

## **ON THE REGULARITY FOR SOME CLASS OF $L^p$ -WELL-POSED LINEAR SYSTEMS**

**FOUAD MARAGH and HAMID BOUNIT**

Laboratory of Applied Mathematics and Applications  
(Functional Analysis Group)  
Faculty of Sciences  
Université Ibn Zohr d'Agadir  
BP. 8106, Cité Dakhla  
Agadir  
Maroc  
e-mail: [fouad.maragh@edu.uiz.ac.ma](mailto:fouad.maragh@edu.uiz.ac.ma)  
[h.bounit@uiz.ac.ma](mailto:h.bounit@uiz.ac.ma)

### **Abstract**

We consider the weak regularity problem of  $L^p$ -well-posed linear systems ( $1 \leq p < \infty$ ) in Banach state spaces when its associated unbounded controllers take values in the extrapolated Favard class. We prove that this type of  $L^p$ -well-posed linear systems are weakly regular and this regularity is output operators independent up to the well-posedness, provided the associated state spaces are non-reflexive and that the adjoint of its associated semigroups are strongly continuous on the dual of its state space. As applications, we consider a class of boundary control systems and a class of well-posed bilinear systems introduced in [2].

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### 1. Introduction

Many infinite-dimensional systems can be described by the equations

$$\begin{cases} \dot{x}(t) = Ax(t) + B_l u(t), \\ y(t) = Cx(t) + Du(t), \quad t \geq 0, \\ x(0) = x_0, \end{cases} \quad (1.1)$$

or

$$\begin{cases} \dot{x}(t) = Ax(t) + v(t)B_b x(t), \\ y(t) = Cx(t), \quad t \geq 0, \\ x(0) = x_0, \end{cases} \quad (1.2)$$

on a triple of Banach spaces, namely, the input space  $U$ , the state space  $X$ , and the output space  $Y$ . We have  $u(t) \in U$ ,  $v(t) \in \mathbb{C}$ ,  $x(t) \in X$ , and  $y(t) \in Y$ . The operators  $A$ ,  $B_l$ ,  $B_b$ , and  $C$  are usually unbounded. It is often convenient to use the “integral” representation of the system, which consists of the four operators from the initial state  $x_0$  and the input function  $u$  (or  $v$ ) to the final state  $x(t)$  and the output function  $y$  by

$$\begin{cases} x(t) = \mathbb{T}(t)x_0 + \Phi_t^l(u), \\ y = \Psi x_0 + \mathbb{L}(u), \end{cases} \quad (1.3)$$

or

$$\begin{cases} x(t) = \mathbb{T}(t)x_0 + \Phi_t^b(v, x), \\ y = \Psi x_0 + \mathbb{F}(v, x). \end{cases} \quad (1.4)$$

Here,  $\mathbb{T}(t)$  is the semigroup generated by  $A$  (which maps the initial state  $x_0$  into the final state  $x(t)$ ),  $\Phi_t^l$  (resp.,  $\Phi_t^b$ ) is the controllability map, and  $\Psi$  is the observability map. The operators  $\mathbb{L}$  and  $\mathbb{F}$  are the input-output

and the “input and state”- output maps, respectively (see the notation in Sections 2 and 4). The well-posedness assumption is that (1.3) (resp., (1.4)) behaves well in an  $L^p$  (resp.,  $L^{(p,q)}$ )-setting for some  $p \in [1, \infty]$  (resp.,  $p, q \in [1, \infty]$ ), i.e.,  $x(t) \in X$  and  $y \in L^p_{\text{loc}}(\mathbb{R}^+; Y)$  depend continuously on  $x_0 \in X$  and  $u \in L^p_{\text{loc}}(\mathbb{R}^+; U)$  (resp.,  $v \in L^p_{\text{loc}}(\mathbb{R}^+)$ ). If this is the case, we call the operators  $(\mathbb{T}, \Phi^l, \Psi, \mathbb{L})$  (resp.,  $(\mathbb{T}, \Phi^b, \Psi, \mathbb{F})$ ) a well-posed linear (resp., bilinear) system on  $(U, X, Y)$  (resp., on  $(X, Y)$ ). A complete reference about well-posed linear systems can be found in [20]. Well-posed bilinear systems were recently introduced in [2] and its set of known properties was studied in [2]. This class of systems has been introduced to allow us to study a class of bilinear systems with unbounded control and observation operators. There is an almost one-to-one correspondence between (1.1) and (1.3) (resp., (1.2) and (1.4)): most well-posed linear (resp., bilinear) systems can be represented as in (1.1) (resp., (1.2)). These linear and bilinear systems are called regular (weakly regular) (see Definitions 2.3 and 4.6). Regular (weakly regular) well-posed linear and bilinear systems are characterized in Weiss [26, 19]; and Bounit-Idrissi [2], respectively. However, there do exist irregular well-posed linear systems, and this may happen, for example, in the most commonly studied case where  $p = 2$  and  $U, X, Y$  are Hilbert spaces. This is not the case for a class of well-posed bilinear systems as we shall see in this paper. The authors in [18, 20] were interested in the regularity of the two extreme cases  $p = 1$  and  $p = \infty$ , which are especially useful in the proof of regularity of the solution to an optimal control problem for a stable parabolic systems [21] and they are well studied in [18-20]. In particular, it was been proved in [20] that all  $L^1$ -well-posed linear systems are weakly regular, and they are regular whenever the state space is reflexive or the output space is finite dimensional (see [20, Theorem 5.6.6 and Lemma 5.7.1 (ii)]). The weak regularity on general Banach state spaces [20, Theorem 5.6.6], was

obtained by the representation theorem on multipliers on  $L^1$  (see [3, 4]). A stronger version of [20, Theorem 5.6.6] was announced in [26, Remark 3.9] but without proof.

Why should the weak regularity problem be considered in more generality than for  $p = 1$ ? One hand, since we prove in Proposition 3.2 that the space of the  $L^1$ -admissible control operators are those which has its range contained in a very small superspace of  $X$  called the extrapolated Favard class, which constitutes an extension of [23, Theorem 4.8]. Secondly, the main type of problems we have in mind, are equations with some kind of boundary control systems (e.g., systems with delay), taking values in, e.g., Hilbert state spaces, frequently just in  $\mathbb{R}^n$ . Typical equations of this type are delay equations, or models of age-structured populations. Semigroup settings can be applied to such equations by blowing up the state space, working with  $C_0$ -semigroups in spaces like  $C([-r, 0]; \mathbb{R}^n)$  or  $L^1([-r, 0]; \mathbb{R}^n)$ . In many examples of this type, we encounter the situation that the control operator, though not bounded from the control space  $U$  into  $X$ , has its range contained in extrapolated Favard class. This is the case for all examples which can be handled by the settings using Greiner's estimate on the Dirichlet operator [22]. Also, the well-posed bilinear systems as introduced in [2] are among systems having this property.

The paper is organized as follows: The next section is devoted to necessary notations and background on well-posed and regular and weakly regular linear systems. Section 3 is devoted to proving our results. In Section 4, we present some applications to a class of boundary control problems and to the well-posed bilinear system introduced in [2].

## 2. Background on Well-Posed Linear System

In this section, we recall in a very sketchy way some definitions and results on well-posed linear systems introduced by Salamon and Weiss (see [16, 26, 27]).

