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ON THE FAVARD CLASSES FOR VOLTERRA EQUATIONS

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Abstract

We introduce the Favard spaces for resolvent family, extending some of the well-known theorems for semigroups.

1. Introduction

Favard class for semigroups was developed as early as 1967 by Butzer and Berens, presented in the monograph [3]. In semigroup theory, the Favard class plays an important role, particularly in perturbation theory. The body of knowledge has increased steadily since then; the recent monograph of [6] gives a good account of modern developments. Applications appear, in particular, in [11, 13, 5], but are certainly not

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restricted to this. However, these concepts have been slightly introduced to Volterra integral equations in [8, 9], although they are closely related to perturbation theory, which play an important role in various fields and have been treated by Favard spaces. The aim of this paper is to give an extension to a Favard classes for Volterra integral equations similar to the one for semigroups. In fact, we recover several well-known theorems for semigroups, if we consider $a(t) = 1$.

In Section 2, we give some preliminaries about the concept of resolvent family, and the relationship between linear integral equation of Volterra type with scalar kernel. It is well-known that for a Cauchy problem, there are strong relations connecting its semigroup solution and its associated generator. Likewise, for a Volterra scalar problem, there are some results connecting its resolvent family and the domain of the associated generator; which will be reviewed in Section 3. There are many results available from semigroup theory concerning the Favard spaces [6]. In Section 4, we define the Favard spaces for scalar Volterra integral equations, and for these spaces, we account for some results which are similar to those of semigroups.

2. Preliminaries

In this section, we collect some elementary facts about scalar Volterra equations and resolvent family. These topics have been covered in detail in [12]. We refer to these works for reference to the literature and further information.

Let $(X, \|\cdot\|)$ be a Banach space, A be a linear closed densely defined operator in X, and $a \in L^1_{\text{loc}}(\mathbb{R}^+)$ is a scalar kernel. We consider the linear Volterra equation

$$\begin{cases} x(t) = \int_0^t a(t-s)Ax(s)ds + f(t), \quad t \geq 0, \\ x(0) = x_0 \in X, \end{cases} \tag{2.1}$$

where $f \in \mathcal{C}(\mathbb{R}^+, X)$.

We denote by $[D(A)]$ the domain of A equipped with the graph-norm. We define the convolution product of the scalar function a with the vector-valued function f by

$$(a * f)(t) := \int_0^t a(t-s)f(s)ds, \quad t \geq 0.$$

Definition 2.1. A function $x \in C(\mathbb{R}^+, X)$ is called

(1) Strong solution of (2.1), if $x \in \mathcal{C}(\mathbb{R}^+, [D(A)])$ and (2.1) is satisfied.

(2) Mild solution of (2.1), if $a * x \in \mathcal{C}(\mathbb{R}^+, [D(A)])$ and

$$x = f(t) + A[a * x](t), \quad t \geq 0. \tag{2.2}$$

Obviously, every strong solution of (2.1) is a mild solution. Conditions under which mild solutions are strong solutions are studied in [12].

Definition 2.2. Equation (2.1) is called well-posed if, for each $v \in D(A)$, there is a unique strong solution $x(t, v)$ on $\mathbb{R}^+$ of

$$x(t, v) = v + (a * Ax)(t), \quad t \geq 0, \tag{2.3}$$

and for a sequence $(x_n) \subset D(A)$, $x_n \to 0$ implies $u(t, x_n) \to 0$ in X, uniformly on compact intervals.

Definition 2.3. Let $a \in L^1_{\text{loc}}(\mathbb{R}^+)$. A strongly continuous family $(S(t))_{t\geq 0} \subset \mathcal{L}(X)$ is called resolvent family for Equation (2.1), if the following three conditions are satisfied:

(S1) $S(0) = I$.

(S2) $S(t)$ commutes with A, which means $S(t)(D(A)) \subset D(A)$ for all $t \geq 0$, and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$.

(S3) For each $x \in D(A)$ and all $t \geq 0,$ the resolvent equations hold

$$S(t)x = x + \int_0^t a(t-s)AS(s)xds.$$

Note that the resolvent for (2.1) is uniquely determined and further information on resolvent can be found in the monograph by Prüss [12]. We also notice that the choice of the kernel a classifies different families of strongly continuous solution operators in $\mathcal{L}(X)$: For instance, when $a(t) = 1,$ then $S(t)$ corresponds to a C_0-semigroup and when $a(t) = t,$ then $S(t)$ corresponds to cosine operator function.

The existence of a resolvent family allows one to find the solution for the Equation (2.1). Several properties of resolvent families has been discussed in [2, 12].

The resolvent family is the central object to be studied in the theory of Volterra equations. The importance of the resolvent family $S(t)$ is that, if it exists, then the solution $x(t)$ of (2.1) is given by the following variation of parameters formula in [12]:

$$x(t) = \frac{d}{dt}\int_0^t S(t-s)f(s)ds, \tag{2.4}$$

for all $t \geq 0,$ and

$$x(t) = S(t)f(0) + \int_0^t S(t-s)f'(s)ds, \tag{2.5}$$

where $t \geq 0$ and $f \in W^{1,1}(\mathbb{R}^+, X),$ gives us a mild solution for (2.1).

The following well-known result [12, Proposition 1.1] establishes the relation between well-posedness and existence of a resolvent family. In what follows, $\mathcal{R}$ denotes the range of a given operator.

