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STABILITY FOR CANONICAL FOLIATIONS ON INOUE SURFACES

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Abstract

The harmonicity and stability for foliations on Riemannian manifolds are studied by Kamber and Tondeur, and are generalized to canonical foliations on locally conformal Kähler manifolds. These results are based on the hypothesis under the metric is bundle-like. In this paper, we study harmonicity and stability for foliations with metrics which is not bundle-like. In fact, we show harmonicity and stability for canonical foliations on Inoue surfaces with Tricerri metric.

1. Introduction

Let (M, J, g_M) be a compact locally conformal Kähler manifold with locally conformal Kähler form Ω, i.e., there exists a closed 1-form ω, called the *Lee form*, satisfying $d\Omega = \omega \wedge \Omega$. A foliation $\mathcal{F}$ on compact Riemannian manifold (N, g_N) is *harmonic* if all leaves of $\mathcal{F}$ are

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minimal submanifolds. It is equivalent to be an extremal of the energy functional under special variations if the metric g_N is bundle-like with respect to $\mathcal{F}$ (Kamber-Tondeur [6]). In [4], we obtained sufficient conditions that a harmonic foliation on a compact locally conformal Kähler manifold is stable: if $\mathcal{F}$ is a harmonic foliation on M with bundle-like metric g_M foliated by complex submanifolds, then $\mathcal{F}$ is stable. This is an analogue of the theorem "a holomorphic map between two compact Kähler manifolds is stable as a harmonic map". In general, satisfactory results for foliations on a Riemannian manifold are under the assumption that the metric is bundle-like with respect to the foliation. In this paper, we show stability for harmonic foliations does not satisfy the assumption.

Inoue [5] constructed non-Kähler complex surfaces S_M, $S^{(+)}_{N,p,q,r;t}$, $S^{(-)}_{N,p,q,r}$ as quotient manifolds $\mathbb{H}\times\mathbb{C}/G$, where G is a group of suitable automorphisms acting on $\mathbb{H}\times\mathbb{C}$, and Tricerri [8] constructed locally conformal Kähler metrics, called the *Tricerri metrics*, on these surfaces (with $t\in\mathbb{R}$). There are foliations on these surfaces foliated by complex submanifolds, which is called *the canonical foliation* (as foliations on locally conformal Kähler manifolds). For S_M each leaf corresponds to the upper half-plane $\mathbb{H}$. The canonical foliations on Inoue surfaces have no compact leaves and Tricerri metrics are not bundle-like with respect to the foliations. The main result is the following:

Theorem 1.1. *The canonical foliations on Inoue surfaces with Tricerri metrics* S_M, $S^{(+)}_{N,p,q,r;t}(t\in\mathbb{R})$, $S^{(-)}_{N,p,q,r}$ *are extremal of the energy functional for special variations and are stable.*

This paper is organized as follows. In Section 2, we review the theory of harmonic foliations by Kamber and Tondeur and notions for locally conformal Kähler manifolds. Then Section 3 is devoted to the definition of Inoue surfaces and compute the Levi-Civita connections associated with Tricerri metrics. Finally, the proof of Theorem 1.1 is given in Section 4.

2. Preliminaries

Let (N, g_N) be an n-dimensional compact Riemannian manifold and let $\mathcal{F}$ be a foliation given by an integrable subbundle $L \subset TN$. We define a torsion free connection ∇ on normal bundle $Q = TN / L$ by

$$\begin{cases} \nabla_X S = \pi[X, Y_S], \text{ for } X \in \Gamma(L),\ S \in \Gamma(Q) \text{ and } Y_S = \sigma(S) \in \Gamma(\sigma(Q)), \\ \nabla_X S = \pi(\nabla^N_X Y_S), \text{ for } X \in \Gamma(\sigma(Q)),\ S \in \Gamma(Q) \text{ and } Y_S = \sigma(S) \in \Gamma(\sigma(Q)), \end{cases} \tag{2.1}$$

