Transnational Journal of Mathematical Analysis and Applications
Vol. 2, Issue 2, 2014, Pages 105-130
ISSN 2347-9086
Published Online on January 19, 2015

http://jyotiacademicpress.net

ANALYTIC FUNCTIONS OF COMPLEX MATRICES

M. A. ABUL-DAHAB and Z. M. KISHKA

Department of Mathematics
Faculty of Science
South Valley University
Qena 83523
Egypt
e-mail: mamabuldahab@yahoo.com

Department of Mathematics
Faculty of Science
Sohag University
Sohag 82524
Egypt
e-mail: zanhomkishka@yahoo.com

Abstract

The purpose of this paper is to introduce and establish some properties of analytic functions of square complex matrices such as Cauchy-Riemann equations. Besides, we discuss matrix real integration, complex integration, and Cauchy integral formula for these functions and for functions of several square commutative complex matrices.

2010 Mathematics Subject Classification: 14H52, 11M20, 11G05.

Keywords and phrases: complex variables, Cauchy-Riemann equations, Cauchy integral formula, matrix functional calculus.

Received October 19, 2014

1. Introduction

Matrix theory is a fundamental area of mathematics with applications not only to many branches of mathematics but also to science and engineering. It is a connection to many different branches of mathematics (c.f., e.g., [1, 6, 7, 9, 10]). In this paper, we study some properties for analytic functions of square complex matrices. We derive a necessary and sufficient condition for the matrix function to be analytic. Besides, we discuss matrix real integration, complex integration of matrix functions, and Cauchy's integral formula for functions of single complex matrix and for functions of several square commutative complex matrices.

We shall now introduce certain symbols which will be useful in our work. Throughout this paper, consider the complex space $\mathbb{C}^{N\times N}$ of complex matrices of common order N. The symbol $|a|$ will denote a matrix, all elements of which are equal to the number a. The symbol $|X|$ will denote a matrix whose elements are equal to the moduli of the elements $x_{ij}(z),\ i,\ j = 1, 2, \ldots, N$ of the matrix X, i.e.,

$$\{|X|\}_{ij} = |\{X\}_{ij}|.$$

If a certain matrix Y has positive elements which are greater than the elements of the matrix $|X|$, we shall write this down in the form of an inequality

$$|X| < Y.$$

In the other words, this inequality is equivalent to the following system of N^2 inequalities:

$$|\{X\}_{ij}| < \{Y\}_{ij}, \quad i,\ j = 1, 2, \ldots, N.$$

The symbol for the quotient of two matrices $\frac{A}{B}$ does not have a definite meaning. We interpret it as in [5] two ways; as the product AB^{-1} or $B^{-1}A$; these products are in general distinct, it is only in an exact significance and this can be obtained when $AB = BA$; B is non-singular.

Definition 1.1. Let the matrix function $f(X)$; $X = [x_{ij}(z)]$; $i, j = 1, 2, 3, \ldots, N$, be a square complex matrix whose its elements are functions of the complex variable z. The limit of this function is defined as follows:

$$\lim_{z \to z_0} f(X) = [\lim_{z \to z_0} f_{ij}(z)]. \tag{1.1}$$

If,

$$\lim_{z \to z_0} f(X) = A; \quad \lim_{z \to z_0} G(X) = B.$$

Then,

$$\lim_{X \to X_0} f(X)G(X) = A.B; \quad \lim_{X \to X_0} G(X)f(X) = B.A,$$

and

$$\lim_{X \to X_0} \{af(X) + bG(X)\} = aA + bB; \quad a, b \in \mathbb{C}.$$

If, $X = [x_{ij}(z)]$, $Y = [y_{ij}(z)]$ are two commutative matrices in region $\mathbb{D}$; $(\mathbb{D} \subset \mathbb{C}^{N \times N})$,

$$\lim_{z \to z_0} x_{ij} = a_{ij}; \quad \lim_{z \to z_0} y_{ij} = b_{ij}.$$

Then

$$AB = \lim_{z \to z_0} XY = [\sum_{s=1}^{N} \lim_{z \to z_0} x_{is} y_{sj}] = [\sum_{s=1}^{N} \lim_{z \to z_0} y_{is} x_{sj}].$$

Definition 1.2. Let $f(X)$ be matrix function of the square complex matrix $X = \{x_{ij}(z)\}$, we say that $f(X)$ is continuous in a region $\mathbb{D}$, if

$$\lim_{h \to 0} \|f(X + hI) - f(X)\| = 0, \tag{1.2}$$

where I is the unit matrix associated with the square complex matrix X.

Suppose that $X = [x_{ij}(z)]$ is a square complex matrix of finite order N, whose elements are functions of the complex variable z. The derivative $\frac{d}{dX} f$ of the matrix function $f(X)$ will be defined as follows (c.f. [5]):

$$\frac{d}{dX} f = \lim_{h \to 0} \frac{f(X + hI) - f(X)}{h}; \quad h = h_1 + ih_2; \quad h_1, h_2 \in \Re. \quad (1.3)$$

Theorem 1.1. *If* $f(X)$ *is differentiable with respect to X in* $\mathbb{D}$, *then* $f(X)$ *is continuous in* $\mathbb{D}$, *but the converse is not true.*

Proof.

$$\lim_{h \to 0}[f(X + hI) - f(X)] = \lim_{h \to 0}\left[\frac{f(X + hI) - f(X)}{h}\right] \times \lim_{h \to 0} h = 0,$$

therefore, the matrix function $f(X)$ is continuous. □

To show that the converse is not true consider the following example:

Example 1.1. $f(X) = \overline{X}$ is continuous but no where differentiable, $\overline{X}$ denotes to the conjugate of X.