We fix some notations which will be used throughout this paper. For a complex Banach space  $E$  and  $p \in [1, +\infty]$ , we denote by  $\mathbb{R}_+ = [0, +\infty[$ ,  $E^p = L^p(\mathbb{R}_+, E)$ ,  $E_{\text{loc}}^p = L_{\text{loc}}^p(\mathbb{R}_+, E)$ , and  $f \diamond_{\tau} g$  the  $\tau$ -concatenation of  $f$  and  $g$  for  $\tau \geq 0$  is defined by

$$(f \diamond_{\tau} g)(t) = \begin{cases} f(t); & 0 \leq t < \tau, \\ g(t - \tau); & t \geq \tau. \end{cases}$$

The norm of any other Banach space, say  $E$ , will be specified by  $\|\cdot\|_E$  and we denote the space of bounded linear operators from  $E$  to another Banach space  $F$  by  $\mathcal{L}(E; F)$ .

**Definition 2.1** ([26]). Let  $U, X, Y$  be Banach spaces and  $1 \leq p \leq +\infty$ , an  $L^p$ -well-posed linear system on  $(U, X, Y)$ , for state space  $X$ , input space  $U$  and output space  $Y$ , is the quadruple  $\Sigma_l = (\mathbb{T}, \Phi^l, \Psi, \mathbb{L})$ , where

- (i)  $\mathbb{T} = (\mathbb{T}(t))_{t \geq 0}$  is a  $C_0$ -semigroup of bounded linear operators on  $X$ .
- (ii)  $\Phi^l = (\Phi_t^l)_{t \geq 0}$  is a family of operators in  $\mathcal{L}(U^p, X)$  such that

$$\Phi_{\tau+t}^l(u \diamond_{\tau} v) = \mathbb{T}(t)\Phi_{\tau}^l u + \Phi_t^l v, \quad (2.1)$$

for any  $u, v \in U^p$  and any  $\tau, t \geq 0$ .

- (iii)  $\Psi = (\Psi_t)_{t \geq 0}$  is a family of operators in  $\mathcal{L}(X, Y^p)$  such that

$$\Psi_{\tau+t} x = \Psi_{\tau} x \diamond_{\tau} \Psi_t \mathbb{T}(\tau)x, \quad (2.2)$$

for any  $x \in X$  and any  $\tau, t \geq 0$ , and  $\Psi_0 = 0$ .

(iv)  $\mathbb{L} = (\mathbb{L}_t)_{t \geq 0}$  is a family of operators in  $\mathcal{L}(U^p, Y^p)$  such that

$$\mathbb{L}_{\tau+t}(u \underset{\tau}{\diamond} v) = \mathbb{L}_{\tau} u \underset{\tau}{\diamond} (\Psi_{\tau} \Phi_{\tau}^l u + \mathbb{L}_{\tau} v), \quad (2.3)$$

for any  $u, v \in U^p$  and any  $\tau, t \geq 0$ , and  $\mathbb{L}_0 = 0$ .

By a well-posed linear system, we mean a system which is  $L^p$ -well-posed for some  $p, 1 \leq p \leq \infty$ . The different components of a well-posed linear system  $(\mathbb{T}, \Phi^l, \Psi, \mathbb{L})$  are called as follows:  $\mathbb{T}$  is the  $C_0$ -semigroup,  $\Phi^l$  is the controllability map,  $\Psi$  is the observability map, and  $\mathbb{L}$  is the input-output map. The state  $x(t) \in X$  at time  $t \in \mathbb{R}^+$  and the output  $y \in Y_{\text{loc}}^p$  of with initial time zero, initial state  $x_0 \in X$  and input function  $u \in U_{\text{loc}}^p$  are given by (1.3). The maps  $\Psi$  and  $\mathbb{L}$  are studied in [24, 26, 27].

Now, if we regard  $\Psi_t$  (resp.,  $\mathbb{L}_t$ ) as elements of  $\mathcal{L}(X, Y_{\text{loc}}^p)$  (resp.,  $\mathcal{L}(U_{\text{loc}}^p, Y_{\text{loc}}^p)$ ), then these operators have a strong limit as  $t \rightarrow +\infty$  denoted by  $\Psi_{\infty}$  (resp.,  $\mathbb{L}_{\infty}$ ) called the extended output (resp., input/output) map of  $\Psi$  (resp., of  $\mathbb{L}$ ). Thus, the operator  $\Psi_{\tau}$  (resp.,  $\mathbb{L}_{\tau}$ ) are obtained by truncation as follows:  $\Psi_{\tau} = \Psi_{\infty} \underset{\tau}{\diamond} 0$  (resp.,  $\mathbb{L}_{\tau} = \mathbb{L}_{\infty} \underset{\tau}{\diamond} 0$ ) for all  $\tau \geq 0$ . Thus, the composition properties (2.2) and (2.3) have the following extension:

$$\Psi_{\infty} x = \Psi_{\infty} x \underset{\tau}{\diamond} \Psi_{\infty} \mathbb{T}(\tau) x,$$

for any  $x \in X$  and any  $\tau \geq 0$ .

$$\mathbb{L}_{\infty}(u \underset{\tau}{\diamond} v) = \mathbb{L}_{\infty} u \underset{\tau}{\diamond} (\Psi_{\infty} \Phi_{\tau}^l u + \mathbb{L}_{\infty} v),$$

for any  $u, v \in U_{\text{loc}}^p$  and any  $\tau \geq 0$ .

Before introducing the operators  $B_l$  and  $C$  in (1.1), we need two auxiliary spaces  $X_1$  and  $X_{-1}$ . The spaces  $X_1$  and  $X_{-1}$  are defined as follows:  $X_1 = (D(A), \|\cdot\|_1)$ , where  $A$  is the generator of  $C_0$ -semigroup  $\mathbb{T}$  and  $\|x\|_1 = \|(\lambda I - A)x\|_X$  (for some  $\lambda$  fixed in the resolvent set  $\rho(A)$  of  $A$ ), and  $X_{-1}$  is the completion of  $X$  with respect to the norm  $\|x\|_{-1} = \|(\lambda I - A)^{-1}x\|_X$ . These spaces are independent of the choice of  $\lambda$  and are related by the following continuous and dense injections:

$$X_1 \overset{d}{\hookrightarrow} X \overset{d}{\hookrightarrow} X_{-1}.$$

The semigroup  $\mathbb{T}$  can be restricted to a  $C_0$ -semigroup  $\mathbb{T}_1$  on  $X_1$ , with generator denoted by  $(A_1, D(A_1))$ , and can be extended to a  $C_0$ -semigroup  $\mathbb{T}_{-1}$  on  $X_{-1}$ , with generator denoted by  $(A_{-1}, D(A_{-1}))$ .

For  $\lambda \in \rho(A)$ , the resolvent operator  $R(\lambda, A) = (\lambda I - A)^{-1}$  and its extension  $R(\lambda, A_{-1}) = (\lambda I - A_{-1})^{-1}$  are isomorphisms from  $X$  to  $X_1$  and from  $X_{-1}$  to  $X$ , respectively. In particular,  $\rho(A_{-1}) = \rho(A)$ ,  $D(A_{-1}) = X$  and the norm of  $X$  is equivalent to the graph norm of  $A_{-1}$ . More information on the spaces  $X_1$  and  $X_{-1}$  can be found, for example, in [9].