where $\sigma : Q \to TN$ is a splitting such that $\sigma(Q)$ coincides with the orthogonal complement $L^{\perp}$ of L in TN with respect to g_N. Here, the torsion T_∇ is defined by $T_\nabla(X, Y) = \nabla_X \pi(Y) - \nabla_Y \pi(X) - \pi[X, Y]$ for any $X, Y \in \Gamma(TM)$. If the normal bundle Q is equipped with a holonomy invariant fiber metric g_Q, i.e., $Xg_Q(S, T) = g_Q(\nabla_X S, T) + g_Q(S, \nabla_X T)$ for all $X \in \Gamma(L)$, the foliation $\mathcal{F}$ is called a *Riemannian foliation* or an *R-foliation*. There is a unique metric g_Q for an R-foliation with a torsion free connection ∇ on the normal bundle Q. A Riemannian metric g_N on N is called a *bundle-like* metric with respect to the foliation $\mathcal{F}$ if the foliation becomes an R-foliation in terms of the fiber metric g_Q induced on Q. From now on, we assume that a fiber metric on Q is always induced from the metric g_N on N.

Tricerri metrics on Inoue surfaces are not bundle-like with respect to the canonical foliations. However, in the case where the metric g_N is not necessarily bundle-like, the connection ∇ defined in (2.1) satisfies the following lemma:

Lemma 2.1. $\nabla_X g_Q = 0$ *for all* $X \in \Gamma(Q)$.

Proof. For $S, T \in \Gamma(Q)$, by setting $Y_S = \sigma(S)$, $Y_T = \sigma(T)$, we have

$$\begin{aligned} Xg_Q(Y_S, Y_T) = Xg_M(Y_S, Y_T) &= g_N(\nabla^N_X Y_S, Y_T) + g_N(Y_S, \nabla^N_X Y_T) \\ &= g_M(\sigma\pi(\nabla^N_X Y_S), \sigma(T)) + g_N(\sigma(S), \sigma\pi(\nabla^N_X Y_T)) \\ &= g_Q(\pi(\nabla^N_X Y_S), T) + g_Q(S, \pi(\nabla^N_X Y_T)) \\ &= g_Q(\nabla_X S, T) + g_Q(S, \nabla_X T). \end{aligned}$$

□

Denoting by $\pi \in \Omega^1(N, Q)$ the canonical projection from TN onto Q, we have $d_\nabla \pi \in \Omega^2(N, Q)$, $d^*_\nabla \pi \in C^\infty(N, Q)$ and so forth. Then we have the following fact (Kamber and Tondeur [6, 3.3]).

Proposition 2.2. *Let $\mathcal{F}$ be a foliation on a Riemannian manifold (N, g_N). Then the following three conditions are equivalent*:

(i) π *is harmonic form*,

(ii) *all leaves for the foliation are minimal submanifolds of N*,

(iii) $d^*_\nabla \pi = 0$.

A foliation is said to be *harmonic* if it satisfies the equivalent conditions as in above proposition. In addition, if N is compact and oriented, g_N is bundle-like, and $\mathcal{F}$ is an R-foliation, then these conditions are equivalent to $\Delta\pi = 0$.

We next see a variational characterization of harmonic foliations on compact Riemannian manifold (N, g_N). We define the *energy* of the foliation $\mathcal{F}$ by

$$E(\mathcal{F}) = \frac{1}{2}\|\pi\|^2 = \frac{1}{2}\langle \pi, \pi \rangle,$$

where π is the canonical projection from TN onto Q and is considered as a Q-valued 1-form on N. Let $\{U_\alpha, f^\alpha, \gamma^{\alpha\beta}\}$ be the Haefliger cocycle representing $\mathcal{F}$. Namely, $\{U_\alpha\}$ is an open cover of N with $f^\alpha : U_\alpha \to \mathbb{R}^q$ such that $\gamma^{\alpha\beta}$ are local diffeomorphisms on $U_\alpha \cap U_\beta (\neq \phi)$ satisfying $f^\alpha = \gamma^{\alpha\beta} f^\beta$. Here q denotes the codimension of $\mathcal{F}$. On U_α, $Q = (f^\alpha)^* T\mathbb{R}^q$. Note that if g_N is not bundle-like, there exist no metrics g_α on $\mathbb{R}^q$ satisfying $(f^\alpha)^* g_\alpha(x) = g_Q(x)$ for all x in U_α.

