Transnational Journal of Mathematical Analysis and Applications
Vol. 2, Issue 1, 2014, Pages 89-103
ISSN 2347-9086
Published Online on July 23, 2014

http://jyotiacademicpress.net

EXISTENCE RESULTS TO A QUASILINEAR PARABOLIC SYSTEMS INVOLVING *p*-LAPLACIAN OPERATORS VIA TIME-DISCRETIZATION METHOD

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Abstract

In this paper, we consider a quasilinear parabolic systems with the singular absorption term

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \Delta_{p_i} u_i = \dfrac{1}{(u_i)^{\alpha_i}} + f_i(x, u_1, u_2) & \text{in } \Omega \times (0, T), \\ u_i = 0 \text{ on } \partial\Omega \times [0, T], \ u_i > 0 & \text{on } Q = \Omega \times (0, T), \\ u_i(x, 0) = \varphi_i(x) & \text{in } \Omega. \end{cases}$$

In particular, we prove the existence of discrete approximate solutions by means of the Rothe discretization in time method under some conditions on α_i, f_i, and p_i, $i = 1, 2$.

2010 Mathematics Subject Classification: 35K55, 35K57, 35K65, 35B40.
Keywords and phrases: quasilinear parabolic systems, existence of solutions, singular nonlinearity, p-Laplacian operator, Rothe method, sub and supersolutions.
Received May 23, 2014

1. Introduction

In this paper, we study a quasilinear parabolic systems involving p_i-Laplacian operators of the type $(\mathcal{S})$

$$\frac{\partial u_1}{\partial t} - \Delta_{p_1} u_1 = \frac{1}{(u_1)^{\alpha_1}} + f_1(x, u_1, u_2) \quad \text{in } Q_T = \Omega \times (0, T), \tag{1.1}$$

$$\frac{\partial u_2}{\partial t} - \Delta_{p_2} u_2 = \frac{1}{(u_2)^{\alpha_2}} + f_2(x, u_1, u_2) \quad \text{in } Q_T = \Omega \times (0, T), \tag{1.2}$$

$$u_1 = u_2 = 0 \text{ on } \partial\Omega \times [0, T],\ u_1 > 0 \text{ and } u_2 > 0 \text{ in } Q_T = \Omega \times (0, T), \tag{1.3}$$

$$(u_1(x, 0), u_2(x, 0)) = (\varphi_1(x), \varphi_2(x)) \qquad \text{in } \Omega, \tag{1.4}$$

where $\Delta_{p_i} u_i = \operatorname{div}(|\nabla u_i|^{p_i - 2} |\nabla u_i|)$, $1 < p_i < \infty$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and the functions α_i, f_i, $i = 1, 2,$ satisfy some conditions specified later.

Systems $(\mathcal{S})$ appears in the study of non-Newtonian flows, chemical heterogeneous catalyst kinetics, combustion. We refer to the survey Hernandez et al. [15], the book Ghergu and Radulescu [6] and the bibliography therein for more details about the corresponding models.

Recently, Badra et al. [3] discussed the existence and long-behaviour of solutions of the quasilinear and singular parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2} |\nabla u|) = \dfrac{1}{u^{\delta}} + f(x, u) & \text{in } Q_T, \\ u_{|t=0} = u_0(x) & \text{in } \Omega, \\ u_{|\partial\Omega} = 0 \text{ on } \partial\Omega \times [0, T], \quad u > 0 & \text{on } Q_T. \end{cases}$$

In this paper, motivated by the ideas in [3], we generalize and extend the results of [3] to systems $(\mathcal{S})$.

This is the plan of paper. We recall our assumptions and state main results in Section 2. In Section 3, we show the existence of discrete scheme. And, after showing some estimates on there approximations, the passage to the limit and the existence results are given in Section 4.

2. Assumptions and Main Results

2.1. Notations and assumptions

Let Ω be a smooth and bounded domain in $\mathbb{R}^N (N \geq 2)$. Set for $t > 0$, $Q_t := \Omega \times (0, t)$, $S_t := \partial\Omega \times (0, t)$.