Theorem 2.4. *Equation* (2.1) *is well-posed if and only if* (2.1) *admits a resolvent family* $(S(t))_{t\geq 0}$. *If this is the case we have in addition* $\mathcal{R}(a * S(t)) \subset D(A)$, *for all* $t \geq 0$ *and*

$$S(t)x = x + A\int_0^t a(t-s)S(s)xds, \tag{2.6}$$

for each $x \in X$, $t \geq 0$.

From this, we obtain that if $(S(t))_{t\geq 0}$ is a resolvent family of (2.1), we have $A(a * S)(\cdot)$ is strongly continuous and the so-called mild solution $x(t) = S(t)x_0$ solves Equation (2.1) for all $x_0 \in X$ with $f = 0$. A resolvent family $(S(t))_{t\geq 0}$ is called exponentially bounded, if there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, and the pair (M, ω) is called type of $(S(t))_{t\geq 0}$. The growth bound of $(S(t))_{t\geq 0}$ is $\omega_0 = \inf\{\omega \in \mathbb{R}, \|S(t)\| \leq Me^{\omega t}, t \geq 0, M > 0\}$. The resolvent family is called exponentially stable if $\omega_0 < 0$.

Note that, contrary to the case of C_0-semigroup, resolvent for (2.1) need not to be exponentially bounded: A counterexample can be found in [4, 12]. However, there is checkable conditions guaranteeing that (2.1) possesses an exponentially bounded resolvent operator.

We will use the Laplace transform at times. Suppose $g : \mathbb{R}^+ \to X$ is measurable and there exists $M > 0, \omega \in \mathbb{R}$, such that $\|g(t)\| \leq Me^{\omega t}$ for almost $t \geq 0$. Then the Laplace transform

$$\hat{g}(\lambda) = \int_0^\infty e^{-\lambda t} g(t)dt,$$

exists for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > \omega$.

A function $a \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ is ω (resp., ω^+)-exponentially bounded, if $\int_0^\infty e^{-\omega s}|a(s)|ds < \infty$ for some $\omega \in \mathbb{R}$ (resp., $\omega > 0$).

The following proposition stated in [12], establishes the relation between resolvent family and Laplace transform.

Proposition 2.5 ([12]). *Let* $a \in L^1_{\text{loc}}(\mathbb{R}^+)$ *be* ω*-exponentially bounded. Then* (2.1) *admits a resolvent family* $(S(t))_{t\geq 0}$ *of type* (M, ω), *if and only if the following conditions hold*:

(1) $\hat{a}(\lambda) \neq 0$ *and* $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$, *for all* $\lambda > \omega$.

(2) $H(\lambda) := \frac{1}{\lambda \hat{a}(\lambda)}\left(\frac{1}{\hat{a}(\lambda)} I - A\right)^{-1}$ *called the resolvent associated to* $S(t)$ *satisfies*

$$\|H^{(n)}(\lambda)\| \leq Mn!\,(\lambda - \omega)^{-(n+1)} \text{ for all } \lambda > \omega \text{ and } n \in \mathbb{N}.$$

Under these assumptions, the Laplace-transform of $S(\cdot)$ is well-defined and it is given by $\hat{S}(\lambda) = H(\lambda)$ for all $\lambda > \omega$.

3. Domains of *A*: A Review

Assuming the existence of a resolvent family $(S(t))_{t\geq 0}$ for (2.1), it is natural to ask how to characterize the domain $D(A)$ of the operator A in terms of the resolvent family. This is important, for instance, in order to study the Favard class in perturbation theory (see [8, 9]). For very special case, the answer to the above question is well-known. For instance, when $a(t) = 1$ or $a(t) = t$, A is the generator of a C_0-semigroup $T(t)$ or a cosine family $C(t)$ and we have

$$D(A) = \left\{x \in X : \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists}\right\},$$

and

$$D(A) = \left\{x \in X : \lim_{t \to 0^+} \frac{C(t)x - x}{t^2} \text{ exists}\right\},$$

respectively (see [12]).

Since A is a closed operator, we may consider the graph of A with the appropriate norm $G(A)$, and $\|(x, Ax)\|_{G(A)} = \|x\| + \|Ax\|$ for $(x, Ax) \in G(A)$. This space is isometrically isomorphic to $D(A)$ with the graph norm. We call this space $(X_1, \|\cdot\|_1)$. It is continuously embedded in X. If the resolvent set $\rho(A)$ of A is nonempty, $A_1 : D(A^2) \to X_1$, with $A_1 x = Ax$, is a closed operator in X_1 and $\rho(A) = \rho(A_1)$. On the other hand, we may consider $(X \times X)/G(A)$ with its natural norm. It is isometrically isomorphic to the completion of X with the norm

$$\|x\|_{-1} = \inf_{y \in D(A)} (\|y\| + \|x - Ay\|),$$

for $x \in X$.

We call this space $(X_{-1}, \|\cdot\|_{-1})$. The operator $A : D(A) \to X_{-1}$ is continuous and densely defined, its (unique) extension to X as domain makes it a closed operator in X_{-1}, and it is called A_{-1}. We have $\rho(A) = \rho(A_{-1})$.

It was observed in [8] that $D(A)$ has the following characterization.