For $\nu \in \Gamma(Q)$, we put

$$\Phi_t^\alpha(x) = f^\alpha(\exp_x(t\nu^\alpha(x))), \quad x \in U_\alpha,\ t \in (-\varepsilon, \varepsilon), \tag{2.2}$$

where $\nu^\alpha = \nu|_{U_\alpha}$. We then have a variation Φ_t^α of $f^\alpha = \Phi_0^\alpha$, where ε is sufficiently small. Since $\Phi_t^\alpha(x) = \gamma^{\alpha\beta}\Phi_t^\beta(x)$ on $U_\alpha \cap U_\beta$, the local variations $\{\Phi_t^\alpha\}$ define a variation $\mathcal{F}_t$ of the foliation $\mathcal{F}$. Moreover, we have

$$\dot{\pi} = \nabla_{\frac{\partial}{\partial t}}|_{t=0}(\Phi_t^\alpha)_* = \nabla\nu^\alpha \in \Omega^1(U_\alpha, Q), \tag{2.3}$$

where we regarded $\dot{\pi}$ as a section in $\Gamma(Q)$ via the identification $Q = (f^\alpha)^* T\mathbb{R}^q$ (cf. Kamber-Tondeur [6] and Eells-Sampson [3]). These variations are called special variations associated to sections of Q. If the metric g_Q on Q is bundle-like, then $\mathcal{F}$ is harmonic if and only if it is an extremal of the energy functional for special variations of $\mathcal{F}$ (Kamber-Tondeur [6, 4.12]). In the case where g_Q is not bundle-like, this does not hold in general. In Section 4, we shall prove the canonical foliations on Inoue surfaces with Tricerri metrics are extremal of special variations.

Note that the definition (2.2) is different form the Kamber-Tondeur's original definition as in [6], because the original definition works out for bundle-like metrics only. Our definition (2.2) is well-defined for metrics which is not necessarily bundle-like, and coincides with original one if the metric is bundle-like.

To obtain the second variation, we need a 2-parameter variation $\mathcal{F}_{s,t}$ of $\mathcal{F}_{0,0} = \mathcal{F}$ defined locally as $\Phi^{\alpha}_{s,t}$, where

$$\Phi^{\alpha}_{s,t}(x) = f^{\alpha}(\exp_x(s\mu^{\alpha}(x) + t\nu^{\alpha}(x))),$$

for $x \in U_{\alpha}$, $s, t \in (-\varepsilon, \varepsilon)$ and $\nu, \mu \in \Gamma(Q)$. By (2.3) and the compactness of N, we have

$$\begin{aligned}\left.\frac{\partial^2}{\partial s \partial t}\right|_{s=0,t=0} E(\mathcal{F}_{s,t}) &= \left.\frac{\partial^2}{\partial s \partial t}\right|_{s=0,t=0} \frac{1}{2}\langle \pi_{s,t}, \pi_{s,t}\rangle = \left.\frac{\partial}{\partial s}\right|_{s=0}\langle \nabla\nu, \pi_{s,0}\rangle \\ &= \langle \nabla_s \nabla\nu, \pi\rangle + \langle \nabla\nu, \nabla\mu\rangle \\ &= \langle \nabla\nabla_s\nu, \pi\rangle + \langle R^{\nabla}(\mu, \pi)\nu, \pi\rangle + \langle \nabla\nu, \nabla\mu\rangle \\ &= \langle \nabla\nabla_s\nu, \pi\rangle - \langle R^{\nabla}(\mu, \pi)\pi, \nu\rangle + \langle d_{\nabla}\nu, d_{\nabla}\mu\rangle,\end{aligned}$$

where R^{∇} denote the curvature operator for Q. Hence the second variation formula is given by

$$\left.\frac{\partial^2}{\partial s \partial t}\right|_{s=0,t=0} E(\mathcal{F}_{s,t}) = -\langle R^{\nabla}(\nu, \pi)\pi, \mu\rangle + \langle \nabla\nabla_{\frac{\partial}{\partial s}}\nu, \pi\rangle + \langle d_{\nabla}\nu, d_{\nabla}\mu\rangle. \quad (2.4)$$