The norm in a space X will be denoted as follows:

$$\|\cdot\|_r \text{ if } X = L^r(\Omega), \quad 1 \leq r \leq +\infty;$$

$$\|\cdot\|_{1,q} \text{ if } X = W^{1,q}(\Omega), \quad 1 \leq q \leq +\infty;$$

$$\|\cdot\|_X \text{ otherwise};$$

and $\langle .,. \rangle$ denotes the duality between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$. For any $p \geq 1$, we define it's conjugate p' by $\frac{1}{p} + \frac{1}{p'} = 1$. On this paper, C_i and C will denote various positive constants.

In the sequel, the same symbol c will be used to indicate some positive constants, possibly different from each other, appearing in the various hypotheses and computations and depending only on data. When we need to fix the precise value of one constant, we shall use a notation like M_i, $i = 1, 2, \ldots,$ instead.

To control the singular term $\frac{1}{u^{\alpha_i}}$, we need to consider solutions in the cone $\mathcal{C}_i$, where $\mathcal{C}_i$ is the set of functions $v \in L^\infty(\Omega)$ such that $\exists c_1, c_2$ with

$$\begin{cases} c_1 d(x) \le v \le c_2 d(x) \quad \text{if } \alpha_i < 1, \\ c_1 d(x) \log^{\frac{1}{p_i}}\left(\frac{k}{d(x)}\right) \le v \le c_2 d(x) \log^{\frac{1}{p_i}}\left(\frac{k}{d(x)}\right) \text{ with } k \text{ large and if } \alpha_i = 1, \\ c_1 d(x)^{\frac{p_i}{\alpha_i + p_i - 1}} \le v \le c_2 d(x)^{\frac{p_i}{\alpha_i + p_i - 1}} \text{ if } \alpha_i > 1, \end{cases}$$

with $d(x) = \text{dist}(x, \partial\Omega)$.

In the sequel, we shall present the following assumptions:

(H1) $f_i \in C^1(\Omega \times \mathbb{R}^+ \times \mathbb{R}^+)$, $(i = 1, 2)$.

(H2) $(u, v) \to f_i(x, u, v)$ is increasing function.

Lemma 2.1 (Theorem 1.3, cf. [2]). *Let* $g \in L^\infty(\Omega)$ *and* $0 < \delta_i < 2 + \frac{1}{r_i - 1}$. *Then for any* $\lambda > 0$, *there exists a unique* w_λ *in* $W_0^{1, p_i}(\Omega) \cap \mathcal{C}_i$ *such that*

$$w - \lambda\left(\Delta_{p_i} w + \frac{1}{w^{\delta_i}}\right) = g \quad in\ \Omega,$$

$$w_{|\partial\Omega} = 0.$$

2.2. Existence theorem

Let us introduce the function space

Definition 1.

$$\mathcal{V}_i(Q_T) = \left\{u_i : u_i \in L^\infty(0, T; W_0^{1, p_i}(\Omega)) \cap L^\infty(Q_T), \frac{\partial u_i}{\partial t} \in L^2(Q_T)\right\}.$$

Then, we define

Definition 2. A pair of functions $u = (u_1, u_2) \in \mathcal{V}_1(Q_T) \times \mathcal{V}_2(Q_T)$ is called a weak solution (resp., subsolution, supersolution) of $(\mathcal{S})$ if

(1) for any compact $K \in Q_T$, $\text{ess}_K \inf u_1 > 0$, $\text{ess}_K \inf u_2 > 0$,

(2) for every test function $\phi = (\phi_1, \phi_2) \in \mathcal{V}_1(Q_T) \times \mathcal{V}_2(Q_T)$,

$$\int_{Q_T} \frac{\partial u_i}{\partial t} \phi_i dz - \int_{Q_T} |\nabla u_i|^{p_i - 2} |\nabla u_i| . |\nabla \phi_i| \, dz$$

$$- \int_{Q_T} \frac{1}{(u_i)^{\alpha_i}} \phi_i dz - \int_{Q_T} f_i(x, u) \phi_i dz = 0 \quad (\leq 0, \geq 0), \quad z = (x, t),$$

(3) $u_i(x, 0) = \varphi_i(x) \quad (\leq 0, \geq 0)$ a.e. in Ω.