Proposition 3.1. *Let* (2.1) *admits a resolvent family with growth bound* ω (*such that the Laplace transform of the resolvent exists for* $\lambda > \omega$) *for* ω*-exponentially bounded* $a \in L^1_{\text{loc}}(\mathbb{R}^+)$. *Set for* $0 < \theta < \frac{\pi}{2}$ *and* $\epsilon > 0$

$$\Omega_\theta^\epsilon := \left\{\frac{1}{\hat{a}(\lambda)} : \text{Re}\lambda > \omega + \epsilon, |\arg \lambda| \le \theta\right\}.$$

Then the following characterization of $D(A)$ *holds*:

$$D(A) = \left\{ x \in X : \lim_{|\mu| \to \infty,\, \mu \in \Omega_\theta^0} \mu A(\mu I - A)^{-1} x \text{ exists} \right\}.$$

Without loss of generality, we may assume that $\int_0^t |a(s)|^p \, ds \neq 0$ *for all* $t > 0$ *and some* $1 \leq p < \infty$. *Otherwise, we would have for some* $t_0 > 0$ *and* $p_0 \geq 1$ *that* $a(t) = 0$ *for almost all* $t \in [0, t_0]$, *and thus by definition of resolvent family* $S(t) = I$ *for* $t \in [0, t_0]$. *This implies that* A *is bounded, which is the trivial case with* $X = D(A)$.

In what follows, we will use in the forthcoming sections the following assumption on $a \in L^p_{\text{loc}}(\mathbb{R}^+)$ with $1 \leq p < \infty$. It corresponds to [8, Assumption 2.3] when $p = 1$.

$(H_{a,p})$ There exist $\epsilon_a > 0$ and $t_a > 0$, such that for all $0 < t \leq t_a$, we have

$$\left| \int_0^{t_a} a(s) ds \right| \geq \epsilon_a \int_0^{t_a} |a(s)|^p \, ds.$$

This is the case for functions a, which are positive (resp., $a(I) \subset \,]0, 1]$) at some interval $I = [0, t_0[$ for $p = 1$, and $p > 1$, respectively. For almost all reasonable functions in applications, it is easy to see that they satisfy this assumption. There are nonetheless examples of functions that do not.

Now, let us define the set $\widetilde{D}(A)$ as follows:

$$\widetilde{D}(A) := \left\{ x \in X : \lim_{t \to 0^+} \frac{S(t)x - x}{(1 * a)(t)} \text{ exists} \right\}.$$

It was proved in [8] that under $(H_{a,1})$,

$$D(A) = \widetilde{D}(A) = \{ x \in X : \lim_{t \to 0^+} \frac{S(t)x - x}{(1 * a)(t)} = Ax \}. \tag{3.1}$$

From now and in view of this result, we say that the pair (A, a) is a generator of a resolvent family $(S(t))_{t\geq 0}$.

Remark 3.2. When $a = 1 + 1 * k$, the Volterra system (2.1) with $f(t) = x_0$ is equivalent to the following integro-differential Volterra system:

$$\dot{x}(t) = Ax(t) + \int_0^t k(t-s)Ax(s)ds, \quad t \geq 0. \tag{3.2}$$

Furthermore, if (3.2) admits a resolvent family $(S(t))_{t\geq 0}$, then it is easy to see that

$$\widetilde{D}(A) = \{x \in X : \lim_{t\to 0^+} \frac{S(t)x - x}{[1 * (1 + 1 * k)](t)} = Ax\}$$

$$= \left\{x \in X : \lim_{t\to 0^+} \frac{S(t)x - x}{t} = Ax\right\}.$$

In the case when $k \in BV_{\text{loc}}(\mathbb{R}^+)$, with $BV_{\text{loc}}(\mathbb{R}^+)$ is the space of functions of locally bounded variation, the operator A becomes a generator of a C_0-semigroup $(T(t))_{t\geq 0}$, which is a necessary and sufficient condition for the existence of a resolvent family (see [12]). Whence $\widetilde{D}(A)$ is also characterized in term of $(T(t))_{t\geq 0}$ and we have

$$\widetilde{D}(A) = \left\{x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t} = Ax\right\} = \left\{x \in X : \lim_{t\to 0^+} \frac{S(t)x - x}{t} = Ax\right\}.$$

4. Favard Spaces with Kernel

In semigroup theory, the Favard space sometimes called the generalized domain is defined for a given semigroup $(T(t))_{t\geq 0}$ (with A as its generator) as

$$\widetilde{F}^{\alpha}(A) := \left\{x \in X : \sup_{t>0} \frac{\|T(t)x - x\|}{t^{\alpha}} < \infty\right\}, \quad 0 < \alpha \leq 1,$$

with norm

$$\|x\|_{\widetilde{F}^{\alpha}(A)} := \|x\| + \sup_{t>0} \frac{\|T(t)x - x\|}{t^{\alpha}},$$

which makes $\widetilde{F}^{\alpha}(A)$ a Banach space, $T(t)$ is a bounded operator on this space but is not necessary strongly continuous on it. X_1 is a closed subspace of $\widetilde{F}^{\alpha}(A)$ and both spaces coincide when $\alpha = 1$, and X is reflexive (see, e.g., [6]). It is natural to ask how to define in a similar way $\widetilde{F}^{\alpha}(A)$ of the operator A in terms of the resolvent family. In fact, these spaces can be defined for general resolvent family in a similar way. In fact, it can be defined for all A, for which there exists a sequence $(\lambda_n)_n$ with $\lambda_n \in \rho(A)$ and $|\lambda_n| \to \infty$ in a similar fashion, as was proved in [8] for resolvent family and in [9] for integral resolvent family and in [10] for (k, a)-resolvent family for the case $\alpha = 1$. Remark that both [8] and [9] have not considered the Favard class of order α. These spaces will be the topic of this section and will be useful for the notion of the admissibility in Section 5.

This leads to the following definition which corresponds to a natural extension, in our context, of the Favard classes frequently used in approximation theory for semigroups.