Definition 2.3. A harmonic foliation $\mathcal{F}$ is said to be *stable* if it is an extremal of the energy functional for special variations and for every $\nu \in \Gamma(Q)$

$$\frac{d^2}{dt^2}\Big|_{t=0} E(\mathcal{F}_t) \geq 0.$$

Remark 2.4. For a harmonic foliation $\mathcal{F}$ with bundle-like g_N, the second variation formula is given by

$$\left.\frac{\partial^2}{\partial s \partial t}\right|_{s=0,t=0} E(\mathcal{F}_{s,t}) = \langle \mathcal{J}_{\nabla}\mu, \nu\rangle,$$

where $\mathcal{J}_\nabla = \Delta - \rho_\nabla$ is the Jacobi operator of $\mathcal{F}$. Here ρ_∇ is the Ricci operator for Q. Let $V_\lambda(\mathcal{F})$ be the eigenspace associated to eigenvalue λ. The *index* of a harmonic foliatrion $\mathcal{F}$ is defined by

$$\mathrm{index}(\mathcal{F}) = \sum_{\lambda_i < 0} \dim V_{\lambda_i}(\mathcal{F}).$$

Then $\mathcal{F}$ is stable if and only if index $(\mathcal{F}) = 0$, i.e., $\langle \mathcal{J}_\nabla \nu, \nu \rangle \geq 0$ for all $\nu \in \Gamma(Q)$.

The remainder of this section we review some definitions and properties of locally conformal Kähler manifolds we need later. For details, see Dragomir-Ornea [2] for instance.

Let (M, J, g) be a Hermitian manifold, Ω be the fundamental 2-form associated with (g_M, J). If there exsits a closed 1-form ω such that $d\Omega = \omega \wedge \Omega$, then Ω is called a *locally conformal Kähler form* and ω is called the *Lee form*. We define the *Lee vector field* and the *anti-Lee vector field* by $B = \omega^\sharp$ and $A := -JB$, respectively. Here $\sharp$ denotes the raising of indices with respect ot g_M. If the distribution $\mathbb{R}A \otimes \mathbb{R}B$ generated by vector fields A and B on M defines a foliation $\mathcal{F}$, then the foliation is called the *canonical foliation* on M. In this case, every leaf is a 1-dimensional complex submanifold of M and is a minimal submanifold by the following lemma (Dragomir-Ornea [2, Theorem 12.1]):

Lemma 2.5. *Any complex submanifold N of a locally conformal Kähler manifold M is minimal if and only if the Lee vector field B for M is tangent to N.*

This lemma together with Proposition 2.2 leads the harmonicity for the canonical foliations on locally conformal Kähler manifolds.

3. Inoue Surfaces and Tricerri Metrics

In this section, we review definitions of Inoue surfaces and Tricerri metrics, and compute Levi-Civita connections associated with Tricerri metrics. Let $\mathbb{C} = \{z = z_1 + \sqrt{-1}z_2\}$ be the set of complex numbers and let $\mathbb{H} = \{w = w_1 + \sqrt{-1}w_2 \in \mathbb{C};\ w_2 > 0\}$ be the upper half-plane.

3.1. Surface S_M. Let $M = (m_{ij}) \in SL(3, \mathbb{Z})$ be a unimodular matrix with one real eigenvalue α and two non-real complex eigenvalues $\beta, \overline{\beta}$. Consider the eigenvectors (a_1, a_2, a_3) and (b_1, b_2, b_3) associated to the eigenvalues α and β, respectively. Let G_M be the group of complex automorphisms of $\mathbb{H} \times \mathbb{C}$ generated by transformations

$$(w, z) \mapsto (\alpha w, \beta z),$$

$$(w, z) \mapsto (w + a_j, z + b_j), \quad j = 1, 2, 3.$$

The quotient space $S_M := \mathbb{H} \times \mathbb{C} / G_M$ is an Inoue surface. The metric $g_M = w_2^{-2} dw \otimes d\overline{w} + w_2 dz \otimes d\overline{z}$ on $\mathbb{H} \times \mathbb{C}$ defines a locally conformal Kähler metric, called the *Tricerri* metric, on S_M with Lee form $\omega = w_2^{-1} dw_2$. Indeed, the fundamental 2-form of g_M is given by

$$\Omega_M = \sqrt{-1}\left(\frac{dw \wedge d\overline{w}}{w_2^2} + w_2 dz \wedge d\overline{z}\right),$$

and satisfies $d\Omega_M = w_2^{-1} dw_2 \wedge \Omega_M$. We now choose an orthonormal frame for the tangent bundle TS_M as follows:

$$E_1 = w_2 \frac{\partial}{\partial w_1}, \quad E_2 = w_2 \frac{\partial}{\partial w_2}, \quad E_3 = \frac{1}{\sqrt{w_2}} \frac{\partial}{\partial z_1}, \quad E_4 = \frac{1}{\sqrt{w_2}} \frac{\partial}{\partial z_2}.$$

We then have

$$[E_1, E_2] = -E_1, \quad [E_2, E_3] = -\frac{1}{2} E_3, \quad [E_2, E_4] = -\frac{1}{2} E_4. \tag{3.1}$$

The distribution generated by $B = E_2$ and $A = -JB = E_1$ defines the canonical foliation $\mathcal{F}$ with complex leaves by (3.1).