Proof.

$$\frac{f(X + hI) - f(X)}{h} = \frac{\overline{(X + hI)} - \overline{(X)}}{h} = \frac{\overline{(hI)}}{h},$$

where $\lim_{h \to 0} \frac{\overline{(hI)}}{h}$ does not exist. Therefore $f(X) = \overline{X}$ is not differentiable. □

Theorem 1.2. *Let* $f(X)$ *be a matrix function differentiable with respect to X in D and invertible. Then* $f^{-1}(X)$ *is also differentiable, and*

$$\frac{d}{dX} f^{-1}(X) = -f^{-1}(X)\left(\frac{d}{dX} f(X)\right)f^{-1}(X).$$

Proof. The following identity can be easily verified:

$$f^{-1}(X + hI) - f^{-1}(X) = f^{-1}(X + hI)[f(X) - f(X + hI)]f^{-1}, \qquad (1.4)$$

where

$$\frac{d}{dX} f(X) = \lim_{h \to 0} \frac{f(X + hI) - f(X)}{h}.$$

Dividing both sides of (1.4) by h and taking the limit as $h \to 0$, we get

$$\lim_{h \to 0} \frac{f^{-1}(X + hI) - f^{-1}(X)}{h} = \lim_{h \to 0} \left[\frac{f^{-1}(X + hI)[f(X) - f(X + hI)]f^{-1}(X)}{h} \right],$$

and

$$\frac{d}{dX} f^{-1}(X) = -f^{-1}(X)\left(\frac{d}{dX} f(X)\right)f^{-1}(X).$$

□

2. Cauchy-Riemann Matrix Equations

Theorem 2.1. *The necessary condition for the matrix function* $f(X)$ *to differentiable in the region* $\mathbb{D}$ *is that*

$$\frac{\partial u}{\partial \alpha} = \frac{\partial v}{\partial \beta} \quad and \quad \frac{\partial u}{\partial \beta} = -\frac{\partial v}{\partial \alpha}, \qquad (2.1)$$

where $f(X) = u(\alpha, \beta) + iv(\alpha, \beta)$, $X = [x_{ij}(z)] = [\alpha_{ij}(x, y) + i\beta_{ij}(x, y)] = \alpha + i\beta$, α *and* β *are commutative matrices for all* $z \in \mathbb{D}$.

Proof. Suppose that the matrix function $f(X)$ is differentiable in the region $\mathbb{D}$, then according to (1.3), we get

$$f'(X) = \lim_{h \to 0} \frac{f(X + hI) - f(X)}{h}; \quad h = h_1 + ih_2; \quad h_1, h_2 \in \Re, \qquad (2.2)$$

and

$$f'(X) = \lim_{h_1 + ih_2 \to 0} \frac{u(\alpha + h_1 I,\ \beta + h_2 I) - u(\alpha,\ \beta)}{h_1 + ih_2}$$

$$+ \ i \lim_{h_1 + ih_2 \to 0} \frac{v(\alpha + h_1 I,\ \beta + h_2 I) - v(\alpha,\ \beta)}{h_1 + ih_2}.$$

Since the limit is assumed to exist, h can approach zero from any convenient direction. In particular, if we choose to let $h \to 0$, through real part direct of the complex matrix X so that $h_2 = 0$ and $h = h_1$, then

$$f'(X) = \lim_{h_1 \to 0} \frac{f(X + h_1 I) - f(X)}{h_1}$$

$$= \lim_{h_1 \to 0} \frac{u(\alpha + h_1 I,\ \beta) - u(\alpha,\ \beta)}{h_1}$$

$$+ \ i \lim_{h_1 \to 0} \frac{v(\alpha + h_1 I,\ \beta) - v(\alpha,\ \beta)}{h_1},$$

$$f'(X) = \frac{\partial u}{\partial \alpha} + i \frac{\partial v}{\partial \alpha}. \tag{2.3}$$

Now, let $h \to 0$ through imaginary part direct of the complex matrix X so that $h_1 = 0$ and $h = h_2$, and $f'(X)$ can be written in the form:

$$f'(X) = \lim_{i h_2 \to 0} \frac{f(X + h_2 I) - f(X)}{ih_2}$$

$$= \lim_{h_2 \to 0} \frac{u(\alpha,\ \beta + h_2 I) - u(\alpha,\ \beta)}{ih_2}$$

$$+ \ i \lim_{h_2 \to 0} \frac{v(\alpha,\ \beta + h_2 I) - v(\alpha,\ \beta)}{i\ h_2}$$

$$= \frac{u_\beta(\alpha,\ \beta)}{i} + i \frac{v_\beta(\alpha,\ \beta)}{i},$$

thus,

$$f'(X) = -i\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \beta}. \tag{2.4}$$

From (2.3) and (2.4), it follows that

$$\frac{\partial u}{\partial \alpha} + i\frac{\partial v}{\partial \alpha} = -i\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \beta},$$

i.e.,

$$\frac{\partial u}{\partial \alpha} = \frac{\partial v}{\partial \beta} \quad \text{and} \quad \frac{\partial u}{\partial \beta} = -\frac{\partial v}{\partial \alpha}.$$

□

To show that the matrix function $f(X)$ may not be differentiable at the matrix $X_0 = [x_{ij}(z_0)]$; $z = z_0$ although Cauchy-Riemann equations are satisfied at X; $z = z_0$, consider the following example:

Example 2.1. Let

$$f(X) = \begin{cases} \dfrac{\alpha\beta}{\alpha^2 + \beta^2}, & \text{if } X \neq 0_{2\times 2}, \\ 0_{2\times 2}, & \text{if } X = 0_{2\times 2}, \end{cases}$$

where

$$\alpha = \begin{bmatrix} x & 3x \\ 3x & x \end{bmatrix}, \ \beta = \begin{bmatrix} 2y & y \\ y & 2y \end{bmatrix}, \ X = \alpha + i\beta,$$

we see that

$$u(\alpha, 0) = u(0, \beta) = v(\alpha, \beta) = v(\alpha, 0) = v(0, \beta) = 0.$$

Thus,

$$u_\alpha(0, 0) = \lim_{h\to 0}\frac{u(\alpha + hI, 0) - u(0, 0)}{h} = 0.$$

Similarly, $u_\beta(0, 0) = 0$, $v_\alpha(0, 0) = 0$, and $v_\beta(0, 0) = 0$.