We refer the readers to [7, 13, 16] for the existence of supersolution and subsolution for systems $(\mathcal{S})$.

Using a time discretization method, and existence of supersolution and subsolution, we prove the following result concerning $(\mathcal{S})$:

Theorem 3. *Let* $p_i > 2, 0 < \alpha_i < 2 + \frac{1}{p_i - 1}, (i = 1, 2)$ *and* $\varphi_i \in W_0^{1, p_i}(\Omega) \cap \mathcal{C}_i$ *be given.*

Suppose that f_i *verify* (H1) *and* (H2) *and that* $(\mathcal{S})$ *has a* $\underline{u}, \overline{u}$ *a supersolution, subsolution. Then, for each* $T > 0$ *given, systems* $(\mathcal{S})$ *has at least one weak solution* $u = (u_1, u_2) \in \mathcal{C}_1 \times \mathcal{C}_2$ *uniformly for* $t \in (0, T)$.

Proof. The main tools in the proof of this theorem are discrete scheme (2.1)

$$\frac{u_i^n - u_i^{n-1}}{\tau} - \Delta_{p_i} u_i^n = \frac{1}{(u_i^n)^{\alpha_i}} + f_i(x, u_1^{n-1}, u_2^{n-1}) \quad \text{in } \Omega, \tag{2.1}$$

$$u_i^n = 0 \quad \text{on } \partial\Omega,$$

$$u_i^0 = \varphi_i \quad \text{in } \Omega,$$

where $N\tau = T$ is a fixed positive real, and $1 \leq n \leq N$.

We can write (2.1) as

$$w - \lambda(\Delta_{p_i} w + \frac{1}{w^{\delta_i}}) = g \quad \text{in } \Omega,$$

$$w_{|\partial\Omega} = 0,$$

where $w = u_i^n$, $\lambda = \tau$, $\delta_i = \alpha_i$, and $g = \tau f(x, u_1^{n-1}, u_2^{n-1}) + u_i^{n-1}$.

From Lemma 2.1, we define by iteration $u_i^n \in W_0^{1, p_i}(\Omega) \cap \mathcal{C}_i$ and $u_i^0 = \varphi_i \in W_0^{1, p_i}(\Omega) \cap \mathcal{C}_i$.

So consequently, $(u_i)_\tau$, $(\tilde{u}_i)_\tau$ set by: For all $n \in \{1, \dots, N\}$,

$$\forall t \in [(n-1)\tau, n\tau] \quad \begin{cases} (u_i)_\tau(t) = u_i^n, \\ (\tilde{u}_i)_\tau(t) = \dfrac{(t - (n-1)\tau)}{\tau}(u_i^n - u_i^{n-1}) + u_i^{n-1}, \end{cases}$$

are well defined and satisfied in addition

$$\frac{\partial(\tilde{u}_i)_\tau}{\partial t} - \Delta_{p_i}(u_i)_\tau = \frac{1}{((u_i)_\tau)^{\alpha_i}} + f_i(x, (u_1)_\tau(. - \tau), (u_2)_\tau(. - \tau)). \tag{2.2}$$

We first establish some energy estimates of $(u_i)_\tau$, $(\tilde{u}_i)_\tau$.

We need several lemmas to complete the proof of Theorem 3. □

Lemma 2.2. *For any* $n \in N^*$, *the relation* $\underline{u}_i \le u_i^{n-1} \le \overline{u}_i$, *imply that* $\underline{u}_i \le u_i^n \le \overline{u}_i$.