The dual frame of $\{E_i\}$ is given by the 1-forms

$$\theta^1 = \frac{dw_1}{w_2}, \quad \theta^2 = \frac{dw_2}{w_2}, \quad \theta^3 = \sqrt{w_2}\,dz_1, \quad \theta^4 = \sqrt{w_2}\,dz_2.$$

Differentiating these relations, we obtain

$$d\theta^1 = \theta^1 \wedge \theta^2, \quad d\theta^2 = 0, \quad d\theta^3 = \frac{1}{2}\theta^2 \wedge \theta^3, \quad d\theta^4 = \frac{1}{2}\theta^2 \wedge \theta^4.$$

By $d\theta^i = -\sum_k \omega^i_k \wedge \theta^k$ and $\omega^i_k + \omega^k_i = 0$, we also have

$$\omega^1_2 = -\omega^2_1 = \theta^1, \quad \omega^2_3 = -\omega^3_2 = \frac{1}{2}\theta^3, \quad \omega^2_4 = -\omega^4_2 = \frac{1}{2}\theta^4.$$

Therefore, we obtain

$$\begin{cases} \nabla_{E_1} E_1 = E_2, \\ \nabla_{E_1} E_2 = -E_1, \end{cases} \quad \begin{cases} \nabla_{E_3} E_2 = \frac{1}{2} E_3, \\ \nabla_{E_3} E_3 = -\frac{1}{2} E_2, \end{cases} \quad \begin{cases} \nabla_{E_4} E_2 = \frac{1}{2} E_4, \\ \nabla_{E_4} E_4 = -\frac{1}{2} E_2, \end{cases} \tag{3.2}$$

by $\nabla_{E_i} E_k = \sum_m \omega^m_k(E_i)E_m$.

3.2. Surface $S^{(+)}_{N,p,q,r;t}$. For a unimodular matrix $N = (n_{ij})$ in $SL(2, \mathbb{Z})$, let $\alpha(> 1)$ and $1/\alpha$ be eigenvalues of N with eigenvectors (a_1, a_2) and (b_1, b_2), respectively. Let $p, q, r (r \neq 0)$ be integers and t be a complex number. Define (c_1, c_2) as the solution of the equation

$$(c_1, c_2) = (c_1, c_2) \cdot {}^tN + (e_1, e_2) + \frac{b_1a_2 - b_2a_1}{r}(p, q),$$

where

$$e_i = \frac{1}{2} n_{i1}(n_{i1} - 1)a_1b_1 + \frac{1}{2} n_{i2}(n_{i2} - 1)a_2b_2 + n_{i1}n_{i2}b_1a_2, \quad i = 1, 2. \tag{3.3}$$

Let $G^{(+)}_{N,p,q,r;t}$ be the group of complex automorphisms of $\mathbb{H} \times \mathbb{C}$ generated by

$$(w,\, z) \mapsto (\alpha w,\, z + t),$$

$$(w,\, z) \mapsto (w + a_i,\, z + b_i w + c_i), \quad i = 1,\, 2,$$

$$(w,\, z) \mapsto (w,\, z + \frac{b_1 a_2 - b_2 a_1}{r}).$$

We define $S^{(+)}_{N,\,p,\,q,\,r;\,t} := \mathbb{H} \times \mathbb{C} \,/\, G^{(+)}_{N,\,p,\,q,\,r;\,t}$.