Definition 4.1. Let (2.1) admits a bounded resolvent family $(S(t)_{t\geq 0})$ on X, for ω^{+}-exponentially bounded $a \in L^1_{\text{loc}}(\mathbb{R}^{+})$. For $0 < \alpha \leq 1$, we define the Favard space of order α associated to (A, a) as follows:

$$F^{\alpha}(A) := \left\{x \in X : \sup_{\lambda>\omega} \left\|\lambda^{\alpha-1} \frac{1}{\hat{a}(\lambda)} A\left(\frac{1}{\hat{a}(\lambda)} I - A\right)^{-1} x\right\| < \infty\right\}$$

$$= \left\{x \in X : \sup_{\lambda>\omega} \|\lambda^{\alpha} AH(\lambda)x\| < \infty\right\}.$$

Similarly, we define the Favard spaces of A_{-1} denoted by $F^{\alpha}(A_{-1})$.

Remark 4.2. (i) As for the semigroups, it is natural to define the following space:

$$\widetilde{F}^{\alpha}(A) := \left\{ x \in X : \sup_{t>0} \frac{\|S(t)x - x\|}{|(1 * a)(t)|^{\alpha}} < \infty \right\},$$

for (A, a) generator of a resolvent family $(S(t)_{t\geq 0})$ on X.

(ii) It is clear that $\widetilde{D}(A) \subset \widetilde{F}^{1}(A)$ and in virtue of Proposition 3.1, we have $D(A) \subset F^{1}(A)$. Moreover, if a satisfies $(H_{a,1})$, then $D(A) \subset \widetilde{F}^{1}(A)$ due to the fact that $F^{1}(A) \subset \widetilde{F}^{1}(A)$ (see [8]). In this way, for different functions $a(t)$, we obtain different Favard class of order α with may be considered as extrapolation spaces between $D(A)$ and X.

(iii) When $a(t) = 1$, we recall that and $(S(t))_{t\geq 0}$ corresponds to a bounded C_0-semigroup generated by A. In this situation, we obtain $F^{\alpha}(A) = \left\{ x \in X : \sup_{\lambda>0} \left\| \lambda^{\alpha} A(\lambda I - A)^{-1} x \right\| < \infty \right\}$ and $F^{\alpha}(A) = \widetilde{F}^{\alpha}(A)$. This case is well-known. See, e.g., [6].

(iv) The Favard class of A with kernel $a(t)$ can be alternatively defined as the subspace of X given by $\left\{ x \in X : \limsup_{\lambda\to\infty} \left\| \lambda^{\alpha-1} \frac{1}{\hat{a}(\lambda)} A\left(\frac{1}{\hat{a}(\lambda)} I - A \right)^{-1} x \right\| < \infty \right\}$. As a consequence of $S(t)$ being bounded, the above space coincides with $F^{\alpha}(A)$ in Definition 4.1 and that $\widetilde{F}^{\alpha}(A) := \left\{ x \in X : \sup_{0<t\leq 1} \frac{\|S(t)x - x\|}{|(1 * a)(t)|^{\alpha}} < \infty \right\}$.

(v) Let $a = 1 + 1 * k$ and (A, a) be a generator of a bounded resolvent family $(S(t)_{t\geq 0})$ on X. In this case, $\widetilde{F}^{\alpha}(A) = \left\{ x \in X : \sup_{0<t\leq 1} \frac{\|S(t)x - x\|}{t^{\alpha}} < \infty \right\}$

(due to $\lim_{t \to 0^+} \frac{(1 * a)(t)}{t} = 1$) and we have $S(t)(F^{\alpha}(A)) \subset F^{\alpha}(A)$, for all $\alpha \in]0, 1]$ and $t \geq 0$ thanks to [7, Theorem 7] ($(\mu I - A)^{-1}$ commutes with $S(t)$ for all $\mu \in \rho(A)$).

The proof of the following is immediate.

Proposition 4.3. *The Favard classes of order* α *of* A *with kernel* $a(t)$, $F^{\alpha}(A)$, *and* $\widetilde{F}^{\alpha}(A)$ *are Banach spaces with respect to the norms*: $\|x\|_{F^{\alpha}(A)} := \|x\| + \sup_{\lambda > \omega} \left\| \lambda^{\alpha - 1} \frac{1}{\hat{a}(\lambda)} A\left(\frac{1}{\hat{a}(\lambda)} I - A\right)^{-1} x \right\|$ *and* $\|x\|_{\widetilde{F}^{\alpha}(A)} := \|x\| + \sup_{0 < t \leq 1} \frac{\|S(t)x - x\|}{|(1 * a)(t)|^{\alpha}}$, *respectively*.

As for the semigroups case, we obtain the natural inclusions between the Favard class for different exponents.

Proposition 4.4. *Let* (2.1) *admits a bounded resolvent family* $(S(t))_{t \geq 0}$ *on* X, *for* ω^+*-exponentially bounded* $a \in L^1_{\text{loc}}(\mathbb{R}^+)$. *For all* $0 < \beta < \alpha \leq 1$, *we have*

(i) $D(A) \subset F^{\alpha}(A) \subset F^{\beta}(A)$.

(ii) $\widetilde{D}(A) \subset \widetilde{F}^{\alpha}(A) \subset \widetilde{F}^{\beta}(A)$.