**Proposition 2.2** ([23]). *Every  $L^p$ -well-posed linear system  $\Sigma_l = (\mathbb{T}, \Phi^l, \Psi, \mathbb{L})$  on  $(U, X, Y)$  has a unique control operator  $B_l \in \mathcal{L}(U, X_{-1})$ , determined by the fact that the state  $x(t)$  of  $\Sigma_l$  defined in (1.3) is given by the standard variation of constants formula for (1.1), i.e.,*

$$x(t) = \mathbb{T}(t)x_0 + \int_0^t \mathbb{T}_{-1}(t-s)B_l u(s)ds.$$

*This formula is valid for all  $x_0 \in X$  and all  $u \in U_{loc}^p$  if  $\Sigma_l$  is  $L^p$ -well-posed for some  $p < \infty$ .*

In the other words,  $x(\cdot)$  is the strong solution of the equation  $\dot{x}(t) = Ax(t) + B_l u(t)$  with initial state  $x_0$  at initial time zero, and input function  $u$ . The existence of a unique ( $L^p$ -admissible) control operator  $B_l$  for  $p < \infty$  associated to  $(\mathbb{T}, \Phi^l)$ , i.e., for all  $u \in U_{\text{loc}}^p$  and  $t \in \mathbb{R}_+$ , one has

$$\Phi_t^l u = \int_0^t \mathbb{T}_{-1}(t-s) B_l u(s) ds,$$

which is proved in [16, 23].

It also has a unique observation operator  $C \in \mathcal{L}(X_1, Y)$  (see [24, Theorem 3.3]) for all values of  $p \in [1; \infty]$ , which is determined by the fact that the state output map  $\Psi$  in (1.3) is given by (for all  $t \in \mathbb{R}^+$ )

$$(\Psi_\infty x_0)(t) = C \mathbb{T}(t) x_0, \quad \forall x_0 \in X_1. \quad (2.4)$$

Conversely, an operator  $C \in \mathcal{L}(X_1, Y)$  is called an admissible observation operator with exponent  $p$ , if the estimate

$$\int_0^t \|C \mathbb{T}(\tau) x\|^p d\tau \leq \gamma^p \|x\|^p,$$

holds for some (hence all)  $t > 0$  and all  $x \in D(A)$  with constant  $\gamma = \gamma(t) > 0$  (see [24]).

We denote by  $\mathcal{A}_p(U, X, \mathbb{T})$  (resp.,  $\mathcal{O}_p(X, Y, \mathbb{T})$ ) the space of  $L^p$ -admissible control (resp., observation) operators for  $A$ . Actually, by saying  $B \in \mathcal{A}_p(U, X, \mathbb{T})$ , we mean that  $B \in \mathcal{L}(U, X_{-1})$  and that

$$\int_0^t \mathbb{T}_{-1}(t-s) B u(s) ds \in X,$$

for any  $t > 0$  and any  $u \in U_{\text{loc}}^p$  (see [23]).



The control operator  $B_l$  is said to be bounded if the range of  $B_l$  lies in  $X$ , in which case  $B \in \mathcal{L}(U, X)$ . The observation operator  $C$  is said to be bounded if it is continuous with respect to the norm of  $X$ , i.e., if it can be extended to an operator in  $\mathcal{L}(X, Y)$ .

For  $p < \infty$ , it is possible to give a similar expression of  $\Psi_\infty x$  as in (2.4) even if  $x \notin X_1$ . More precisely, if we introduce the  $L$ -extension of  $C$ ,

$$C_L x_0 := \lim_{\tau \searrow 0} \frac{C}{\tau} \int_0^\tau \mathbb{T}(\sigma) x_0 d\sigma. \quad (2.5)$$

with domain  $(D(C_L, \|\cdot\|_{D(C_L)}))$  consisting of all  $x \in X$  for which the limit in (2.5) exists in  $Y$  with

$$\|x\|_{D(C_L)} := \|x\|_X + \sup_{\tau \in (0,1]} \left\| \frac{1}{\tau} C \int_0^\tau \mathbb{T}(\sigma) x d\sigma \right\|_Y$$

(see [24, 27] for details), then for almost every  $t \geq 0$ , the function  $\Psi_\infty$  from (1.3) is given by

$$(\Psi_\infty x_0)(t) = C_L \mathbb{T}(t) x_0.$$

The weak  $L$ -extension of  $C$ , denoted by  $C_{L_w}$ , as introduced in [19] is defined in the same way, but the strong limit is replaced by the weak limit.

Although the range of  $B_l$  is contained in a space larger than  $X$ , its admissibility guarantees only the existence of a solution  $x \in C(\mathbb{R}^+; X)$  for (1.1) (see Theorem 2.2) for a given function  $u \in U_{\text{loc}}^p$ . Since the  $L^p$ -admissible output operator  $C$  is not defined on all of  $X$ , additional hypotheses will be needed to make sense of the output  $y(t)$ , which is the strong (or weak) regularity in the sense of Salamon-Weiss as introduced in [26], or equivalently, the triple  $(A, B_l, C)$  generates a (weakly) regular  $L^p$ -well-posed linear system.

The following definition has been introduced in Weiss [23, 26].

**Definition 2.3.** A well-posed linear system  $\Sigma_l = (\mathbb{T}, \Phi^l, \Psi, \mathbb{L})$  is called regular if and only if for  $u \in U$ , the following limit exists in  $Y$ :

$$\lim_{\tau \searrow 0} \frac{1}{\tau} \int_0^\tau \mathbb{L}_\infty(1 \otimes u)(s) ds = Du, \quad (2.6)$$

where  $1 \otimes u$  is the constant function on  $\mathbb{R}_+$  equal to  $u$  and the limit is taken in  $Y$ .

$\Sigma_l$  is called weakly-regular, if the above limit exists only in the weak topology. In either case, the operator  $D$  defined by (2.6) associated to  $\mathbb{L}_\infty$  is called feed through operator and that  $D \in \mathcal{L}(U, Y)$  follows from the uniform boundedness principle.

This concept of regularity was studied in [26], and several equivalent characterizations of it are available, see [26]. We consider the following characterization to be the most basic one.

**Theorem 2.4.** *Let  $\Sigma_l = (\mathbb{T}, \Phi^l, \Psi, \mathbb{L})$  be a well-posed linear system on  $(U, X, Y)$  and  $B_l$  be the control operator associated to  $(\mathbb{T}, \Phi^l)$ , and  $C$  be the observation operator associated to  $(\mathbb{T}, \Psi)$ , then the following statements are equivalent:*

- (i)  $\Sigma_l$  is regular (resp., weakly regular).
- (ii) There exists  $s \in \rho(A)$  such that  $R(s, A_{-1})B_l U \subset D(C_L)$  (resp.,  $R(s, A_{-1})B_l U \subset D(C_{Lw})$ ).

If  $p < \infty$ , then regularity and weak regularity imply that the output function can be expressed as in (1.1) (see [26] and [19], respectively). More precisely, for almost every  $t \geq 0$ , the function  $y(t)$  from (1.1) is given by

$$y(t) = C_L x(t) + Du(t) \text{ (resp., } y(t) = C_{Lw} x(t) + Du(t)\text{)}.$$

### 3. Main Results

In this section, we will show how to obtain the weak regularity of an  $L^p$ -well-posed linear system if its controller takes its range in the extrapolated Favard class. As a direct consequence, we will prove in the next section that a class of well-posed bilinear system introduced in [2] is always weakly regular.