3.3. Surface $S^{(-)}_{N,\,p,\,q,\,r}$. Let $N = (n_{ij}) \in GL(2,\, \mathbb{Z})$ with $\det\, N = -1$ and two eigenvalues $\alpha(> 1),\, -1/\alpha$. Let $(a_1,\, a_2),\, (b_1,\, b_2)$ be eigenvectors associated to $\alpha,\, -1/\alpha$, and let $p,\, q,\, r\, (r \neq 0)$ be integers. Define $(c_1,\, c_2)$ as the solution of the equation

$$-(c_1,\, c_2) = (c_1,\, c_2) \cdot {}^t N + (e_1,\, e_2) + \frac{b_1 a_2 - b_2 a_1}{r}(p,\, q).$$

Here $e_i,\, i = 1,\, 2,$ are as in (3.3). Let $G^{(-)}_{N,\,p,\,q,\,r}$ be the group of complex automorphisms of $\mathbb{H} \times \mathbb{C}$ generated by the

$$(w,\, z) \mapsto (\alpha w,\, -z),$$

$$(w,\, z) \mapsto (w + a_i,\, z + b_i w + c_i), \quad i = 1,\, 2,$$

$$(w,\, z) \mapsto (w,\, z + \frac{b_1 a_2 - b_2 a_1}{r}).$$

We set $S^{(-)}_{N,\,p,\,q,\,r} := \mathbb{H} \times \mathbb{C} \,/\, G^{(-)}_{N,\,p,\,q,\,r}$. It is well-known that $G^{(-)}_{N,\,p,\,q,\,r}$ coincides with $G^{(+)}_{N^2,\,p_1,\,q_1,\,r;\,0}$ for some $p_1,\, p_2$ and then $S^{(-)}_{N,\,p,\,q,\,r}$ has $S^{(+)}_{N^2,\,p_1,\,q_1,\,r;\,0}$ as its unramified double covering.

3.4. Tricerri metrics on $S^{(+)}_{N,\,p,\,q,\,r;\,t}$ and $S^{(-)}_{N,\,p,\,q,\,r}$. If we consider a Hermitian metric

$$g_N = \frac{1 + (z_2)^2}{(w_2)^2}\, dw \otimes d\overline{w} - \frac{z_2}{w_2}(dw \otimes d\overline{z} + dz \otimes d\overline{w}) + dz \otimes d\overline{z}$$

on $\mathbb{H} \times \mathbb{C}$, then the fundamental 2-form Ω_N is given by

$$\Omega_N = \sqrt{-1}\left(\frac{1+(z_2)^2}{(w_2)^2}\, dw \wedge d\overline{w} - \frac{z_2}{w_2}\,(dw \wedge d\overline{z} + dz \wedge d\overline{w}) + dz \wedge d\overline{z}\right),$$

and satisfies $d\Omega_N = w_2^{-1} dw_2 \wedge \Omega_N$. The following holds (Tricerri [8, Lemma 3.2]):

Proposition 3.1. *The metric g_N on $\mathbb{H} \times \mathbb{C}$ defined above is invariant under the action of $G^{(-)}_{N,p,q,r}$. It is invariant under the action of $G^{(+)}_{N,p,q,r;t}$ if and only if t is real.*

Hence the metric g_N induce locally conformal Kähler metrics on $S^{(+)}_{N,p,q,r;t} (t \in \mathbb{R})$, $S^{(-)}_{N,p,q,r}$, which is denoted by the same notation g_N. Easy computation gives us the Lee form ω of g_N is $w_2^{-1} dw_2$. Note that on $S^{(+)}_{N,p,q,r;t}$ with $t \in \mathbb{C} \setminus \mathbb{R}$ there are no locally conformal Kähler metrics (see Belgun [1]).

If we choose an orthonormal frame of $\mathbb{H} \times \mathbb{C}$

$$E_1 = w_2 \frac{\partial}{\partial w_1} + z_2 \frac{\partial}{\partial z_1},\ E_2 = w_2 \frac{\partial}{\partial w_2} + z_2 \frac{\partial}{\partial z_2},\ E_3 = -\frac{\partial}{\partial z_1},\ E_4 = -\frac{\partial}{\partial z_2},$$

then

$$[E_1, E_2] = -E_1, \quad [E_2, E_4] = -E_4, \quad [E_1, E_4] = -E_3. \tag{3.4}$$

The dual frame of $\{E_i\}$ is given by the 1-forms

$$\theta^1 = \frac{dw_1}{w_2}, \quad \theta^2 = \frac{dw_2}{w_2}, \quad \theta^3 = \frac{z_2}{w_2} dw_1 - dz_1, \quad \theta^4 = \frac{z_2}{w_2} dw_2 - dz_2.$$

We then have

$$d\theta^1 = \theta^1 \wedge \theta^2, \quad d\theta^2 = 0, \quad d\theta^3 = \theta^1 \wedge \theta^4, \quad d\theta^4 = \theta^2 \wedge \theta^4.$$