Thus, the Cauchy-Riemann equations are satisfied at $X = [x_{ij}(0)] = 0$; i.e., when $z = 0$. Now, it can be easily seen that the matrix function $f(X)$ is not differentiable at the matrix $X = [x_{ij}(0)] = 0$ because

$$\lim_{X \to 0} f(X) = \lim_{z \to 0} f(x_{ij}(z)) = \lim_{x+iy \to 0} \frac{\alpha\beta}{\alpha^2 + \beta^2}$$

$$= \lim_{x+iy \to 0} \frac{\begin{bmatrix} x & 3x \\ 3x & x \end{bmatrix} \cdot \begin{bmatrix} 2y & y \\ y & 2y \end{bmatrix}}{\begin{bmatrix} 10x^2 & 6x^2 \\ 6x^2 & 10x^2 \end{bmatrix} + \begin{bmatrix} 5y^2 & 4y^2 \\ 4y^2 & 5y^2 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2m & m \\ m & 2m \end{bmatrix}}{\begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} + \begin{bmatrix} 5m^2 & 4m^2 \\ 4m^2 & 5m^2 \end{bmatrix}},$$

we have used the general path $y = mx$. Thus, the matrix function $f(X)$ given in Example 2.1 is not differentiable at the matrix $X = 0$ as required.

In the following theorem, we prove that Cauchy-Riemann equations with the continuity of the first partial derivatives give a sufficient condition for differentiability of the complex matrix function

$$f(X) = u(\alpha, \beta) + iv(\alpha, \beta).$$

Theorem 2.2. *Let* $f(X) = u(\alpha, \beta) + iv(\alpha, \beta)$ *be a matrix function defined in a domain* $\mathbb{D}$ *such that the first order partial derivatives* $u_\alpha, u_\beta, v_\alpha,$ *and* v_β *are continuous in* $\mathbb{D}$. *If the first order partial derivatives of* u, v *satisfy the Cauchy-Riemann equations at* X_0 *in* $\mathbb{D}$, *then f is differentiable at* X_0 *in* $\mathbb{D}$.

Proof. Since $u(\alpha, \beta)$ and its first order partial derivatives are continuous in D, we have

$$u(\alpha + h_1 I, \beta + h_2 I) - u(\alpha, \beta)$$

$$= h_1 u_\alpha + h_1 \epsilon_1 + h_2 u_\beta + h_2 \epsilon_2, \tag{2.5}$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $h_1 \to 0$ and $h_2 \to 0$.

Similarly,

$$v(\alpha + h_1 I, \beta + h_2 I) - v(\alpha, \beta)$$

$$= h_1 v_\alpha + h_1 \epsilon_3 + h_2 v_\beta + h_2 \epsilon_4, \tag{2.6}$$

where $\epsilon_3 \to 0$ and $\epsilon_4 \to 0$ as $h_1 \to 0$ and $h_2 \to 0$.

Then

$$\frac{f(X + hI) - f(X)}{h} = \frac{1}{h}[u(\alpha + h_1 I, \beta + h_2 I) - u(\alpha, \beta)$$

$$+ iv(\alpha + h_1 I, \beta + h_2 I) - v(\alpha, \beta)].$$

Using (2.5) and (2.6), we get

$$\frac{f(X + hI) - f(X)}{h} = \frac{1}{h}\left[\{h_1 u_\alpha + h_2 u_\beta + h_1 \epsilon_1(\alpha, h_1) + h_2 \epsilon_2(\alpha, h_2)\}\right.$$

$$\left. + i\{h_1 v_\alpha + h_2 v_\beta + h_1 \epsilon_3(\alpha, h_1) + h_2 \epsilon_4(\alpha, h_2)\}\right]$$

$$= \frac{1}{h}\left[h_1(u_\alpha + iv_\alpha) + h_2(u_\beta + iv_\beta) + h_1 \zeta + h_2 \eta\right],$$

where $\zeta = \epsilon_1 + i\epsilon_3$ and $\eta = \epsilon_2 + i\epsilon_4$

$$\lim_{h_1 \to 0, h_2 \to 0} \zeta = 0, \qquad \lim_{h_1 \to 0, h_2 \to 0} \eta = 0.$$

Using Cauchy-Riemann equations, we get

$$\frac{\partial u}{\partial \alpha} = \frac{\partial v}{\partial \beta} \quad \text{and} \quad \frac{\partial u}{\partial \beta} = -\frac{\partial v}{\partial \alpha},$$

$$\frac{f(X+hI)-f(X)}{h} = \frac{1}{h}[h(u_\alpha + iv_\alpha) + h_1\zeta + h_2\eta]$$

$$= u_\alpha + iv_\alpha + \frac{h_1}{h}\zeta + \frac{h_2}{h}\eta.$$

Now, since $|\frac{h_1}{h}| \le 1$, $|\frac{h_2}{h}| \le 1$ and

$$\| \frac{h_1\zeta + h_2\eta}{h} \| \le \|\zeta\| + \|\eta\| \to 0 \text{ as } \{ {}^{h_1 \to 0}_{h_2 \to 0}.$$

Hence,

$$\lim_{h\to 0} \frac{f(X+hI)-f(X)}{h} = u_\alpha + iv_\alpha.$$

Therefore $f(X)$ is differentiable. □

3. Matrix Real Integration

Definition 3.1. Suppose that $X = [x_{ij}(t)]$ is a square matrix; $a \le t \le b$, $a, b \in \Re$ such that

$$X.\frac{dX}{dt} = \frac{dX}{dt}.X; \quad t \in [a, b],$$

$$f'(X) = \lim_{h\to 0} \frac{f(X+hI)-f(X)}{h}; \quad t \in [a, b].$$

Then

$$\int_a^b f'(X)dX = \int_a^b f'(X)\frac{dX}{dt}dt = \int_a^b \frac{d}{dt} f(X)dt$$

$$= \int_a^b [\frac{d}{dt} f_{ij}(t)]dt = [f_{ij}(t)]|_{t=a}^{b} = f(x_{ij}(b)) - f(x_{ij}(a))$$

$$= F(B) - F(A).$$

Thus, we can write

$$\int_a^b f'(X)dX = \int_A^B f'(X)dX = F(B) - F(A),$$

where $A = [x_{ij}(a)]$ and $B = [x_{ij}(b)]$.