Let  $(X, \|\cdot\|_X)$  be a Banach space and let  $\mathbb{T} := (\mathbb{T}(t))_{t \geq 0}$  be a  $C_0$ -semigroup on  $X$ , with generator  $(A, D(A))$ . A Favard class of order  $\alpha$  with  $0 < \alpha \leq 1$  associated to  $A$  is

$$F^\alpha(A) = \left\{ x \in X \mid \sup_{0 < t \leq 1} \frac{1}{t^\alpha} \|e^{-\omega t} \mathbb{T}(t)x - x\|_X < +\infty \right\}, \quad (3.1)$$

for some  $\omega \geq \omega_0$ , where  $\omega_0$  is the growth bound of  $\mathbb{T}$ ,  $F^\alpha(A)$  becomes a Banach space when is equipped with each of the following equivalent norms:

$$\|x\|_{F^\alpha(A)} := \|x\|_X + \sup_{t > 0} \frac{1}{t^\alpha} \|e^{-\omega t} \mathbb{T}(t)x - x\|_X,$$

or

$$\|x\|_{F^\alpha(A)} := \|x\|_X + \sup_{s > \omega} \|s^\alpha AR(s, A)x\|_X.$$

Similarly, we define the Favard class  $F^\alpha(A_{-1})$  associated to  $A_{-1}$ . We notice that  $F^\alpha(A)$  is independent of the choice of  $\omega$ , invariant under  $\mathbb{T}$ , and we have these following embeddings:

$$X_1 \hookrightarrow F^\alpha(A) \hookrightarrow X,$$

and

$$X \hookrightarrow F^\alpha(A_{-1}) \hookrightarrow X_{-1}$$

(see [9, 5, 15]), and  $R(\lambda, A_{-1})$  is also an isomorphism from  $F^\alpha(A_{-1})$  to  $F^\alpha(A)$  for  $\lambda \in \rho(A)$ , then  $F^\alpha(A) = R(\lambda, A_{-1})F^\alpha(A_{-1})$ .

In the sequel, we use the notation  $F(A)$  in the place of  $F^1(A)$ . If  $X$  is reflexive, then  $F(A) = D(A)$  (so that the results of our paper are interesting only in non-reflexive state spaces).

We will need the following crucial lemma, which is due to ([6, Theorem 9]). A similar version can be found in ([14, Proposition 3.3]) and ([15, Lemma 4.3.9]).

**Lemma 3.1.** *Let  $(\mathbb{T}(t))_{t \geq 0}$  be a  $C_0$ -semigroup of bounded linear operators on  $X$ , with generator  $(A, D(A))$ . For all  $f \in L^1_{\text{loc}}(0, +\infty; F(A))$  and  $t \geq 0$ , we set*

$$(\mathbb{T} * f)(t) := \int_0^t \mathbb{T}(t-s)f(s)ds.$$

Then one has

(i)  $(\mathbb{T} * f)(t) \in D(A)$ .

(ii) For  $\omega > \omega_0(\mathbb{T})$ , there is a constant  $K$ , independent of  $(t, f)$ , such that

$$\|(\mathbb{T} * f)(t)\|_{D(A)} \leq Ke^{(\omega)t} \|f\|_{L^1(0,t; F(A))},$$

where  $\|\cdot\|_{D(A)}$  denotes the graph norm of  $A$ , i.e.,  $\|x\|_{D(A)} = \|x\| + \|Ax\|$ ,  $x \in D(A)$ .

The following proposition will be needed throughout the paper, which is an extension of [23, Theorem 4.8].

**Proposition 3.2.** *Let  $U, X$  be Banach spaces, and  $A$  generate a  $C_0$ -semigroup  $\mathbb{T}$  on  $X$ , and  $F(A_{-1})$  Favard space defined in (3.1) Then we have*

$$\mathcal{A}_1(U, X, \mathbb{T}) = \mathcal{L}(U, F(A_{-1})).$$

**Proof.** Without loss of generality, we assume that,  $A$  is invertible (indeed, otherwise we can replace  $A$  with  $A - \lambda$  for a certain  $\lambda \in \rho(A)$ ). Let  $B \in \mathcal{L}(U, F(A_{-1}))$  and  $u \in U_{\text{loc}}^1$ , then  $Bu(\cdot) \in L_{\text{loc}}^1(\mathbb{R}^+, F(A_{-1}))$  and by Lemma 3.1 (i), we have

$$\int_0^t \mathbb{T}_{-1}(t-s)Bu(s)ds \in D(A_{-1}) = X,$$

for all  $t \geq 0$ , which implies that  $B \in \mathcal{A}_1(U, X, \mathbb{T})$  and thanks to Lemma 3.1 (ii), there is  $K := K_t > 0$  such that

$$\left\| \int_0^t \mathbb{T}_{-1}(t-s)Bu(s)ds \right\| \leq K \|B\|_{\mathcal{L}(U, F(A_{-1}))} \|u\|_{L^1(0,t;U)},$$

for all  $u \in U_{\text{loc}}^1$ .

On the other hand, let  $B \in \mathcal{A}_1(U, X, \mathbb{T})$ . It is enough thanks to the closed graph theorem to show that  $\text{Range}(B) \subset F(A_{-1})$  since  $F(A_{-1}) \hookrightarrow X_{-1}$ . Recall that in [20, Proposition 4.2.9], it has been proved that for  $p \in [1, +\infty]$ , the following  $p$ -resolvent condition:

$$\|(sI - A_{-1})^{-1}B\|_{\mathcal{L}(U, X)} \leq \frac{K}{\text{Re}(s)^{1/p}}, \quad s \in \mathbb{C}_\omega \text{ with } (\omega > \omega_0(A)), \quad (3.2)$$

is necessary for the  $L^p$ -admissibility of the operator  $B$ .

In our case (i.e.,  $p = 1$ ), (3.2) means that for all  $u \in U$

$$\|s(sI - A_{-1})^{-1}Bu\|_X \leq K \|u\| \text{ for all } s > 0.$$

Furthermore, it is easy to observe that

$$\|s(sI - A_{-1})^{-1}Bu\|_X = \|sA_{-1}(sI - A_{-1})^{-1}Bu\|_{X_{-1}}. \quad (3.3)$$

Then for all  $B \in \mathcal{A}_1(U, X, \mathbb{T})$  and  $u \in U$ , we have

$$\sup_{s>0} \|sA_{-1}(sI - A_{-1})^{-1}Bu\|_{X_{-1}} < \infty,$$

accordingly to (3.2) and (3.3). The fact that  $\sup_{s>0} \|sA_{-1}R(s, A_{-1})x\|_{X_{-1}}$  is an equivalent norm on  $F(A_{-1})$ , we deduce that  $Bu \in F(A_{-1})$ , which end the proof.  $\square$

**Proposition 3.3.** *Let  $X, Y$  be Banach spaces and  $A$  generate a  $C_0$ -semigroup  $\mathbb{T}$  on  $X$  and  $\mathbb{C} \in \mathcal{L}(X_1, Y)$  be an observation operator. Then we have*

$$D(A_{Lw}) \subset D(C_{Lw}).$$

**Proof.** Without loss of generality, we assume that,  $A$  is invertible. Consider the operator  $C_\tau \in \mathcal{L}(X, Y)$  defined by

$$C_\tau x := \frac{C}{\tau} \int_0^\tau \mathbb{T}(t)x dt,$$

for all  $x \in X$  and  $0 < \tau \leq 1$ .