By $d\theta^i = -\sum_k \omega^i_k \wedge \theta^k$ and $\omega^i_k + \omega^k_i = 0$,

$$\begin{cases} \omega^1_2 = -\omega^2_1 = -\theta^1, \\ \omega^1_3 = -\omega^3_1 = -\frac{1}{2}\theta^4, \\ \omega^1_4 = -\omega^4_1 = -\frac{1}{2}\theta^3, \end{cases} \quad \begin{cases} \omega^2_3 = -\omega^3_2 = 0, \\ \omega^2_4 = -\omega^4_2 = -\theta^4, \\ \omega^3_4 = -\omega^4_3 = -\frac{1}{2}\theta^1. \end{cases}$$

Therefore, we obtain

$$\begin{cases} \nabla_{E_1} E_1 = E_2, \\ \nabla_{E_1} E_2 = -E_1, \\ \nabla_{E_1} E_3 = \frac{1}{2} E_4, \\ \nabla_{E_1} E_4 = -\frac{1}{2} E_3, \end{cases} \quad \begin{cases} \nabla_{E_3} E_1 = \frac{1}{2} E_4, \\ \nabla_{E_3} E_4 = -\frac{1}{2} E_1, \end{cases} \quad \begin{cases} \nabla_{E_4} E_1 = \frac{1}{2} E_3, \\ \nabla_{E_4} E_2 = E_4, \\ \nabla_{E_4} E_3 = -\frac{1}{2} E_1, \\ \nabla_{E_4} E_4 = -E_2, \end{cases} \tag{3.5}$$

by $\nabla_{E_i} E_k = \sum_m \omega^m_k (E_i) E_m$.

4. Proof of Main Theorem

Let (S, g) be one of the surfaces (S_M, g_M), $(S^{(+)}_{N,p,q,r;t}, g_N)(t \in \mathbb{R})$ or $(S^{(-)}_{N,p,q,r}, g_N)$. By $JE_1 = E_2$ and (3.1) or (3.4), the distribution $\mathbb{R}E_1 \otimes \mathbb{R}E_2$ on S define the canonical foliation $\mathcal{F}$ on S. By (2.3), Lemma 2.1 and Stokes theorem,

$$\begin{aligned} \frac{d}{dt} E(\pi_t)\big|_{t=0} &= \langle \dot{\pi}, \pi \rangle = \langle \nabla \nu, \pi \rangle = \sum_{k=3}^{4} \langle \nabla_k \nu, \pi(E_k) \rangle \\ &= \sum_{k=3}^{4} \{ \langle \nu, \nabla_k E_k \rangle + \int_S \nabla_k \{ g_Q(\nu, E_k) \} dv \} = 0, \end{aligned} \tag{4.1}$$

where we set $\nabla_k := \nabla_{E_k}$ and use $\nu \in \Gamma(Q)$ and (3.2) or (3.5). Note that the connection ∇ is defined in (2.1) and is the composition of the canonical projection π and the Levi-Civita connection ∇ associated to the Tricceri metric for $k = 3, 4$. It follows that the canonical foliations on Inoue surfaces are extremal of the energy functional under special variations.

We next compute the right-hand side in (2.4). As (4.1), we have

$$\langle \nabla\nabla_s \nu, \pi\rangle = -\sum_{k=3}^{4}\{\langle \nabla_s \nu, \nabla_k E_k\rangle + \int_S \nabla_k\{g_Q(\nabla_s \nu, E_k)\} dv\} = 0,$$

by (3.2) or (3.5) and Stokes theorem. We also have

$$\nabla_{E_3} E_3 = \nabla_\nu E_3 = \nabla_{[\nu, E_3]} E_3 = 0 \text{ and } \nabla_{E_4} E_4 = \nabla_\nu E_4 = \nabla_{[\nu, E_4]} E_4 = 0,$$

by (3.2) or (3.5). Hence,

$$R^{\nabla}(\nu, \pi)\pi = \sum_{k=3}^{4}(\nabla_\nu \nabla_k E_k - \nabla_k \nabla_\nu E_k - \nabla_{[\nu, E_k]} E_k) = 0.$$

We now conclude that

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(\mathcal{F}_t) = \langle d_\nabla \nu, d_\nabla \nu\rangle \geq 0,$$

as required.

□

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