Example 3.1. Evaluate the following integration:

$$\int_{t=\frac{\pi}{2}}^{\pi} e^{\begin{bmatrix} \sin t & t+1 \\ t+1 & \sin t \end{bmatrix}} . \begin{bmatrix} \cos t & 1 \\ 1 & \cos t \end{bmatrix} dt.$$

Solution.

$$\int_{t=\frac{\pi}{2}}^{\pi} e^{\begin{bmatrix} \sin t & t+1 \\ t+1 & \sin t \end{bmatrix}} . \begin{bmatrix} \cos t & 1 \\ 1 & \cos t \end{bmatrix} dt = \left\{ e^{\begin{bmatrix} \sin t & t+1 \\ t+1 & \sin t \end{bmatrix}} \right\}_{t=\frac{\pi}{2}}^{\pi}$$

$$= e^{\begin{bmatrix} 0 & \pi+1 \\ \pi+1 & 0 \end{bmatrix}} - e^{\begin{bmatrix} 1 & \frac{\pi}{2}+1 \\ \frac{\pi}{2}+1 & 1 \end{bmatrix}}.$$

If

$$X.\frac{dX}{dt} \neq \frac{dX}{dt}.X; \quad t \in [a, b].$$

Then

$$\int_a^b f'(X)dX = \int_A^B f'(X)dX \neq F(B) - F(A).$$

According to the following example:

Example 3.2.

$$X = \begin{bmatrix} 1 & 1 \\ t^4 & t^4 \end{bmatrix}; \quad \frac{dX}{dt} = \begin{bmatrix} 1 & 1 \\ 4t^3 & 4t^3 \end{bmatrix},$$

$$X.\frac{dX}{dt} \neq \frac{dX}{dt}.X,$$

$$X^2 = \begin{bmatrix} 1+t^4 & 1+t^4 \\ t^4+t^8 & t^4+t^8 \end{bmatrix},$$

$$\{X^2\}_{t=0}^{1} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix},$$

$$2X.\frac{dX}{dt} = 2\begin{bmatrix} 1 & 1 \\ t^4 & t^4 \end{bmatrix}.\begin{bmatrix} 1 & 1 \\ 4t^3 & 4t^3 \end{bmatrix} = 2\begin{bmatrix} 4t^3 & 4t^3 \\ 4t^7 & 4t^7 \end{bmatrix},$$

$$\int_0^1 2X.\frac{dX}{dt}\,dt = \int_0^1 2XdX = \begin{bmatrix} 2t^4 & 2t^4 \\ t^8 & t^8 \end{bmatrix}\Bigg|_{t=0}^{1} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix},$$

i.e.,

$$\int_a^b f'(X)dX = \int_A^B f'(X)dX \neq F(B) - F(A).$$

Now, we will derive some properties of definite integration of matrix functions.

(i) Suppose that $X = x_{ij}(t)$ is a square real matrix whose elements are odd functions, i.e.,

$$x_{ij}(-t) = -x_{ij}(t); \quad i, j = 1, 2, 3, \ldots, N.$$

Then

$$\int_{-a}^{a} f(X)dX = \int_{-a}^{0} f(X)dX + \int_0^a f(X)dX$$

$$= \int_{-A}^{0} f(X)dX + \int_0^A f(X)dX$$

$$= \begin{cases} 0; & f(X) \text{ is odd function in } X, \\ 2\int_0^A f(X)dX; & f(X) \text{ is even function in } X. \end{cases}$$

Example 3.3. Evaluate the following integrations:

(i)

$$\int_{-1}^{1} X^{2n} dX = \int_{-A}^{A} X^{2n} dX; \quad X = \begin{pmatrix} t & t^3 & t^5 \\ t^3 & t^5 & t \\ t^5 & t & t^3 \end{pmatrix}, A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

(ii)

$$\int_{-1}^{1} X^{2n} \sin X dX; \quad X = \begin{pmatrix} \sinh t & t^5 & \sin 3t \\ t^5 & \sin 3t & \sinh t \\ \sin 3t & \sinh t & t^5 \end{pmatrix}.$$

Solution. (i)

$$\int_{-1}^{1} X^{2n} dX = \int_{-A}^{A} X^{2n} dX = 2\{ \frac{X^{2x+1}}{2n+1} \}_0^A$$

$$= \frac{2}{2n+1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^{2n+1} = \frac{2.3^{2n}}{2n+1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Solution. (ii)

$$\int_{-1}^{1} X^{2n} \sin X dX = 0,$$

where 0 is the zero matrix associated with the square complex matrix X.

(ii) $$\int_a^b f(X) dX = \int_A^B f(X) dX = -\int_B^A f(X) dX.$$

(iii) $$\int_a^b f(X) dX = \int_A^B f(X) dX = \int_A^D f(X) dX + \int_D^B f(X) dX,$$

where $A = [x_{ij}(a)]$, $D = [x_{ij}(d)]$, and $B = [x_{ij}(b)]$; $a \leq d \leq b$.

(iv) $$\int_0^a f(X)dX = \int_0^A f(A - X)dX,$$

where A, X, and $\frac{dX}{dt}$ are commutative matrices.

4. Complex Integration of Matrix Functions

Let $X = [x_{ij}(z)]$ be a square complex matrix; $z = x + iy$, X and $\frac{dX}{dz}$ are commutative matrices on the curve Γ in $\mathbb{D}$, then

$$\int_\Gamma f(X)dX = [\lim_{n\to\infty} S_{n,ij}];$$

$$S_{n,ij} = \sum_{k=1}^{n} G_{ij}(\zeta_k)\Delta Z_k; \quad [G_{ij(z)}] = f(X)\frac{dX}{dz}.$$

Therefore, we obtain

$$\int_\Gamma f(X)dX = \int_\Gamma \{u(\alpha, \beta) + iv(\alpha, \beta)\}(d\alpha + id\beta)$$

$$= \int_\Gamma \{u\, d\alpha - vd\beta\} + i\int_\Gamma \{u\, d\alpha + vd\beta\},$$

where α, β, $\frac{\partial\alpha}{\partial x}$, $\frac{\partial\alpha}{\partial y}$, $\frac{\partial\beta}{\partial x}$, and $\frac{\partial\beta}{\partial y}$ are commutative matrices on the curve Γ.