Proof. (i) Let $x \in F^{\alpha}(A)$, then for all $\lambda > \omega$, we have

$$\begin{aligned}
\left\| \lambda^{\beta - 1} \frac{1}{\hat{a}(\lambda)} A\left(\frac{1}{\hat{a}(\lambda)} I - A\right)^{-1} x \right\| &= \left\| \lambda^{\beta - \alpha} \lambda^{\alpha - 1} \frac{1}{\hat{a}(\lambda)} A\left(\frac{1}{\hat{a}(\lambda)} I - A\right)^{-1} x \right\| \\
&= \lambda^{\beta - \alpha} \left\| \lambda^{\alpha - 1} \frac{1}{\hat{a}(\lambda)} A\left(\frac{1}{\hat{a}(\lambda)} I - A\right)^{-1} x \right\| \\
&\leq \frac{1}{\lambda^{\alpha - \beta}} \sup_{\lambda > \omega} \left\| \lambda^{\alpha - 1} \frac{1}{\hat{a}(\lambda)} A\left(\frac{1}{\hat{a}(\lambda)} I - A\right)^{-1} x \right\| \\
&\leq \frac{1}{\omega^{\alpha - \beta}} \sup_{\lambda > \omega} \left\| \lambda^{\alpha - 1} \frac{1}{\hat{a}(\lambda)} A\left(\frac{1}{\hat{a}(\lambda)} I - A\right)^{-1} x \right\|,
\end{aligned}$$

which implies that $x \in F^{\beta}(A)$ and from Remark 4.2 (2), we deduce that $D(A) \subset F^{\alpha}(A)$.

(ii) Let $x \in \widetilde{F}^{\alpha}(A)$, and $0 < t \leq 1$. We have

$$\frac{\|S(t)x - x\|}{|(1 * a)(t)|^{\beta}} = \frac{1}{\left|\int_0^t a(s)ds\right|^{\beta-\alpha}} \frac{\|S(t)x - x\|}{\left|\int_0^t a(s)ds\right|^{\alpha}}$$

$$\leq \|a\|_{L^1[0,1]}^{\alpha-\beta} \cdot \sup_{0<t\leq 1} \frac{\|S(t)x - x\|}{\left|\int_0^t a(s)ds\right|^{\alpha}}.$$

Hence $x \in \widetilde{F}^{\beta}(A)$ and that $\widetilde{D}(A) \subset \widetilde{F}^{\alpha}(A)$ due to Remark 4.2 (2).

□

Note that under $(H_{a,1})$ we have (i) $F^1(A) \subset \widetilde{F}^1(A)$ (see Remark 4.2 (2)), where as the inclusion (ii) $\widetilde{F}^1(A) \subset F^1(A)$ was proved under the strong assumption in [8, Assumption 3.1].

Now we will prove that (ii) holds for all nonnegative $a \in L^1_{\text{loc}}(\mathbb{R}^+)$.

Proposition 4.5. *Let* (2.1) *admits a bounded resolvent family* $(S(t))_{t\geq 0}$ *on X, for* ω^+*-exponentially bounded nonnegative* $a \in L^1_{\text{loc}}(\mathbb{R}^+)$. *Then, we have* $F^1(A) = \widetilde{F}^1(A)$.

Proof. Since $a(t)$ is a nonnegative, $(H_{a,1})$ is satisfied and by [8], we have $F^1(A) \subset \widetilde{F}^1(A)$. Now, let $x \in \widetilde{F}^1(A)$ and set $\sup_{0<t\leq 1} \frac{\|S(t)x - x\|}{|(1 * a)(t)|}$ $:= J_x < \infty$. We write $\frac{1}{\hat{a}(\lambda)} A\left(\frac{1}{\hat{a}(\lambda)} I - A\right)^{-1} = \lambda A H(\lambda)$, for all $\lambda > \omega$.

Using the integral representation of the resolvent (see Proposition 2.5), we obtain

$$\lambda AH(\lambda)x = \frac{\lambda}{\hat{a}(\lambda)} H(\lambda)x - \frac{1}{\hat{a}(\lambda)} x$$

$$= \frac{\lambda}{\hat{a}(\lambda)} \left[H(\lambda)x - \frac{1}{\lambda} x \right]$$

$$= \frac{\lambda}{\hat{a}(\lambda)} \int_0^\infty e^{-\lambda s} (S(s)x - x) ds$$

$$= \frac{\lambda}{\hat{a}(\lambda)} \int_0^\infty e^{-\lambda s}.(1 * a)(s). \frac{S(s)x - x}{(1 * a)(s)} ds.$$

The resolvent family $(S(t))_{t \geq 0}$ being bounded; $\|S(t)\| \leq M$ for some $M > 0$ and for all $t \geq 0$. Then we obtain

$$\|\lambda AH(\lambda)x\| \leq \frac{\lambda}{\hat{a}(\lambda)} \int_0^\infty e^{-\lambda s} (1 * a)(s) ds. \sup_{t > 0} \frac{\|S(t)x - x\|}{(1 * a)(t)}$$

$$\leq \frac{\lambda}{\hat{a}(\lambda)} \int_0^\infty e^{-\lambda s} (1 * a)(s) ds. (L\|x\| + \sup_{0 < t \leq 1} \frac{\|S(t)x - x\|}{(1 * a)(t)})$$

$$= \frac{\lambda}{\hat{a}\,(\lambda)} \widehat{1 * a}\,(\lambda). (\, L \, \|x\| + J_x \,)$$

$$= L\|x\| + J_x,$$

with $L = \frac{1 + M}{(1 * a)(1)}$. This implies that $\sup_{\lambda > \omega} \|\lambda AH(\lambda)x\| < \infty$, which ends the proof. □

Note that in the semigroup case, i.e., $a(t) = 1$, we have the well-known result that $\widetilde{F}^{\alpha}(A) = F^{\alpha}(A)$ [6]. In what follows, we investigate conditions on the kernel a to prove that this is the case for the (A, a) generator of the resolvent families.