Let  $x_0 \in D(A_{Lw})$  and  $x^* \in X^*$ . Since  $CA^{-1} \in \mathcal{L}(X, Y)$ , for all  $y^* \in Y^*$ , we have

$$\begin{aligned} \lim_{\tau \rightarrow 0} \langle C_\tau x_0, y^* \rangle &= \lim_{\tau \rightarrow 0} \langle CA^{-1}A_\tau x_0, y^* \rangle \\ &= \lim_{\tau \rightarrow 0} \langle A_\tau x_0, x^* \rangle, \text{ with } x^* := (CA^{-1})^* y^* \in X^*, \end{aligned}$$

which exists.

Finally, this implies that  $x_0 \in D(C_{Lw})$  and we have

$$C_{Lw}x_0 = CA^{-1}A_{Lw}x_0,$$

which end the proof.  $\square$

It is clear from definition of  $A_L$  that  $D(A_L)$  and  $D(A)$  coincide. Thus, if  $X$  is reflexive, then we obtain  $F(A) = D(A_L) = D(A)$  (defined in (3.1)), which are contained in  $D(A_{Lw})$  with  $A_{Lw}$  is the weak  $L$ -extension of  $A$ . We will be interested in question for what semigroups (on non-reflexive) Banach spaces, the Favard class  $F(A)$  is contained in  $D(A_{Lw})$ .

The following lemma will be very useful to state our principal result:

**Lemma 3.4.** *Let  $X$  be a non reflexive Banach space, and  $A$  generate a  $C_0$ -semigroup  $(\mathbb{T}(t))_{t \geq 0}$  on  $X$  verifying the following condition:*

**(H)** : “( $\mathbb{T}^*(t)$ ) <sub>$t \geq 0$</sub>  is a  $C_0$ -semigroup on  $X^*$ ”. Then we have

$$F(A) \subset D(A_{Lw}).$$

**Proof.** Without loss of generality, we suppose  $(\mathbb{T}(t))_{t \geq 0}$  is exponentially stable (i.e.,  $F(A)$  is independent to the shifting of the semigroup). Consider the operator  $A_\tau \in \mathcal{L}(X)$  defined by

$$A_\tau x := \frac{A}{\tau} \int_0^\tau \mathbb{T}(t)x dt,$$

for  $0 < \tau \leq 1$  and  $x \in X$ .

Let  $x_0 \in X$ . Since there is a bounded sequence  $(x_n)_{n \in \mathbb{N}} \subset D(A)$  such that  $\lim_{n \rightarrow \infty} x_n = x_0$  in  $X$ , we have  $\lim_{n \rightarrow \infty} A_\tau x_n = A_\tau x_0$ . Therefore, for  $x^* \in D(A^*)$ , we have

$$\begin{aligned} \lim_{\tau \rightarrow 0} \langle A_\tau x_0, x^* \rangle &= \lim_{\tau \rightarrow 0} \left\langle \lim_{n \rightarrow \infty} A_\tau x_n, x^* \right\rangle \\ &= \lim_{\tau \rightarrow 0} \lim_{n \rightarrow \infty} \langle A_\tau x_n, x^* \rangle \\ &\stackrel{(*)}{=} \lim_{n \rightarrow \infty} \lim_{\tau \rightarrow 0} \langle A_\tau x_n, x^* \rangle \end{aligned}$$

$$\begin{aligned}
x_n &\in \underline{D}(A) \lim_{n \rightarrow \infty} \langle Ax_n, x^* \rangle \\
&= \lim_{n \rightarrow \infty} \langle x_n, A^* x^* \rangle \\
&= \langle x_0, A^* x^* \rangle.
\end{aligned}$$

Now, we define the following linear functionals  $(L_\tau)_{\tau > 0}$  defined on  $X^*$  by

$$L_\tau x^* := \langle A_\tau x_0, x^* \rangle, \text{ with } x_0 \in F(A).$$

One has

$$\begin{aligned}
|L_\tau x^*| &= |\langle A_\tau x_0, x^* \rangle| \\
&= \left| \left\langle \frac{\mathbb{T}(\tau)x_0 - x_0}{\tau}, x^* \right\rangle \right| \\
&\leq \|x_0\|_{F(A)} \|x^*\|,
\end{aligned}$$

which implies that  $L_\tau \in \mathcal{L}(X^*; \mathbb{C})$  is uniformly bounded. Since the condition **(H)** is equivalent to  $D(A^*)$  is dense in  $X^*$  and  $\lim_{\tau \rightarrow 0} L_\tau x^*$  exists for all  $x^* \in D(A^*)$ , the uniform boundedness theorem implies that  $\lim_{\tau \rightarrow 0} L_\tau x^*$  exists for all  $x^*$  in  $X^*$ .

The proof is completed by justifying the above interchange of limit operations (\*). Indeed, for  $\tau > 0$ , we have

$$\begin{aligned}
\sup_{n \in \mathbb{N}} |\langle A_\tau x_n - Ax_n, x^* \rangle| &= \sup_{n \in \mathbb{N}} \left| \left\langle \frac{\mathbb{T}(\tau)x_n - x_n}{\tau} - Ax_n, x^* \right\rangle \right| \\
&= \sup_{n \in \mathbb{N}} \left| \left\langle x_n, \frac{\mathbb{T}^*(\tau)x^* - x^*}{\tau} - A^* x^* \right\rangle \right|
\end{aligned}$$



$$\begin{aligned} &\leq \sup_{n \in \mathbb{N}} \|x_n\| \left\| \frac{\mathbb{T}^*(\tau)x^* - x^*}{\tau} - A^*x^* \right\| \\ &\leq M \left\| \frac{\mathbb{T}^*(\tau)x^* - x^*}{\tau} - A^*x^* \right\|, \end{aligned}$$

for some  $M > 0$ .

Now the fact that  $x^* \in D(A^*)$  and that  $(\mathbb{T}^*(t))_{t \geq 0}$  is a  $C_0$ -semigroup on  $X^*$ , we obtain

$$\lim_{\tau \rightarrow 0} \sup_{n \in \mathbb{N}} |\langle A_\tau x_n - Ax_n, x^* \rangle| = 0.$$

□

The following result is a direct combination of Proposition 3.3 and Lemma 3.4.

**Proposition 3.5.** *Let  $X, Y$  be Banach spaces and  $A$  generate a  $C_0$ -semigroup  $\mathbb{T}$  on  $X$  satisfying **(H)** and  $C \in \mathcal{L}(X_1, Y)$  be an observation operator. Then we have*

$$F(A) \subset D(C_{Lw}).$$

Now we state the main result of this paper.

**Theorem 3.6.** *Every  $L^p$ -well-posed linear system  $\Sigma_l$  on  $(U, X, Y)$  ( $1 \leq p < \infty$ ) with condition **(H)** and its associated control operator  $B_l \in \mathcal{L}(U, F(A_{-1}))$  is weakly regular.*

**Proof.** Since  $B_l \in \mathcal{L}(U, F(A_{-1}))$  and  $R(s, A_{-1}) \in \mathcal{L}(F(A_{-1}), F(A))$  for all  $s \in \rho(A)$ , we obtain  $R(s, A_{-1})B_l U \subset F(A)$ . Appealing to Proposition 3.5 and Theorem 2.4, we deduce that  $\Sigma_l$  is weakly regular.

□

**Remark 3.7.** For the  $L^1$ -well-posed linear systems, the situation is different since their weak regularity was been proved early in [20, Theorem 5.6.6] by an entirely different method and without the condition **(H)**. Thanks to Proposition 3.2, for every  $L^1$ -well-posed linear systems, we have  $B_l \in \mathcal{L}(U, F(A_{-1}))$ , so if we can dispense of the condition **(H)**, then Theorem 3.6 may constitutes an extension and a new proof of [20, Theorem 5.6.6].