Proof. By the above assumptions, we have

$$\frac{u_i^n - u_i^{n-1}}{\tau} - (\Delta_{p_i} u_i^n - \Delta_{p_i} \overline{u}_i) - \left(\frac{1}{(u_i^n)^{\alpha_i}} - \frac{1}{(\overline{u}_i)^{\alpha_i}} \right)$$

$$\le f_i(x, u_1^{n-1}, u_2^{n-1}) - f_i(x, \overline{u}_1, \overline{u}_2).$$

We obtain with $u_i^{n-1} \leq \overline{u}_i$ and (H2)

$$f_i(x, u_1^{n-1}, u_2^{n-1}) - f_i(x, \overline{u}_1, \overline{u}_2) = f_i(x, u_1^{n-1}, u_2^{n-1})$$

$$- f_i(x, \overline{u}_1, u_2^{n-1}) + f_i(x, \overline{u}_1, u_2^{n-1}) - f_i(x, \overline{u}_1, \overline{u}_2) \leq 0.$$

We therefore obtain

$$\frac{u_i^n - \overline{u}_i}{\tau} - (\Delta_{p_i} u_i^n - \Delta_{p_i} \overline{u}_i) - \left(\frac{1}{(u_i^n)^{\alpha_i}} - \frac{1}{(\overline{u}_i)^{\alpha_i}} \right)$$

$$\leq f_i(x, u_1^{n-1}, u_2^{n-1}) - f_i(x, \overline{u}_1, \overline{u}_2). \tag{2.3}$$

Multiplying (2.1) by $(u_i^n - \overline{u}_i)_+$, the monotonicity of $w \to -(\Delta_{p_i} w - w^{-\alpha_i})$ implies

$$u_i^n \geq \overline{u}_i.$$

Similarly, we obtain $\underline{u}_i \leq u_i^n$. □

Lemma 2.3. *There exists a positive constant* $C(T, \varphi_1, \varphi_2)$ *such that, for all* $n = 1, \ldots, N$

$$u_i^n \in L^\infty(0, T; L^\infty(\Omega)), \tag{2.4}$$

$$(u_i)_\tau, (\tilde{u}_i)_\tau \in \mathcal{C}_i, \tag{2.5}$$

$$(u_i)_\tau, (\tilde{u}_i)_\tau \textit{ are bounded in } L^{p_i}(0, T; W_0^{1, p_i}(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), \tag{2.6}$$

$$\frac{\partial (\tilde{u}_i)_\tau}{\partial t} \textit{ is bounded in } L^2(Q_T), \tag{2.7}$$

and

$$(u_i)_\tau, (\tilde{u}_i)_\tau \textit{ are bounded in } L^\infty(0, T; W_0^{1, p_i}(\Omega)). \tag{2.8}$$

Proof. (a) By Lemma 2.3, for any $n \in N$, $u_i^n (i = 1, 2)$ are bounded; whence (2.4).

(b) Multiplying (2.1) by τu_i^n, summing from $n = 1$ to N and integrating over Ω, we obtain

$$\tau \sum_{n=1}^{N} \int_{\Omega} \left(\frac{u_i^n - u_i^{n-1}}{\tau} \right) u_i^n dx + \tau \sum_{n=1}^{N} \int_{\Omega} |\nabla u_i^n|^{p_i} dx$$
$$= \tau \sum_{n=1}^{N} \int_{\Omega} (u_i^n)^{1-\alpha_i} dx + \tau \sum_{n=1}^{N} \int_{\Omega} f_i(x, u_1^{n-1}, u_2^{n-1}) u_i^n dx. \tag{2.9}$$

By Young inequality, for $\epsilon > 0$ small, there exists $C_\epsilon(T)$ such that

$$\tau \sum_{n=1}^{N} \int_{\Omega} f_i(x, u_1^{n-1}, u_2^{n-1}) u_i^n dx \le \epsilon \tau \sum_{n=1}^{N} \int_{\Omega} |\nabla u_i^n|^{p_i} dx + C_\epsilon(T). \tag{2.10}$$

With the aid of the identity $2a(a - b) = a^2 - b^2 + (a - b)^2$, we get

$$\tau \sum_{n=1}^{N} \int_{\Omega} \left(\frac{u_i^n - u_i^{n-1}}{\tau} \right) u_i^n dx = \frac{1}{2} \sum_{n=1}^{N} \int_{\Omega} \left(|u_i^n|^2 - |u_i^{n-1}|^2 + |u_i^n - u_i^{n-1}|^2 \right) dx$$
$$= \frac{1}{2} \sum_{n=1}^{N} \int_{\Omega} \left(|u_i^n|^2 - \left|u_i^{n-1}\right|^2 \right) dx + \frac{1}{2} \int_{\Omega} |u_i^N|^2 dx - \frac{1}{2} \int_{\Omega} |\varphi_i|^2 dx.$$