Theorem 4.1. *If the elements of matrix* $X = [x_{ij}(z)]$ *are analytic functions on and inside a simple closed contour* Γ, X, $\frac{dX}{dz}$ *are commutative matrices in* Γ, $f(X)$ *is differentiable in* Γ, *then*

$$\oint_\Gamma f(X)dX = 0. \tag{4.1}$$

Proof.

$$\oint_\Gamma f(X)dX = \oint_\Gamma f(X)\frac{dX}{dz}dz = \oint_\Gamma \{G_{ij}(z)dz\} = 0.$$

□

Example 4.1.

$$\oint_{|z|=1} e^X dX = 0; \quad X = \begin{pmatrix} z+1 & e^z & 3z & \sin z \\ e^z & z+1 & \sin z & 3z \\ 3z & \sin z & z+1 & e^z \\ \sin z & 3z & e^z & z+1 \end{pmatrix}.$$

Example 4.2. Find the analytic matrix function in C whose real part is given by

$$u(\alpha, \beta) = 2\alpha\beta + e^\alpha \cos\beta; \quad X = \alpha + i\beta,$$

where

$$\alpha = \begin{pmatrix} x+1 & e^x\cos y & 3x & \sin x\cosh y \\ e^x\cos y & x+1 & \sin x\cosh y & 3x \\ 3x & \sin x\cosh y & x+1 & e^x\cosh y \\ \sin x\cosh y & 3x & e^x\cosh y & x+1 \end{pmatrix},$$

and

$$\beta = \begin{pmatrix} y & e^x\sin y & 3y & \cos x\sinh y \\ e^x\sin y & y & \cos x\sinh y & 3y \\ 3y & \cos x\sinh y & y & e^x\sin y \\ \cos x\sinh y & 3y & e^x\sin y & y \end{pmatrix}.$$

Solution.

$$f(X) = \int f'(X)dX;$$

$$f'(X) = u_\alpha + iv_\alpha.$$

According to Cauchy-Riemann equations, we have

$$f'(X) = u_\alpha - iu_\beta = 2\beta + e^\alpha \cos\beta - i[2\alpha - e^\alpha \sin\beta]$$

$$= e^X - 2iX.$$

Thus

$$f(X) = \int (e^X - 2iX)dX = e^X - iX^2.$$

Remark 4.1. In the previous example, the matrices $X, \frac{dX}{dz}, \alpha,$ and β are supposed to be commutative matrices.

Lemma 4.1.

$$|\oint_\Gamma f(X)dX| = [|\oint_\Gamma \{f(X)dX\}_{ij}|] \le [M_{ij}L] = \|ML\|, \tag{4.2}$$

where

$$\mathbb{M} = \|M\| = \max_{i,j}[M_{ij}] = \max_{i,j}\{|f(X)_{ij}|;\ X = [x_{ij}(z)],\ z \in \mathbb{C}\},$$

and L is the length of Γ.

Proof. By definition of M, we have

$$|\{f(X)\}_{ij}| \le M \text{ for all } X, \quad z \in \mathbb{C}.$$

Now,

$$|\oint_\Gamma f(X)dX| = [|\oint_\Gamma \{f(X)dX\}_{ij}|]$$

$$\leq \oint_{\Gamma} |\{f(X)\frac{dX}{dz}\, dz\}_{ij}|$$

$$= \oint_{\Gamma} |\{f(X)\}_{ij}||\{\frac{dX}{dz}\}_{ij}|dz$$

$$\leq \oint_{\Gamma} M_{ij}|\{\frac{dX}{dz}\}_{ij}|dz = \|ML\|.$$

□

5. Cauchy Integral Formula for Complex Matrix Functions

Theorem 5.1. *Suppose that* $X = [x_{ij}(z)];\ i,\ j = 1, 2, 3, \ldots, N,$ *and let* $f(X)$ *be function analytic on and inside the closed contour* Γ *that encloses* z_0, *then*

(i) $X = [x_{ij}(z)];\ x_{ij}(z),\ i,\ j = 1, 2, \ldots, N$ *are analytic functions of the complex variable z in* $\mathbb{D}$.

(ii) $\lim_{h\to 0} \frac{f(X + hI) - f(X)}{h}$ *exists for all z in* $\mathbb{D}$.

(iii) *X and* $\frac{dX}{dz}$ *are commutative matrices in* $\mathbb{D}$, *we have*

$$f(X_0) = \frac{1}{2\pi i}\oint_{\Gamma} \frac{f(X)}{z - z_0}\, dz. \tag{5.1}$$

In general,

$$f^{(n)}(X_0) = \left[\frac{d^n}{dz^n}\left(f_{ij}(X)\right)\right]_{z=z_0} = \frac{n!}{2\pi i}\oint_{\Gamma} \frac{f(X)}{(z - z_0)^{n+1}}\, dz, \tag{5.2}$$

where Γ *is a simple closed contour in the domain* $\mathbb{D}$.