## 4. Applications

### 4.1. Application to boundary control systems

Consider

- (1) three Banach spaces  $X$ ,  $\partial X$ , and  $U$ , called the state, boundary, and control space, respectively;
- (2) a closed, densely defined system operator  $A_m : D(A_m) \subseteq X \rightarrow X$ ;
- (3) a boundary operator  $Q \in \mathcal{L}(D(A_m), \partial X)$ ;
- (4) a boundary control operator  $B_\partial \in \mathcal{L}(U, \partial X)$ .

For these operators and spaces and a control function  $u \in L^p_{\text{loc}}(\mathbb{R}^+, U)$  and  $x_0 \in X$ , we consider the abstract Cauchy problem with boundary control system

$$\begin{cases} \dot{x}(t) = A_m x(t), & t \geq 0, \\ Qx(t) = B_\partial u(t), & t \geq 0, \\ x(0) = x_0, \end{cases} \quad (4.1)$$

coupled with the observation equation

$$y(t) = Kx(t), \quad (4.2)$$

where  $K \in \mathcal{L}(D(A_m), Y)$  and  $Y$  is an other Banach (output) space.

In order to investigate (4.1), we make the following assumption ensuring in particular that the uncontrolled abstract Cauchy problem, i.e., (4.1) with  $B_{\partial} = 0$ , is well-posed.

In the spirit of Greiner's approach [22] and based on some results in [10], we assume

(H1) There is a norm  $\|\cdot\|_m$  on  $D(A_m)$  such that  $X_m := (D(A_m), \|\cdot\|_m)$  is a complete space, which is continuously embedded into  $X$  and  $A_m \in \mathcal{L}(X_m, X)$ .

(H2) The operator  $A := A_m|_{\ker Q}$  defined as the restriction of  $A_m$  to  $\ker Q$  generates a strongly continuous semigroup  $(\mathbb{T}(t))_{t \geq 0}$  on a non-reflexive  $X$ .

(H3) The boundary operator  $Q : D(A_m) \rightarrow \partial X$  is onto.

Under these assumptions, the following properties have been shown by [22, Lemmas 1.2 and 1.3]:

(1) For each  $\lambda \in \rho(A)$ ,  $D(A_m) = D(A) \oplus \ker(\lambda - A_m)$ ;

(2)  $Q|_{\ker(\lambda - A_m)}$  is invertible and the operator  $Q_\lambda := (Q|_{\ker(\lambda - A_m)})^{-1} : \partial X \rightarrow \ker(\lambda - A_m) \subseteq X$  is bounded;

(3)  $P_\lambda := Q_\lambda Q \in \mathcal{L}(D(A_m))$  is a projection onto  $\ker(\lambda - A_m)$  along  $D(A)$ ;

(4)  $R(\mu, A)Q_\lambda = \frac{1}{\lambda - \mu}(Q_\mu - Q_\lambda) = R(\lambda, A)Q_\mu$  for all  $\lambda, \mu \in \rho(A)$ ,  $\lambda \neq \mu$ .

The operator  $Q_\lambda$  is called the Dirichlet operator and we have

$$(A_m - A_{-1}) = (\lambda - A_{-1})Q_\lambda Q \text{ on } D(A_m) \text{ for } \lambda \in \rho(A). \quad (4.3)$$

By defining the operators  $B_\lambda := Q_\lambda B_\partial$ , we obtain an explicit representation of the boundary control system (4.1) as follows:

$$\begin{cases} \dot{x}(t) = A(x(t) - B_\lambda u(t)) + \lambda B_\lambda u(t), & t \geq 0, \\ x(0) = x_0. \end{cases} \quad (4.4)$$

Using (4.3), in the bigger space  $X_{-1}$ , the last system can be rewritten in the form

$$\begin{cases} \dot{x}(t) = A_{-1}x(t) + (\lambda - A_{-1})B_\lambda u(t), & t \geq 0, \\ x(0) = x_0, \end{cases} \quad (4.5)$$

when the control operator  $(\lambda - A_{-1})B_\lambda$  does not depend on the choice of  $\lambda \in \rho(A)$  accordingly to (4).

By a mild solution of (4.5), we mean a continuous function  $x(t)$  on  $[0, T]$  satisfying (variation of parameters formula)

$$\begin{aligned} x(t) &:= \mathbb{T}(t)x_0 + \int_0^t \mathbb{T}_{-1}(t-s)(\lambda - A_{-1})B_\lambda u(s) ds \\ &= \mathbb{T}(t)x_0 + (\lambda - A_{-1}) \int_0^t \mathbb{T}(t-s)B_\lambda u(s) ds. \end{aligned} \quad (4.6)$$

Remark that solution of (4.5) is always well-defined in  $X_{-1}$ . In [10, Proposition 2.7], it was proved that a classical solution of (4.4) which means that  $x(\cdot) = x(\cdot, x_0, u)$  is continuously differentiable in  $X$  and  $x(t) \in D(A_m)$  satisfying (4.1) is also a mild solution.

In order to obtain from (4.5) solutions having values in  $X$ , we impose the following strong assumption concerning the behaviour of  $Q_\lambda$  for large  $\lambda$  (see [6]):

(H4) There exist  $\lambda_0, c$  such that

$$\|Q_\lambda\|_{\mathcal{L}(\partial X, X)} \leq \frac{c}{\lambda - \lambda_0},$$

for all  $\lambda > \lambda_0$ .

On can see that the boundary system (4.1)-(4.2) is equivalent to the following distributed parameter system:

$$\begin{cases} \dot{x}(t) = A_{-1}x(t) + (\lambda - A_{-1})B_\lambda u(t), & t \geq 0, \quad x(0) = x_0, \\ y(t) = Cx(t), \end{cases} \quad (4.7)$$

with  $C := Ki$ , where  $i$  is the canonical injection from  $D(A)$  to  $D(A_m)$ .

We are now ready to state the following result:

**Proposition 4.1.** *Assume that (H1)-(H4) are satisfied. If  $(\mathbb{T}(t))_{t \geq 0}$  satisfies condition (H) and that the observation operator  $C \in \mathcal{O}_p(X, Y, \mathbb{T})$  for some  $1 \leq p < \infty$ , then the triple  $(A, (\lambda - A_{-1})B_\lambda, C)$  generates a weakly regular  $L^p$ -well-posed linear system on  $X, U, Y$ .*

**Proof.** It is well-known that condition (H4) is equivalent to  $X_m$  is continuously embedded into  $F(A)$  (see [7]), which in turns equivalent to  $\text{Range}[(\lambda - A_{-1})B_\lambda] \subseteq F(A_{-1})$  implying that the control operator  $(\lambda - A_{-1})B_\lambda \in \mathcal{A}_q(U, X, \mathbb{T})$  for all  $1 \leq q < \infty$  and in particular for  $q = p$ . The fact that  $C$  is  $L^p$ -admissible for some  $1 \leq p < \infty$ , it remains only to construct an input-output operator associated to the triple  $(A, (\lambda - A_{-1})B_\lambda, C)$ , which becomes weakly regular accordingly to Theorem 3.6. Thanks to the [8, Lemma 3.4], for all  $u \in W_{\text{loc}}^{1,p}(\mathbb{R}^+, U)$ , we have  $\int_0^t \mathbb{T}_{-1}(t-s)(\lambda - A_{-1})B_\lambda u(s)ds \in F(A)$ . Appealing to Proposition 3.5, the following linear operator:

$$\mathbb{L}_\infty : W_{\text{loc}}^{1,p}(\mathbb{R}^+, U) \rightarrow Y_{\text{loc}}^p$$

$$u \mapsto \mathbb{L}_\infty(u)(\cdot) := C_{Lw} \int_0^\cdot \mathbb{T}_{-1}(\cdot - s)(\lambda - A_{-1})B_\lambda u(s)ds, \quad (4.8)$$

is well defined.