Since $\alpha_i < 2 + \frac{1}{p_i - 1}$ and $\underline{u}_i \le u_i^n \le \overline{u}_i$, we have

$$\tau \sum_{n=1}^{N} \int_{\Omega} (u_i^n)^{1-\alpha_i} dx \le \begin{cases} T \int_{\Omega} (\overline{u}_i)^{1-\alpha_i} dx < +\infty, & \text{if } \alpha_i \le 1, \\ T \int_{\Omega} (\underline{u}_i)^{1-\alpha_i} dx < +\infty, & \text{if } \alpha_i > 1. \end{cases} \tag{2.11}$$

Gathering the above estimates, we get (2.5) and (2.6).

(c) Multiplying the Equation (2.1) by $u_i^n - u_i^{n-1}$ and summing from $n = 1$ to N, wet get

$$\tau \sum_{n=1}^{N} \int_{\Omega} \left(\frac{u_i^n - u_i^{n-1}}{\tau} \right)^2 dx + \sum_{n=1}^{N} \int_{\Omega} |\nabla u_i^n|^{p_i - 2} \nabla u_i^n \cdot \nabla (u_i^n - u_i^{n-1}) dx$$

$$- \sum_{n=1}^{N} \int_{\Omega} \left(\frac{u_i^n - u_i^{n-1}}{(u_i^n)^{\alpha_i}} \right) dx = \sum_{n=1}^{N} \int_{\Omega} f_i(x, u_1^{n-1}, u_2^{n-1})(u_i^n - u_i^{n-1}) dx. \tag{2.12}$$

By Young inequality, we get

$$\sum_{n=1}^{N} \int_{\Omega} f_i(x, u_1^{n-1}, u_2^{n-1})(u_i^n - u_i^{n-1}) dx$$

$$\leq C_\epsilon(T) + \frac{\tau}{2} \sum_{n=1}^{N} \int_{\Omega} \left(\frac{u_i^n - u_i^{n-1}}{\tau} \right)^2 dx. \tag{2.13}$$

From the convexity of the expressions $\int_{\Omega} |\nabla w|^{p_i} dx$ and $-\frac{1}{1-\alpha_i}$ $\int_{\Omega} w^{1-\alpha_i} x$, we get the following inequality:

$$\frac{1}{p_i} \int_{\Omega} |\nabla u_i^n|^{p_i} dx - \frac{1}{p_i} \int_{\Omega} |\nabla u_i^{n-1}|^{p_i} dx \leq \int_{\Omega} |\nabla u_i^n|^{p_i - 2} \nabla u_i^n \cdot \nabla (u_i^n - u_i^{n-1}) dx, \tag{2.14}$$

and

$$\frac{1}{1-\alpha_i} \int_{\Omega} (u_i^{n-1})^{1-\alpha_i} dx - \frac{1}{1-\alpha_i} \int_{\Omega} (u_i^n)^{1-\alpha_i} dx \leq -\int_{\Omega} \left(\frac{u_i^n - u_i^{n-1}}{(u_i^n)^{\alpha_i}} \right) dx, \tag{2.15}$$

which imply with (2.12), (2.14), and (2.15) that

$$\frac{\tau}{2}\sum_{n=1}^{N}\int_{\Omega}\left(\frac{u_i^n - u_i^{n-1}}{\tau}\right)^2 dx + \frac{1}{p_i}\int_{\Omega}|\nabla u_i^N|^{p_i}\, dx$$

$$\leq \frac{1}{1-\alpha_i}\int_{\Omega}\frac{1}{(u_i^n)^{\alpha_i - 1}}\, dx + C. \tag{2.16}$$