Proof. Choose a circle Γ_1 with center z_0 and radius r_0 such that Γ_1 lies in the interior of Γ, it follows that

$$\oint_{\Gamma}\frac{f(X)}{z-z_0}dz = \oint_{\Gamma_1}\frac{f(X)}{z-z_0}dz$$

$$= \oint_{\Gamma_1}\left(\left\{\frac{f(X)-f(X_0)+f(X_0)}{z-z_0}\right\}_{ij}\right)dz$$

$$= \oint_{\Gamma_1}\left(\left\{\frac{f(X)-f(X_0)}{z-z_0}\right\}_{ij}\right)dz + \oint_{\Gamma_1}\frac{f_{ij}(X_0)}{z-z_0}dz$$

$$= \oint_{\Gamma_1}\left(\left\{\frac{f(X)-f(X_0)}{z-z_0}\right\}_{ij}\right)dz + f_{ij}(X_0)\oint_{\Gamma_1}\frac{dz}{z-z_0}$$

$$= \oint_{\Gamma_1}\left(\left\{\frac{f(X)-f(X_0)}{z-z_0}\right\}_{ij}\right)dz + f_{ij}(X_0)(2\pi i).$$

Thus,

$$\oint_{\Gamma}\frac{f(X)}{z-z_0}dz = \oint_{\Gamma_1}\left(\left\{\frac{f(X)-f(X_0)}{z-z_0}\right\}_{ij}\right)dz + f_{ij}(X_0)(2\pi i). \qquad (5.3)$$

Since $f(X)$ is function analytic in and on Γ, it is continuous at $X_0 = [x_{ij}(z_0)]$.

From [5] given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$[|\{X - X_0\}_{ij}|] < \|\delta\| \Rightarrow [|\{f(X) - f(X_0)\}_{ij}|] < \|\varepsilon\|.$$

If we choose $r_0 < \delta$, then

$$[|\{X - X_0\}_{ij}|] < \|r_0\| \Rightarrow [|\{f(X) - f(X_0)\}_{ij}|] < \|\varepsilon\|.$$

By Lemma 4.1, we have

$$\left|\oint_{\Gamma_1}\left(\left\{\frac{f(X)-f(X_0)}{z-z_0}\right\}_{ij}\right)dz\right| < \left(\frac{\varepsilon}{r_0}\right)(2\pi r_0) = 2\pi\varepsilon.$$

Since ε is arbitrary, we have

$$\oint_{\Gamma_1}\left(\left\{\frac{f(X)-f(X_0)}{z-z_0}\right\}_{ij}\right)dz = 0.$$

From (5.3), we get

$$f(X_0) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(X)}{z-z_0}\,dz,$$

therefore

$$\frac{f(X_0+Ih)-f(X_0)}{h} = \frac{1}{2\pi i}\oint_\Gamma \frac{1}{h}\left[\frac{1}{z-(z_0+h)} - \frac{1}{z-z_0}\right]f(X)dz$$

$$= \frac{1}{h(2\pi i)}\oint_\Gamma \left[\frac{h}{(z-(z_0+h))(z-z_0)}\right]f(X)dz$$

$$= \frac{1}{(2\pi i)}\oint_\Gamma \left[\frac{(z-z_0)}{(z-(z_0+h))(z-z_0)^2}\right]f(X)dz$$

$$= \frac{1}{(2\pi i)}\oint_\Gamma \left[\frac{(z-z_0-h+h)}{(z-(z_0+h))(z-z_0)^2}\right]f(X)dz$$

$$= \frac{1}{(2\pi i)}\oint_\Gamma \left[\frac{[(z-z_0-h)+(h)]}{(z-(z_0+h))(z-z_0)^2}\right]f(X)dz$$

$$= \frac{1}{(2\pi i)}\oint_\Gamma \frac{f(X)}{(z-z_0)^2}\,dz + \frac{h}{(2\pi i)}\oint_\Gamma \frac{f(X)}{(z-(z_0+h))(z-z_0)^2}\,dz.$$

Thus

$$\frac{f(X_0+hI)-f(X_0)}{h} - \frac{1}{(2\pi i)}\oint_\Gamma \frac{f(X)}{(z-z_0)^2}\,dz$$

$$= \frac{h}{(2\pi i)}\oint_\Gamma \frac{f(X)}{(z-(z_0+h))(z-z_0)^2}\,dz. \tag{5.4}$$

Now, let $\mathbb{M}$ denote the maximum value of $|f(X)|$ on Γ. Let L be the length of Γ and we choose h so small such that $|h| < \frac{\varepsilon}{2}$, we have

$$|z - z_0 - h| \geq |z - z_0| - |h| > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Hence

$$\left| \frac{h}{(2\pi i)} \oint_\Gamma \frac{f(X)}{(z - (z_0 + h))(z - z_0)^2} dz \right| \leq \frac{|h| ML}{2\pi(\varepsilon^2)(\frac{\varepsilon}{2})}.$$

From (5.4), we have

$$\left| \frac{f(X_0 + hI) - f(X_0)}{h} - \frac{1}{(2\pi i)} \oint_\Gamma \frac{f(X)}{(z - z_0)^2} dz \right| \leq \frac{|h| ML}{\pi(\varepsilon^3)},$$

taking limit as $h \to 0$, we get

$$\lim_{h \to 0} \left\{ \frac{f(X_0 + hI) - f(X_0)}{h} - \frac{1}{(2\pi i)} \oint_\Gamma \frac{f(X)}{(z - z_0)^2} dz \right\} = 0,$$

$$\lim_{h \to 0} \frac{f(X_0 + hI) - f(X_0)}{h} = \frac{1}{(2\pi i)} \oint_\Gamma \frac{f(X)}{(z - z_0)^2} dz = f'(X_0).$$

Similarly, it can be shown that $f''(X_0)$ is analytic function of X_0, we get

$$f''(X_0) = \frac{2!}{(2\pi i)} \oint_\Gamma \frac{f(X)}{(z - z_0)^3} dz.$$

By using mathematical induction on n, we can prove that for any positive integer n, that

$$f^{(n)}(X_0) = \left[\frac{d^n}{dz^n} (f_{ij}(X)) \right]_{z = z_0} = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(X)}{(z - z_0)^{n+1}} dz. \tag{5.5}$$

□

Lemma 5.1. *The higher-order chain rule or Faa de Bruno formula established* (c.f. [2, 8]). *We will define the higher-order chain rule of complex matrix functions provided that the following conditions are satisfied*:

(i) $X = [x_{ij}(z)];\ x_{ij}(z),\ i,\ j = 1, 2, \ldots, N$ *are analytic functions of the complex variable z in* $\mathbb{D}$.

(ii) $\lim_{h\to 0} \dfrac{f(X+hI)-f(X)}{h}$ *exists for all z in* $\mathbb{D}$.