Now, using Hölder's inequality and Fubini's theorem, one can obtain

$$\|\mathbb{L}_\infty(u)\|_{Y^p[0,T]} \leq \kappa(T) \|u\|_{U^p[0,T]} \text{ for all } T > 0.$$

Hence, by the density of  $W_{\text{loc}}^{1,p}(\mathbb{R}^+, U)$  in  $L_{\text{loc}}^p(\mathbb{R}^+, U)$ , by continuity  $\mathbb{L}_\infty$  possess an extension to linear, bounded operator from  $L_{\text{loc}}^p(\mathbb{R}^+, U)$  to  $Y_{\text{loc}}^p$ . We denote this extension again by  $\mathbb{L}_\infty$ . Now to verify (2.3), we fix  $\tau$  and define the canonical space  $V_\tau^U := \{(f, g) \in W_{\text{loc}}^{1,p}(\mathbb{R}^+, U) \times W_{\text{loc}}^{1,p}(\mathbb{R}^+, U), f(\tau) = g(0) = 0\}$ . This space is dense in  $L_{\text{loc}}^p(\mathbb{R}^+, U) \times L_{\text{loc}}^p(\mathbb{R}^+, Y)$  and  $f \diamond_\tau g \in W_{\text{loc}}^{1,p}(\mathbb{R}^+, U)$  for all  $(f, g) \in V_\tau^U$ . By means of (4.8), the property of composition (2.3) holds for  $(u, v) \in V_\tau^U$ . Since the concatenation is continuous with respect to both terms, it follows by approximation that (2.3) holds for all  $u, v \in L_{\text{loc}}^p(\mathbb{R}^+, U)$ . By the comment after Theorem 2.4, the expression (4.8) holds for all  $u$  in  $L_{\text{loc}}^p(\mathbb{R}^+, U)$  and a.e.,  $t > 0$ .  $\square$

#### 4.2. Application to well-posed bilinear systems

In view of the above results, in this section, we will show that every well-posed bilinear system as introduced in [2] with the condition **(H)** is always weakly regular. First of all, let us recall the basic definitions of well-posed bilinear systems and formulate some resulting properties, which can be found in [2]. Our definitions agree with those given in [2] if the reels  $p$  and  $q$  are not conjugates.

From now on, the reels  $p$  and  $q$  are in  $[1, +\infty]$  and satisfy  $0 < p^{-1} + q^{-1} \leq 1$  remain fixed. When  $p$  and  $q$  are conjugate (i.e.,  $p^{-1} + q^{-1} = 1$ ), we will write the conjugate  $q$  of  $p$  as  $\bar{p}$ .

**Definition 4.2.** Let  $X$  and  $Y$  be Banach spaces, an  $L^{(p,q)}$ -well-posed bilinear system on  $(X, Y)$ , for state space  $X$ , control space  $\mathbb{C}$  and output space  $Y$ , is the quadruple  $\Sigma_b = (\mathbb{T}, \Phi^b, \Psi, \mathbb{F})$ , where

(i)  $\mathbb{T} = (\mathbb{T}(t))_{t \geq 0}$  and  $\Psi := (\Psi_t)_{t \geq 0}$  satisfy (i) and (iii) of Definition 2.1, respectively.

(ii)  $\Phi^b = (\Phi_t^b)_{t \geq 0}$  is a family of (bilinear) operators in  $\mathcal{L}(\mathbb{C}^q \times X^p, X)$  such that

$$\Phi_{\tau+t}^b \left( (u \underset{\tau}{\diamond} v), (x \underset{\tau}{\diamond} y) \right) = \mathbb{T}(t) \Phi_{\tau}^b(u, x) + \Phi_t^b(v, y), \quad (4.9)$$

for any  $u, v \in \mathbb{C}^q$ ,  $x, y \in X^p$ , and  $\tau, t \geq 0$ .

(iii)  $\mathbb{F} = (\mathbb{F}_t)_{t \geq 0}$  is a family of (bilinear) operators in  $\mathcal{L}(\mathbb{C}^q \times X^p, Y^p)$  such that

$$\mathbb{F}_{\tau+t} \left( (u \underset{\tau}{\diamond} v), (x \underset{\tau}{\diamond} y) \right) = \mathbb{F}_{\tau}(u, x) \underset{\tau}{\diamond} (\Psi_t \Phi_{\tau}^b(u, x) + \mathbb{F}_t(v, y)), \quad (4.10)$$

for any  $u, v \in \mathbb{C}^q$ ,  $x, y \in X^p$ , and  $\tau, t \geq 0$ , and  $\mathbb{F}_0 = 0$ .

By a well-posed bilinear system, we mean a system which is  $L^{(p,q)}$ -well-posed for some  $1 \leq p, q \leq \infty$  satisfying the above condition. Similarly as in Section 2,  $\mathbb{F}_t$  can be extended in  $\mathcal{L}(\mathbb{C}_{\text{loc}}^q \times U_{\text{loc}}^p, Y_{\text{loc}}^p)$  to  $\mathbb{F}_{\infty}$  called the extended input/output map of  $\mathbb{F}$ . Thus, the operator  $\mathbb{F}_{\tau}$  are obtained as follows:  $\mathbb{F}_{\tau} = \mathbb{F}_{\infty} \underset{\tau}{\diamond} 0$  for all  $\tau \geq 0$  (see [2] for more details).

The composition property (4.10) has the following extension:

$$\mathbb{F}_{\infty} \left( (u \underset{\tau}{\diamond} v), (x \underset{\tau}{\diamond} y) \right) = \mathbb{F}_{\infty}(u, x) \underset{\tau}{\diamond} (\Psi_{\infty} \Phi_{\tau}^b(u, x) + \mathbb{F}_{\infty}(v, y)),$$

for any  $u, v \in \mathbb{C}^q$ ,  $x, y \in X^p$ , and any  $\tau \geq 0$ .

**Proposition 4.3.** *Every  $L^{(p,q)}$ -well-posed bilinear system  $\Sigma_b = (\mathbb{T}, \Phi^b, \Psi, \mathbb{F})$  for some  $1 < p, q < \infty$  has a unique control operator  $B_b \in \mathcal{L}(X; X_{-1})$ , determined by the fact that the state  $x(t)$  of  $\Sigma_b$  defined in (1.4) exists and it is given by the following functional equation for (1.2), i.e.,*

$$x(t) = \mathbb{T}(t)x_0 + \int_0^t \mathbb{T}_{-1}(t-s)v(s)B_b x(s)ds.$$

*This formula is valid for all  $x_0 \in X$  and all  $v \in \mathbb{C}_{\text{loc}}^p$  if  $\Sigma_b$  is  $L^{(p,q)}$ -well-posed for some  $1 < p < \infty$ .*

In the other words,  $x(\cdot)$  is the strong solution of the equation  $\dot{x}(t) = Ax(t) + v(t)B_b x(t)$  with initial time zero, initial state  $x_0$ , and scalar input function  $v$ . The existence of (an admissible) control operator  $B_b$  for  $1 < p < \infty$  associated to  $(\mathbb{T}, \Phi^b)$  with  $q = \bar{p}$ , i.e., for all  $v \in \mathbb{C}_{\text{loc}}^p$  and  $x \in X_{\text{loc}}^q$  and  $t \in \mathbb{R}_+$ , one has

$$\Phi_t^b(v, x) = \int_0^t \mathbb{T}_{-1}(t-s)v(s)B_b x(s)ds,$$

is proved in [11], and for all  $1 < p, q < \infty$  in Maragh et al. [12].

