifman, R. Rochberg and G. Weiss, Factorization theorems for Hardy spaces in several variables, Ann. of Math. 103 (1976), 611-635.

[6] G. Di Fazio and M. A. Ragusa, Commutators and Morrey spaces, Boll. Un. Mat. Ital. 5-A (1991), 323-332.

[7] G. Di Fazio and M. A. Ragusa, Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients, J. Func. Anal. 112 (1993), 241-256.

[8] Y. Ding and S. Z. Lu, Weighted boundedness for a class rough multilinear operators, Acta Math. Sinica 17 (2001), 517-526.

[9] J. Garcia-Cuerva, Weighted H^p spaces, Dissert. Math. 162 (1979).

[10] J. Garcia-Cuerva and J. L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland Math., 16, Amsterdam, 1985.

[11] D. J. Grubb and C. N. Moore, A variant of Hörmander's condition for singular integrals, Colloq. Math. 73 (1997), 165-172.

[12] Y. X. He and Y. S. Wang, Commutators of Marcinkiewicz integrals and weighted BMO, Acta Math. Sinica (Chinese Series) 54 (2011), 513-520.

[13] B. Hu and J. Gu, Necessary and sufficient conditions for boundedness of some commutators with weighted Lipschitz spaces, J. Math. Anal. Appl. 340 (2008), 598-605.

[14] S. Janson, Mean oscillation and commutators of singular integral operators, Ark. Math. 16 (1978), 263-270.

[15] L. Z. Liu, Interior estimates in Morrey spaces for solutions of elliptic equations and weighted boundedness for commutators of singular integral operators, Acta Math. Scientia 25(B) (2005), 89-94.

[16] Daqing Lu and L. Z. Liu, M^2-type sharp estimates and weighted boundedness for commutators related to singular integral operators satisfying a variant of Hörmander's condition, Boletim da Sociedade Paranaense de Matematica 31 (2013), 99-114.

[17] T. Mizuhara, Boundedness of some classical operators on generalized Morrey spaces, in "Harmonic Analysis", Proceedings of a Conference held in Sendai, Japan, (1990), 183-189.

[18] M. Paluszynski, Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss, Indiana Univ. Math. J. 44 (1995), 1-17.

[19] J. Peetre, On convolution operators leaving $L^{p,\lambda}$-spaces invariant, Ann. Mat. Pura. Appl. 72 (1966), 295-304.

[20] J. Peetre, On the theory of $L^{p,\lambda}$-spaces, J. Func. Anal. 4 (1969), 71-87.

[21] C. Pérez, Endpoint estimate for commutators of singular integral operators, J. Func. Anal. 128 (1995), 163-185.

[22] C. Pérez and R. Trujillo-Gonzalez, Sharp weighted estimates for multilinear commutators, J. London Math. Soc. 65 (2002), 672-692.

[23] E. M. Stein, Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals, Princeton Univ. Press, Princeton NJ, 1993.

[24] A. Torchinsky, Real Variable Methods in Harmonic Analysis, Pure and Applied Math. 123, Academic Press, New York, 1986.

[25] R. Trujillo-Gonzalez, Weighted norm inequalities for singular integral operators satisfying a variant of Hörmander's condition, Comment. Math. Univ. Carolin. 44 (2003), 137-152.

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