(iii) *X and* $\dfrac{dX}{dz}$ *are commutative matrices in* $\mathbb{D}$.

Thus, the complex matrix function $f(X)$ *has the n-th derivative with respect to z in the general form*

$$(D^\star)^n f(X) = \sum \frac{n!}{k_1!\,k_2!\ldots k_n!}((D^\star)^n f)(X)\left(\frac{D^\star(X)}{1!}\right)^{k_1}\ldots\left(\frac{(D^\star)^k(X)}{n!}\right)^{k_n};\ D^\star = \frac{d}{dz}. \tag{5.6}$$

It can also be expressed in terms of Bell polynomial $\mathbf{B}_{n,s}$ *as*

$$\frac{d^n f_{ij}([x_{ij}(z)])}{dz^n} = \sum_{s=1}^{n} \frac{d^s f_{ij}([x_{ij}(z)])}{dX^s}.\mathbf{B}_{n,s}(X', X'', X''', \ldots, X^{n+1-s}), \tag{5.7}$$

where

$$\mathbf{B}_{n,s} = \sum_{|k|=s,\|k\|=n} \frac{n!}{k!}\left(\frac{X_1}{1!}\right)^{k_1}\left(\frac{X_2}{2!}\right)^{k_2}\ldots\left(\frac{X_n}{n!}\right)^{k_n};\quad s = 1, 2, 3, \ldots, n, \tag{5.8}$$

$$k = (k_1, k_2, \ldots, k_n),$$

$$|k| = k_1 + k_2 + \ldots + k_n,$$

$$\|k\| = k_1 + 2k_2 + \ldots + nk_n,$$

$$k! = k_1!\,k_2!\ldots k_n!.$$

Example 5.1. Evaluate the following integrals:

(i)

$$\oint_\Gamma \frac{e^X}{z^2+4}\,dz,$$

where Γ is $|z-i| = 2$, $X = \begin{pmatrix} \sin z & z^2+1 \\ z^2+1 & \sin z \end{pmatrix}$.

(ii)

$$\oint_\Gamma \frac{X^2+5I}{z-3}\,dz,$$

where Γ is $|z| = 4$, $X = \begin{pmatrix} \cos z & e^{z^2+3} \\ e^{z^2+3} & \cos z \end{pmatrix}$, I is the unit matrix associated with the square complex matrix X.

(iii)

$$\oint_\Gamma \frac{X}{(9-z^2)(z-1)}\,dz,$$

where Γ is $|z| = 2$, $X = \begin{pmatrix} \cos \pi z^2 & \sin \pi z^2 \\ \sin \pi z^2 & \cos \pi z^2 \end{pmatrix}$.

Solution. (i)

$$\frac{1}{z^2+4} = \frac{1}{z+2i}\frac{1}{z-2i}$$

$$= \frac{1}{4i}\left(\frac{1}{z-2i} - \frac{1}{z+2i}\right).$$

Now, $2i$ lies inside Γ, then we get

$$\oint_{\Gamma} \frac{e^X}{z - 2i}\, dz = 2\pi i e^{\begin{bmatrix} \sin 2i & -3 \\ -3 & \sin 2i \end{bmatrix}}.$$

Also, $-2i$ lies outside Γ and hence

$$\oint_{\Gamma} \frac{e^X}{z + 2i}\, dz,$$

is analytic inside and on Γ, then

$$\oint_{\Gamma} \frac{e^X}{z + 2i}\, dz = 0,$$

i.e.,

$$\oint_{\Gamma} \frac{e^X}{z^2 + 4}\, dz = \frac{1}{4i}\left(2\pi i e^{\begin{bmatrix} \sin 2i & -3 \\ -3 & \sin 2i \end{bmatrix}} - 0 \right) = \frac{\pi}{2}\, e^{\begin{bmatrix} \sin 2i & -3 \\ -3 & \sin 2i \end{bmatrix}}.$$

Solution. (ii) Since $f(X) = X^2 + 5I$ is analytic inside and on $|z| = 4$ and $z = 3$ lies inside it, then

$$\oint_{\Gamma} \frac{X^2 + 5I}{z - 3}\, dz = 2\pi i \begin{pmatrix} \cos^2(3) + e^{24} & 2\cos(3)e^{12} \\ 2\cos(3)e^{12} & \cos^2(3) + e^{24} \end{pmatrix}.$$

Solution. (iii) Let

$$f(X) = \frac{X}{9 - z^2}.$$

Clearly $f(X)$ is analytic within and on Γ, then

$$\oint_{\Gamma} \frac{X}{(9 - z^2)(z - 1)}\, dz = \oint_{\Gamma} \frac{f(X)}{z - 1}\, dz$$

$$= -\frac{\pi i}{4}\, I.$$

Example 5.2. Evaluate

$$\oint_{\Gamma} \frac{e^X}{z^3}\, dz,$$

where Γ is $|z| = \frac{1}{2}$, $X = \begin{pmatrix} e^{z^2+3} & \cosh z \\ \cosh z & e^{z^2+3} \end{pmatrix}.$

Solution. We can identify, $z_0 = 0$, $n = 2$, and $f(X) = e^X$. Clearly $f(X)$ is analytic within and on Γ. Then we get

$$\left[\frac{d^2 f(X)}{dz^2}\right]_{z=z_0} = \left[e^X \left(\frac{dX}{dz}\right)^2 + e^X \frac{d^2X}{dz^2}\right]_{z=z_0},$$

where $\left(\frac{dX}{dz}\right)^2 = \begin{pmatrix} 4z^2 e^{2(z^2+3)} + \sinh^2 z & 4ze^{z^2+3}\sinh z \\ 4ze^{z^2+3}\sinh z & 4z^2 e^{2(z^2+3)} + \sinh^2 z \end{pmatrix},$

$$\frac{d^2X}{dz^2} = \begin{pmatrix} 2e^{z^2+3}(2z^2+1) & \cosh z \\ \cosh z & 2e^{z^2+3}(2z^2+1) \end{pmatrix}.$$