In the sequel, we denote by  $\mathcal{A}_{p,q}(X, \mathbb{T})$  the space of all  $(p, q)$ -admissible control operators for  $A$  (or  $\mathbb{T}$ ). Actually, by saying  $\mathbb{B} \in \mathcal{A}_{p,q}(X, \mathbb{T})$ , we means that  $B \in \mathcal{L}(X, X_{-1})$  and that

$$\int_0^t \mathbb{T}_{-1}(t-s)v(s)Bx(s)ds \in X,$$

for any  $t > 0$  and any  $(v, x) \in \mathbb{C}_{\text{loc}}^p \times X_{\text{loc}}^q$  (see [11, 1]).



We can now state the analogue of Proposition 3.2, which links  $\mathcal{A}_{p,q}(X, \mathbb{T})$  to extrapolated Favard class and covers a result in [11]. More information concerning  $\mathcal{A}_{p,\bar{p}}(X, \mathbb{T})$  can be found in [11].

**Proposition 4.4** (Proposition 22, [12]). *Let  $X$  be a Banach spaces and  $A$  generate a  $C_0$ -semigroup  $\mathbb{T}$  on  $X$ . Then with  $\alpha := p^{-1} + q^{-1}$  in the range  $(0, 1]$ , we have*

$$\mathcal{L}(X, F(A_{-1})) \hookrightarrow \mathcal{A}_{p,q}(X, \mathbb{T}) \hookrightarrow \mathcal{L}(X, F^\alpha(A_{-1})).$$

**Remark 4.5.** In the case  $q = \bar{p}$ , we rediscover  $\mathcal{A}_{p,\bar{p}}(X, \mathbb{T}) = \mathcal{L}(X, F(A_{-1}))$ , which is not depending on  $p$ . We note that this result was previously proved in [11].

**Definition 4.6.** A well-posed bilinear system  $\Sigma_b := (\mathbb{T}, \Phi^b, \Psi, \mathbb{F})$  is called regular if and only if for  $x \in X$ , the following limit exists in  $Y$ :

$$\lim_{\tau \searrow 0} \frac{1}{\tau} \int_0^\tau \mathbb{F}_\infty(\mathbf{1}, \mathbf{1} \otimes x)(s) ds = 0,$$

where  $\mathbf{1} \otimes x$  (resp.,  $\mathbf{1}$ ) is the constant function on  $\mathbb{R}_+$  equal to  $x$  (resp., 1).

$\Sigma_b$  is called weakly-regular, if the above limit exists only in the weak topology.

**Remark 4.7.** To an  $L^{(p,\bar{p})}$  well-posed bilinear system  $\Sigma_b$  on  $(X, Y)$ , we associate the well-posed linear system  $\Sigma_l := (\mathbb{T}, \Phi^b(\mathbf{1}, \cdot), \Psi, \mathbb{L}(\mathbf{1}, \cdot))$  on  $(X, X, Y)$  (see [2]), and say that the regularity is equivalent to regularity of  $\Sigma_l$  in  $L^{\bar{p}}$  (see [2, Theorem 4.8]).

In light of Theorem 2.4, we have the following result [2]:

**Theorem 4.8.** *Let  $\Sigma_b = (\mathbb{T}, \Phi^b, \Psi, \mathbb{F})$  be as in Definition 4.2 and  $B_b$  be the control operator associated to  $(\mathbb{T}, \Phi^b)$ , and  $C$  be the observation operator associated to  $(\mathbb{T}, \Psi)$ , then the following statements are equivalent:*

(i)  $\Sigma_L$  is weakly regular (resp., regular).

(ii) There exists  $s \in \rho(A)$  such that  $R(s, A_{-1})B_l U \subset D(C_{Lw})$  (resp.,  $R(s, A_{-1})B_l U \subset D(C_L)$ ).

It was been proved in [2] that for  $1 < p < \infty$ , the regularity implies that the output function can be expressed as in (1.2).

**Theorem 4.9** (Theorem 4.11, [2]). Let  $\Sigma_b = (\mathbb{T}, \Phi^b, \Psi, \mathbb{F})$  be a regular (resp., a weakly regular) bilinear system in  $L^{(p, \bar{p})}$  on  $(X, Y)$ . Let  $B_b$  be the control operator associated to  $(\mathbb{T}, \Phi^b)$  and  $C$  be the observation operator associated to  $(\mathbb{T}, \Psi)$ . Then, for any  $x_0 \in X$ ,  $v \in \mathbb{C}_{\text{loc}}^p$ , there is a unique solution  $(x, y) \in \mathcal{C}(\mathbb{R}_+, X) \times Y_{\text{loc}}^{\bar{p}}$  of the functional equations

$$\begin{cases} x(t) = \mathbb{T}(t)x_0 + \Phi_t^b(v, x), \\ y = \Psi_\infty x_0 + \mathbb{F}_\infty(v, x), \end{cases} \quad (4.11)$$

such that  $x(t) \in D(C_L)$  (resp.,  $x(t) \in C_{Lw}x(t)$ ) for a.e.,  $t \geq 0$  and

$$\begin{cases} \dot{x}(t) = A_{-1}x(t) + v(t)B_b x(t) & (\text{in } X_{-1}), \\ y(t) = C_L x(t) \text{ (resp., } y(t) = C_{Lw}x(t)) & \text{a.e., } t \geq 0, \\ x(0) = x_0. \end{cases} \quad (4.12)$$

Combining Theorems 3.6 and 4.8, we receive the following regularity result:

**Theorem 4.10.** Every  $L^{(p, \bar{p})}$ -well-posed bilinear system with  $1 < p < \infty$  with the condition **(H)** is weakly regular.

**Proof.** Let  $\Sigma_b = (\mathbb{T}, \Phi^b, \Psi, \mathbb{F})$  be an  $L^{(p, \bar{p})}$ -well-posed bilinear system on  $(X, Y)$  and  $\Sigma_l$  be the well-posed linear system in  $L^{\bar{p}}$  on  $(X, X, Y)$  associated to  $\Sigma_b$  (see Remark 4.7). By virtue of Propositions

4.3 and Remark 4.5, we know that there is a unique control operator  $B_b \in \mathcal{L}(X, F(A_{-1}))$  representing the “bilinear” couple  $(\mathbb{T}, \Phi^b)$ , and it is exactly the control operator associated to the “linear” couple  $(\mathbb{T}, \Phi^b(1, \cdot))$ . Theorem 3.6 now shows that  $\Sigma_l$  is weakly regular and the proof is complete.  $\square$

Theorems 4.9 and 4.10 may be summarized by saying that every  $L^{(p, \bar{p})}$ -well-posed bilinear systems ( $1 < p < \infty$ ), which satisfies the condition **(H)** has a representation of the form (1.2).

Again, if we can dispense of the condition **(H)**, then all  $L^{(p, \bar{p})}$ -well-posed bilinear systems become weakly regular.

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