Note that for all a; ω^+-exponentially bounded function, it is clear that $(1 * a)^\alpha$ is also ω^+-exponentially bounded (due to $x^\alpha \leq 1 + x$ for $x \geq 0$ and $\alpha \in]0, 1]$).

We will consider the following assumption on $a \in L^1_{\text{loc}}(\mathbb{R}^+)$ and $0 < \alpha \leq 1$:

(H_a^α) : a is ω^+-exponentially bounded and there exists $\epsilon_{a,\alpha} > 0$, such that for all $\lambda > \omega$,

$$|\hat{a}(\lambda)| \geq \epsilon_{a,\alpha}.\lambda^\alpha.\int_0^\infty e^{-\lambda t}|(1 * a)(t)|^\alpha dt.$$

Note that conditions (H_a^α) and $\lambda\hat{a}(\lambda)$ is bounded are independent (e.g., see Example 4.6 (2)).

Example 4.6. (i) The famous case $a(t) = 1$ satisfies the condition (H_a^α) for all $\alpha \geq 0$ due to

$$\frac{\lambda^\alpha}{\hat{a}(\lambda)} \cdot \int_0^\infty e^{-\lambda t}((1 * 1)(t))^\alpha dt = \Gamma(\alpha + 1) \quad \text{for all } \lambda > 0,$$

which corresponds to the semigroup case (here Γ denotes the Gamma function).

(ii) Consider the standard kernel $a(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ for $\beta \in [0, 1[$. a is nonnegative and for all $\lambda > 0$,

$$\frac{\lambda^\alpha}{\hat{a}(\lambda)} \cdot \int_0^\infty e^{-\lambda t}((1 * a)(t))^\alpha dt = \frac{\lambda^{\alpha+\beta-\alpha\beta-1}}{\beta^\alpha \cdot (\Gamma(\beta))^\beta \cdot \Gamma(\alpha\beta + 1)}$$

$$= \frac{\lambda^{(\alpha-1)\cdot(1-\beta)}}{\beta^\alpha \cdot (\Gamma(\beta))^\beta \cdot \Gamma(\alpha\beta + 1)}.$$

Thus a satisfy (H_a^α).

(iii) Let $a(t) = \mu + \nu t^{\beta}$, $0 < \beta < 1$, $\mu > 0$, $\nu > 0$. Then we have $\hat{a}(\lambda) = \frac{\mu}{\lambda} + \frac{\nu}{\lambda^{\beta+1}} \Gamma(\beta + 1)$ for $\lambda > 0$ and $(1 * a)(t) = \mu t + \nu \frac{t^{\beta+1}}{\beta + 1}$. Further, for $\alpha \in \,]0, 1]$, we have

$$\frac{\lambda^{\alpha}}{\hat{a}(\lambda)} \int_0^{\infty} e^{-\lambda t} ((1 * a)(t))^{\alpha} dt$$

$$= \frac{\lambda^{\alpha}}{\hat{a}(\lambda)} \int_0^{\infty} e^{-\lambda t} (\mu t + \nu \frac{t^{\beta+1}}{\beta + 1})^{\alpha} dt$$

$$= \frac{\lambda^{\alpha}}{\hat{a}(\lambda)} \int_0^{1} e^{-\lambda t} (\mu t + \nu \frac{t^{\beta+1}}{\beta + 1})^{\alpha} dt + \frac{\lambda^{\alpha}}{\hat{a}(\lambda)} \int_1^{\infty} e^{-\lambda t} (\mu t + \nu \frac{t^{\beta+1}}{\beta + 1})^{\alpha} dt$$

$$\leq (\mu + \frac{\nu}{\beta + 1})^{\alpha} \frac{\Gamma(\alpha + 1)}{\mu} + (\mu + \frac{\nu}{\beta + 1})^{\alpha} \frac{\Gamma(\alpha\beta + \alpha + 1)}{\mu} \cdot \lambda^{-\alpha\beta}.$$

Then (H_a^{α}) is satisfied. Note that, in the particular case of $\beta = 1$, $a(t) = \mu + \nu t$, Equation (2.1) corresponds to the model of a solid of Kelvin-Voigt (see [12]).

(iv) Let $a = 1 + 1 * k$ with $k(t) = e^{-t}$. We have $\hat{a}(\lambda) = \frac{\lambda + 2}{\lambda(\lambda + 1)}$ for all $\lambda > 0$ and $(1 * a)(t) = 2t + e^{-t} - 1 \leq 2t$ for all $t \geq 0$. Hence

$$\frac{\lambda^{\alpha}}{\hat{a}(\lambda)} \int_0^{\infty} e^{-\lambda t} ((1 * a)(t))^{\alpha} dt \leq \frac{\lambda^{\alpha}}{\hat{a}(\lambda)} \int_0^{\infty} e^{-\lambda t} (2t)^{\alpha} dt$$

$$= \frac{\lambda + 1}{\lambda + 2} \cdot 2^{\alpha} \cdot \Gamma(\alpha + 1).$$

Then a satisfy (H_a^{α}).

(v) Let $a = 1 + 1 * k$ with $k(t) = -e^{-t}$. We have $\hat{a}(\lambda) = \frac{1}{\lambda + 1}$ for all $\lambda > 0$ and that $(1 * a)(t) = 1 - e^{-t} \leq t$ for all $t \geq 0$. Hence

$$\frac{\lambda^{\alpha}}{\hat{a}(\lambda)}\int_0^{\infty} e^{-\lambda t}((1*a)(t))^{\alpha}dt \leq \lambda^{\alpha}(\lambda+1)\int_0^{\infty} e^{-\lambda t}t^{\alpha}dt$$

$$= \frac{\lambda+1}{\lambda}\cdot\Gamma(\alpha).$$

Then a satisfy (H_a^{α}).