Then

$$\oint_{\Gamma} \frac{e^X}{z^3}\, dz = \frac{2\pi i}{2!}\left[e^{\begin{bmatrix} e^3 & 1 \\ 1 & e^3 \end{bmatrix}} \begin{pmatrix} 2e^3 & 1 \\ 1 & 2e^3 \end{pmatrix}\right].$$

6. Cauchy's Integral Formula for Functions of Several Complex Matrices

In order to simplify notation, we introduce multi-indices: Let n_i, $1 \le i \le k$ be non-negative integers and let $X_1, X_2, X_3, \ldots, X_k$ be commutative matrices in $\mathbb{C}^{N\times N}$. We define

$$\boldsymbol{n} = (n_1, n_2, \ldots, n_k), \quad |\boldsymbol{n}| = n_1 + n_2 + \ldots + n_k,$$

$$\boldsymbol{X} = (X_1, X_2, \ldots, X_k) \text{ and } \boldsymbol{z} = (z_1, z_2, \ldots, z_k).$$

Theorem 6.1. *Let* $f(\boldsymbol{X})$ *be a function; of the several square commutative complex matrices* $X_1, X_2, \ldots, X_k$ *which and their elements are analytic functions of the complex variables* ξ_i, $i = 1, 2, \ldots, k$ *in the domain* $D \subset \mathbb{C}^{N\times N}$, *which is the product of the domains* $D_1, D_2, \ldots, D_k$ *and* X_i, $\frac{\partial X_i}{\partial \xi_i}$ *are commutative matrices in* D_i; $i = 1, 2, \ldots, k$. *Then we have*

$$f^n(A) = \left[\frac{\partial^{|\boldsymbol{n}|}}{\partial \xi_i^{\boldsymbol{n}}}\left(f_{ij}(\boldsymbol{X})\right)\right]_{\xi_i = z_i} = \frac{\prod_{i=1}^k n_i!}{(2\pi i)^k} \oint_{\Gamma_1}\oint_{\Gamma_2}\cdots\oint_{\Gamma_k} \frac{f(\boldsymbol{X})}{\prod_{i=1}^k (\xi_i - z_i)^{n_i+1}} d\xi_1 d\xi_2 \ldots d\xi_k,$$

where Γ_i *is a simple closed contour containing* z_i *and entirely in the domain* D_i; $i = 1, 2, \ldots, k$, $A = ([X_{1,ij}(z_1)], [X_{2,ij}(z_2)], \ldots, [X_{k,ij}(z_k)])$.

Proof. From Cauchy's integral formula for functions of single complex matrix, we get

$$f(A) = \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f(X_1, X_2^0, \ldots, X_k^0)}{\xi_1 - z_1} d\xi_1$$

$$= \frac{1}{2\pi i}\oint_{\Gamma_1} \left\{\frac{1}{\xi_1 - z_1}\frac{1}{2\pi i}\oint_{\Gamma_2} \frac{f(X_1, X_2, X_2^0, \ldots, X_k^0)}{\xi_2 - z_2} d\xi_2\right\} d\xi_1,$$

and after k-steps, we get

$$f(A) = \left(\frac{1}{2\pi i}\right)^k \oint_{\Gamma_1}\oint_{\Gamma_2}\cdots\oint_{\Gamma_k} \frac{f(X_1, X_2, \ldots, X_k)}{(\xi_1 - z_1)(\xi_2 - z_2)\ldots(\xi_k - z_k)} d\xi_1 d\xi_2 \ldots d\xi_k.$$

Therefore, partial differentiation of the function of the several square commutative complex matrices $X_1, X_2, \ldots, X_k$ with respect to the complex variables ξ_i; as in the case of single complex matrix, lead to

$$f^{n}(A)=\left[\frac{\partial^{|\boldsymbol{n}|}}{\partial\xi_i^{\boldsymbol{n}}}\left(f_{ij}(\boldsymbol{X})\right)\right]_{\xi_i=z_i}=\frac{\prod_{i=1}^{k}n_i!}{(2\pi i)^k}\oint_{\Gamma_1}\oint_{\Gamma_2}\cdots\oint_{\Gamma_k}\frac{f(\boldsymbol{X})}{\prod_{i=1}^{k}(\xi_i-z_i)^{n_i+1}}\,d\xi_1 d\xi_2\ldots d\xi_k,$$

$n_i = 0, 1, 2, \ldots,$ and $i = 1, 2, \ldots, k.$

References

[1] R. Brualdi and D. Cvetković, A Combinatorial Approach to Matrix Theory and its Applications, New York, 2009.

[2] A. Bernardini, P. Natalini and P. E. Ricci, Multidimensional Bell polynomials of higher order, Comput. Math. Appl. 50 (2005), 1697-1708.

[3] G. H. Golud and C. F. Van Loan, Matrix Computations, The Johns Hopkins Press Ltd., London, 1996.

[4] Z. M. G. Kishka, Generalized Cauchy integral formula, Sohag Pure Appl. Sci. Bull. Fac. Sci. 9 (1993), 23-34.

[5] Z. M. G. Kishka, Functions of several square commutative complex matrices, S. S. Q. J. Pure Appl. Math. 66 (1992), 69-77.

[6] S. P. Maclachlan and C. W. Oosterlee, Algebraic multigrid solvers for complex-valued matrices, Siam Sci. Compt. 30 (2008), 1548-1571.

[7] S. Z. Rida, M. Abul-Dahab, M. A. Saleem and M. T. Mohammed, On Humbert matrix function $\Psi_1(A, B; C, C'; z, w)$ of two complex variables under differential operator, Int. J. Indus. Math. 2 (2010), 167-179.

[8] P. Natalini and P. E. Ricci, An extension of the Bell polynomials, Comput. Math. Appl. 47 (2004), 719-725.

[9] W. H. Steeb, Problems and Solution in Introduction and Advanced Matrix Calculus, World Scientific, London, 2006.

[10] M. Taylor, A. I. Díaz and L. A. Jódar, A matrix generalization of dimensional analysis: New similarity transforms to address the problem of uniqueness, Adv. Studies Theor. Phys. 2 (2008), 979-995.

■