The above expression together with

$$\int_{\Omega}\frac{1}{(u_i^n)^{\alpha_i-1}}\, dx \leq \max\left\{\int_{\Omega}(\overline{u}_i)^{1-\alpha_i}\, dx, \int_{\Omega}(\underline{u}_i)^{1-\alpha_i}\, dx\right\}$$

yields (2.7) and (2.8). □

By Lemma 2.3, there exists $M_i > 0$ independent of τ such that

$$\|(u_i)_\tau - (\tilde{u}_i)_\tau\|_{L^\infty(0, T; L^2(\Omega))} \leq \max_{1 \leq n \leq N}\|u_i^n - u_i^{n-1}\|_{L^2(\Omega)} \leq M_i\sqrt{\tau}. \tag{2.17}$$

Therefore, taking $\tau \to 0^+$, and up to subsequence, wet get that there exists $u_i, v_i \in L^\infty(0, T; W_0^{1, p_i}(\Omega) \cap L^\infty(\Omega))$ such that $\frac{\partial u_i}{\partial t} \in L^2(Q_T)$, $u_i, v_i \in \mathcal{C}_i$ uniformly and as $\tau \to 0^+$,

$$(u_i)_\tau \overset{*}{\to} u_i \text{ in } L^\infty(0, T; W_0^{1, p_i}(\Omega) \cap L^\infty(\Omega)),$$

$$(\tilde{u}_i)_\tau \overset{*}{\to} v_i \text{ in } L^\infty(0, T; W_0^{1, p_i}(\Omega) \cap L^\infty(\Omega)),$$

$$\frac{\partial(\tilde{u}_i)_\tau}{\partial t} \to \frac{\partial u_i}{\partial t} \text{ in } L^2(Q_T).$$

From (2.17), it follows that $u_i = v_i$. From (2.17), from Lemma 2.3, compactness Sobolev imbedding, the interpolation inequality and Ascoli Arzela theorem, we get that

$$(u_i)_\tau, (\tilde{u}_i)_\tau \to u_i \text{ in } L^\infty(0, T; L^{q_i}(\Omega)), \quad \forall q_i > 1. \tag{2.18}$$

Now, multiplying (2.1) by $(u_i)_\tau - u_i$ and using (2.19), we get by straightforward calculations

$$\int_0^T \int_\Omega \left(\frac{\partial (\tilde{u}_i)_\tau}{\partial t} - \frac{\partial u_i}{\partial t} \right) ((\tilde{u}_i)_\tau - u_i) dxdt - \int_0^T < \Delta_{p_i} (u_i)_\tau, (u_i)_\tau - u_i > dt$$

$$= \int_0^T \int_\Omega (u_i)_\tau^{1-\alpha_i} ((u_i)_\tau - u_i) dxdt$$

$$+ \int_0^T \int_\Omega f_i(x, (u_1)_\tau(. - \tau), (u_2)_\tau(. - \tau)) dxdt + o_\tau(1), \tag{2.19}$$

where $o_\tau(1) \to 0$ as $\tau \to 0^+$.

From convexity of the term $-\int_\Omega (u_i)^{1-\alpha_i} dx$ and since $(u_i)_\tau \to u_i$ in $L^{p_i}(0, T; W_0^{1, p_i}(\Omega))$, we get that

$$\int_\Omega |(\tilde{u}_i)_\tau(T) - u_i(T)|^2 dx - \int_0^T < \Delta_{p_i} (u_i)_\tau - \Delta_{p_i} u_i, (u_i)_\tau - u_i > dt$$

$$- \int_0^T \int_\Omega (u_i)_\tau^{1-\alpha_i} ((u_i)_\tau - u_i) dxdt$$

$$\leq \int_0^T \int_\Omega f_i(x, (u_1)_\tau(. - \tau), (u_2)_\tau(. - \tau)) dxdt + o_\tau(1), \tag{2.20}$$

and from (2.19), we have

$$\int_0^T \int_\Omega f_i(x, (u_1)_\tau(. - \tau), (u_2)_\tau(. - \tau)) dx\, dt = o_\tau(1).$$