The following result establishes the relation between the spaces $\widetilde{F}^{\alpha}(A)$ and $F^{\alpha}(A)$ and therefore generalizes [6, Proposition 5.12].

Proposition 4.7. *Let* (2.1) *admits a bounded resolvent family* $(S(t))_{t\geq 0}$ *on X, for* ω^{+}*-exponentially bounded* $a \in L^1_{\text{loc}}(\mathbb{R}^{+})$ *and* $0 < \alpha \leq 1$.

(i) *If a satisfies* $(H_{a,1})$ *and* $\lambda\hat{a}(\lambda)$ *is bounded, then* $F^{\alpha}(A) \subset \widetilde{F}^{\alpha}(A)$.

(ii) *If a is nonnegative satisfying* (H_a^{α}), *then* $\widetilde{F}^{\alpha}(A) \subset F^{\alpha}(A)$.

Proof. (i) Let $x \in F^{\alpha}(A)$ and $0 < t \leq 1$. Then $\sup_{\lambda>\omega}\|\lambda^{\alpha}AH(\lambda)x\| =: K_x < \infty$. Using the integral representation of the resolvent (see Proposition 2.5), we obtain

$$x = \lambda H(\lambda)x - \lambda\hat{a}(\lambda)AH(\lambda)x \quad \text{for } \lambda > \omega$$

$$=: x_{\lambda} - y_{\lambda}.$$

Since $x_{\lambda} \in D(A)$ and using (S2)-(S3), we have

$$\|S(t)x_{\lambda} - x_{\lambda}\| = \left\|\int_0^t a(t-s)S(s)Ax_{\lambda}ds\right\|$$

$$\leq \int_0^t |a(t-s)|\cdot\|S(s)\|\cdot\|Ax_{\lambda}\|ds$$

$$\leq M \cdot \|Ax_\lambda\| \cdot \int_0^t |a(s)| ds$$

$$= M \cdot \|\lambda^\alpha AH(\lambda)x\| \cdot \lambda^{1-\alpha} \cdot (1 * |a|)(t)$$

$$\leq MK_x . \lambda^{1-\alpha} \cdot (1 * |a|)(t).$$

On the other hand, $(S(t))_{t\geq 0}$ is bounded by M and we have

$$\begin{aligned}
\|S(t)y_\lambda - y_\lambda\| &\leq \|S(t)y_\lambda\| + \|y_\lambda\| \\
&\leq \|S(t)\| \cdot \|y_\lambda\| + \|y_\lambda\| \\
&\leq (M+1) \cdot \|y_\lambda\| \\
&= (M+1) \cdot \|\lambda\hat{a}(\lambda)AH(\lambda)x\| \\
&= (M+1) \cdot |\hat{a}(\lambda)| \cdot \|\lambda^\alpha AH(\lambda)x\| \cdot \lambda^{1-\alpha} \\
&\leq (M+1)K_x \cdot |\hat{a}(\lambda)| \cdot \lambda^{1-\alpha}.
\end{aligned}$$

This implies

$$\frac{\|S(t)x - x\|}{|(1 * a)(t)|^\alpha} \leq \frac{MK_x \lambda^{1-\alpha}(1 * |a|)(t)}{|(1 * a)(t)|^\alpha} + \frac{(M+1)K_x \cdot |\hat{a}(\lambda)|\lambda^{1-\alpha}}{|(1 * a)(t)|^\alpha}$$

$$\leq \frac{MK_x}{\epsilon_a^\alpha} . \lambda^{1-\alpha}((1 * |a|)(t))^{1-\alpha} + \frac{(M+1)K_x}{\epsilon_a^\alpha} \cdot |\lambda\hat{a}(\lambda)| \cdot \lambda^{-\alpha}((1 * |a|)(t))^{-\alpha}$$

$$\leq \frac{MK_x}{\epsilon_a^\alpha} . \lambda^{1-\alpha}((1 * |a|)(t))^{1-\alpha} + \frac{(M+1)K_x K'}{\epsilon_a^\alpha} \cdot \lambda^{-\alpha}((1 * |a|)(t))^{-\alpha}.$$

The third inequality is realized under $(H_{a,1})$: $|(1 * a)(t)| \geq \epsilon_a (1 * |a|)(t)$ and that $|\lambda\hat{a}(\lambda)| \leq K'$ for some $K' > 0$. Substituting $\lambda = \dfrac{N_\omega}{(1 * |a|)(t)} > \omega$ for $t \in]0, 1]$ with $N_\omega = 1 + \omega(1 * |a|)(1)$, we obtain

$$\frac{\|S(t)x - x\|}{|(1 * a)(t)|^{\alpha}} \le \frac{MK_x N_{\omega}^{1-\alpha}}{\epsilon_a^{\alpha}} + \frac{(M+1)K_x K' N_{\omega}^{-\alpha}}{\epsilon_a^{\alpha}},$$

for all $0 < t \le 1$. Thus $\sup_{0<t\le 1} \frac{\|S(t)x - x\|}{|(1 * a)(t)|^{\alpha}} < \infty$, and hence $x \in \widetilde{F}^{\alpha}(A)$.