By Lebesgue theorem and Lemma 2.1,

$$\int_0^T \int_\Omega (u_i)_\tau^{1-\alpha_i} ((u_i)_\tau - u_i) dx\, dt = o_\tau(1).$$

Then

$$\frac{1}{2}\int_{\Omega}|(\tilde{u}_i)_\tau - u_i|^2(T)dx - \int_0^T < \Delta_{p_i}(u_i)_\tau - \Delta_{p_i}u_i,\, (u_i)_\tau - u_i > dt = o_\tau(1). \tag{2.21}$$

Thus,

$$(u_i)_\tau \to u_i \text{ in } L^{p_i}(0, T; W_0^{1, p_i}(\Omega)), \quad \text{as} \quad \tau \to 0^+,$$

and consequently,

$$\Delta_{p_i}(u_i)_\tau \to \Delta_{p_i}u_i \text{ in } L^{p_i'}(0, T; W^{-1, p_i'}(\Omega)).$$

Moreover, from Lemma 2.2 and Lebesgue theorem, we obtain

$$\frac{1}{(u_i)_\tau^{\alpha_i}} \to \frac{1}{(u_i)^{\alpha_i}} \text{ in } L^{+\infty'}(0, T; W^{-1, p_i'}(\Omega)).$$

Therefore, $u_i \in \mathcal{V}_i(Q_T)$ and satisfies $(\mathcal{S})$.

2.3. Uniqueness

Let $p_i > 2$, $0 < \alpha_i < 2 + \frac{1}{p_i - 1}$, $(i = 1, 2)$ and $\varphi_i \in W_0^{1, p_i}(\Omega) \cap \mathcal{C}_i$ be given. Let (H1) to (H2) be satisfied. Then $(\mathcal{S})$ has a unique solution (u_1, u_2) in Q_T.

Proof. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be solutions of $(\mathcal{S})$ satisfying $(u_1, u_2), (v_1, v_2) \in \mathcal{V}_1(Q_T) \times \mathcal{V}_2(Q_T)$, $\forall t \in [0, T]$, we have

$$\int_0^T\int_{\Omega}\frac{\partial(u_i - v_i)}{\partial t}(u_i - v_i)dxdt - \int_0^T < \Delta_{p_i}u_i - \Delta_{p_i}v_i,\, u_i - v_i > dt$$

$$= \int_0^T\int_{\Omega}(u_i^{-\alpha_i} - v_i^{-\alpha_i})(u_i - v_i)dxdt$$

$$+ \int_0^T\int_{\Omega}(f_i(x, u_1, u_2) - f_i(x, u_1, u_2))(u_i - v_i)dxdt.$$

Since $f_i(x, ., .)$ is locally Lipschitz uniformly in Ω, the difference $w_i = u_i - v_i$ satisfies

$$\frac{1}{2}\sum_{i=1}^{2}|w_i|^2_{L^2(\Omega)} + \sum_{i=1}^{2}\int_0^T < \Delta_{p_i}u_i - \Delta_{p_i}v_i, w_i > dt$$

$$- \sum_{i=1}^{2}\int_0^T\int_\Omega (u_i^{-\alpha_i} - v_i^{-\alpha_i})(w_i)dxdt$$

$$\leq c\sum_{i=1}^{2}\int_0^T\int_\Omega |w_i|^2 dt,$$

we observe that if $\alpha_i < 2 + \frac{1}{p_i - 1}$, then $w \to -(\Delta_{p_i}w - w^{-\alpha_i})$ is monotone from $W_0^{1, p_i}(\Omega) \cap \mathcal{C}_i$ to $W^{-1, p_i'}(\Omega)$

$$\frac{1}{2}\sum_{i=1}^{2}|w_i|^2 \leq c\sum_{i=1}^{2}\int_0^T |w_i|^2 dt. \tag{2.22}$$

We finally deduce from Gronwall's lemma,

$$\sum_{i=1}^{2}|w_i|^2 \leq \sum_{i=1}^{2}|w_i(0)|^2 \exp(2cT), \quad \forall t \in (0, T).$$

Thus, we deduce that $u_1 = v_1$ and $u_2 = v_2$.

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