(ii) Let $x \in \widetilde{F}^{\alpha}(A)$ be given, then $\sup_{0<t\le 1} \frac{\|S(t)x - x\|}{|(1 * a)(t)|^{\alpha}} := J_x < \infty$. For $\lambda > \omega$, we write $\lambda H(\lambda)x - x = \lambda\hat{a}(\lambda)AH(\lambda)x$, then

$$\lambda AH(\lambda)x = \frac{\lambda}{\hat{a}(\lambda)}\left(H(\lambda)x - \frac{1}{\lambda}x\right)$$

$$= \frac{\lambda}{\hat{a}(\lambda)}\int_0^{\infty} e^{-\lambda t}(S(t)x - x)dt;$$

$$\lambda^{\alpha}AH(\lambda)x = \frac{\lambda^{\alpha}}{\hat{a}(\lambda)}\int_0^{\infty} e^{-\lambda t}(1 * a)^{\alpha}(t)\left(\frac{S(t)x - x}{(1 * a)^{\alpha}(t)}\right)dt.$$

The fact that a is nonnegative and satisfying (H_a^{α}), we obtain

$$\|\lambda^{\alpha}AH(\lambda)x\| \le \frac{(L_{\alpha}\|x\| + J_x)}{\epsilon_{a,\alpha}} \text{ with } L_{\alpha} = \frac{1 + M}{(1 * a)^{\alpha}(1)}.$$

Therefore, $\sup_{\lambda>\omega}\|\lambda^{\alpha}AH(\lambda)x\| < \infty$, which ends the proof.

□

Remark 4.8. Let $\alpha \in]0, 1]$.

(i) $a(t) = 1$. Then $\lambda\hat{a}(\lambda)$ is bounded for all $\lambda > 0$ and a satisfies $(H_{a,1})$. Furthermore, a satisfies (H_a^{α}) (see Example 4.6 (1)) and by virtue of Proposition 4.7, we obtain $F^{\alpha}(A) = \widetilde{F}^{\alpha}(A)$. Hence, we recover a result for C_0-semigroups case, which corresponds to [6, Proposition 5.12].

(ii) $a(t) = t$ satisfies $(H_{a,1})$ and we have $\lambda\hat{a}(\lambda) = \frac{1}{\lambda}$ is bounded for all $\lambda > 0$. By virtue of Proposition 4.7 (1), we obtain $F^{\alpha}(A) \subset \widetilde{F}^{\alpha}(A)$.

(iii) Let a be a completely positive function. Then (see [12]) a is nonnegative and $\lambda\hat{a}(\lambda) = \frac{1}{k_0 + \frac{1}{k_\infty} + \widehat{k_1}(\lambda)}$, for all $\lambda > 0$, where $k_0 \geq 0$, $k_\infty \geq 0$, and k_1 is nonnegative decreasing function tending to 0 as $t \to \infty$. That is $\lambda\hat{a}(\lambda)$ is bounded and by Proposition 4.7 (1), we obtain $F^{\alpha}(A) \subset \widetilde{F}^{\alpha}(A)$.

(iv) Consider the standard kernel $a(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, with $\beta \in [1, 2[$. Then a satisfies $(H_{a,1})$ and that $\lambda \cdot \hat{a}(\lambda) = \lambda \cdot \lambda^{-\beta} = \lambda^{1-\beta}$, for all $\lambda > 0$ is bounded, thus from Proposition 4.7 (1), $F^{\alpha}(A) \subset \widetilde{F}^{\alpha}(A)$.

(v) $a(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, with $\beta \in [0, 1[$. Then a is nonnegative and

$$\frac{\lambda^{\alpha}}{\hat{a}(\lambda)} \cdot \int_0^{\infty} e^{-\lambda t}((1 * a)(t))^{\alpha} dt = \frac{\lambda^{\alpha+\beta-\alpha\beta-1}}{\beta^{\alpha} \cdot (\Gamma(\beta))^{\beta} \cdot \Gamma(\alpha\beta + 1)}$$

$$= \frac{\lambda^{(\alpha-1)\cdot(1-\beta)}}{\beta^{\alpha} \cdot (\Gamma(\beta))^{\beta} \cdot \Gamma(\alpha\beta + 1)},$$

which is bounded, for all $\lambda > 0$ due to $\beta \in [0, 1[$, and according to Proposition 4.7 (2), we can conclude that $\widetilde{F}^{\alpha}(A) \subset F^{\alpha}(A)$.

(vi) Let $a(t) = \mu + \nu t^{\beta}, 0 < \beta < 1, \mu > 0, \nu > 0$. In virtue of Proposition 4.7, we have $\widetilde{F}^{\alpha}(A) = F^{\alpha}(A)$ according to the Example 4.6 (3).

(vii) Let $a = 1 + 1 * k$. With $k(t) = \pm e^{-t}$, Proposition 4.7 yields $\widetilde{F}^{\alpha}(A) = F^{\alpha}(A)$ according to the Example 4.6 (4)-(5). In general, for $k \in L^1_{\text{loc}}(\mathbb{R}^+)$, ω^+-exponentially bounded, we have $\lambda\hat{a}(\lambda) = 1 + \hat{k}(\lambda)$, which is bounded for all $\lambda > 0$, according to the Riemann-Lebesgue lemma (see [1]). If in addition, a satisfies $(H_{a,1})$, Proposition 4.7 (1) asserts that $F^{\alpha}(A) \subset \widetilde{F}^{\alpha}(A)$. Now, if $k(t)$ is negative with $\hat{k}(0) \geq -1$, then we obtain a nonnegative kernel a satisfying $0 \leq (1 * a)(t) \leq t$. Hence, both $(H_{a,1})$ and (H_a^{α}) are satisfied (see Example 4.6 (4)). Thanks to Proposition 4.7, we have $\widetilde{F}^{\alpha}(A) = F^{\alpha}(A)$.

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