Transnational Journal of Mathematical Analysis and Applications
Vol. 2, Issue 1, 2014, Pages 21-36
ISSN 2347-9086
Published Online on May 17, 2014

http://jyotiacademicpress.net

SHARP WEIGHTED INEQUALITIES FOR MULTILINEAR COMMUTATORS OF MARCINKIEWICZ OPERATORS

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Abstract

In this paper, we prove the sharp inequalities for the multilinear commutators related to the Marcinkiewicz operators. By using the sharp inequalities, we obtain the boundedness of the commutators from $L^p(R^n)$ to $L^q(R^n)$.

1. Introduction

As the development of singular integral operators, their commutators have been well studied. Let T be the Calderón-Zygmund singular integral operator, we know that the commutator $[b, T](f) = T(bf) - bT(f)$ (where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$ (see [3]). In [8], the sharp estimates for some multilinear commutators of the Calderón-Zygmund singular integral operators are obtained. The main purpose of

2010 Mathematics Subject Classification: 42B20, 42B25.
Keywords and phrases: multilinear commutator, Marcinkiewicz operator, BMO, sharp inequality.
Communicated by Lanzhe Liu, Editor, RCMMS.
Received March 5, 2014

this paper is to prove some sharp inequalities for the multilinear commutators related to the Marcinkiewicz operators. By using the sharp inequalities, we obtain the boundedness of the commutators from $L^p(R^n)$ to $L^q(R^n)$.

2. Notations and Results

First, let us introduce some notations (see [1], [8], and [9]). In this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a locally integrable function b, let $b_Q = \frac{1}{|Q|}\int_Q b(x)dx$. The sharp function of b is defined by, for $x \in R^n$,

$$b^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |b(y) - b_Q|dy.$$

It is well-known that (see [9])

$$b^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|}\int_Q |b(y) - c|dy.$$

We say that b belongs to $BMO(R^n)$ if $b^{\#}$ belongs to $L^\infty(R^n)$ and define $\|b\|_{BMO} = \|b^{\#}\|_{L^\infty}$. It has been known that (see [8])

$$\|b - b_{2^k Q}\|_{BMO} \leq Ck\|b\|_{BMO}.$$

For $b_j \in BMO(R^n)(j = 1, \cdots, m)$, set

$$\|\vec{b}\|_{BMO} = \prod_{j=1}^{m} \|b_j\|_{BMO}.$$

Given a positive integer m and $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $\|\vec{b}_\sigma\|_{BMO} = \|b_{\sigma(1)}\|_{BMO} \cdots \|b_{\sigma(j)}\|_{BMO}$.

Let M be the Hardy-Littlewood maximal operator, that is,

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

We write that $M_p(f) = (M(|f|^p))^{\frac{1}{p}}$ for $0 < p < \infty$. Let $0 < \delta < n$, $0 < r < \infty$, set

$$M_{r,\delta}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-\delta r/n}} \int_Q |f(y)|^r dy \right)^{1/r}.$$

If $0 < r \leq p < n/\delta$, $1/q = 1/p - \delta/n$, we know $M_{r,\delta}$ is type of (p, q), that is,

$$\|M_{r,\delta}(f)\|_q \leq C\|f\|_p.$$

In this paper, we will study some multilinear commutators as following:

We denote $\Gamma(x) = \{(y, t) \in R_+^{n+1} : |x - y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$.

Definition. Let b_j $(j = 1, \cdots, m)$ be the fixed locally integrable functions on R^n, $0 < \delta < n$ and $0 < \gamma \leq 1$. Suppose that S^{n-1} is the unit sphere of R^n $(n \geq 2)$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω be homogeneous of degree zero and satisfy the following two conditions:

(i) $\Omega(x)$ is continuous on S^{n-1} and satisfies the Lip_γ condition on S^{n-1}, i.e.,

$$|\Omega(x') - \Omega(y')| \leq M|x' - y'|^\gamma, \quad x', y' \in S^{n-1};$$

(ii) $\int_{S^{n-1}} \Omega(x')dx' = 0.$

The Marcinkiewicz multilinear commutator is defined by

$$\mu_{s,\delta}^{\vec{b}}(f)(x) = \left[\iint_{\Gamma(x)} |F_t^{\vec{b}}(f)(x, y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x, y) = \int_{|y-z|\leq t} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} \left[\prod_{j=1}^m (b_j(x) - b_j(z))\right] f(z)dz.$$

Set

$$F_t(f)(y) = \int_{|y-z|\leq t} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f(z)dz.$$

We also define that

$$\mu_{s,\delta}(f)(x) = \left(\iint_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2},$$

which is the Marcinkiewicz operator (see [5], [6], and [10]).

Remark. Fixed $\lambda > \max(1, 2n/(n+2-2\delta))$. Another Marcinkiewicz multilinear operators is defined by

$$\mu_{\lambda,\delta}^{\vec{b}}(f)(x) = \left[\iint_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t^{\vec{b}}(f)(x, y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x, y) = \int_{|y-z|\leq t} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} \left[\prod_{j=1}^m (b_j(x) - b_j(z))\right] f(z)dz.$$

Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy.$$

We also define

$$\mu_\lambda(f)(x) = \left(\iint_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2},$$

which is another Marcinkiewicz operators.

Let H be the Hilbert space $H = \left\{h : \|h\| = \left(\iint_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+3}\right)^{1/2} < \infty\right\}$.

Then for each fixed $x \in R^n$, $F_t^{\tilde{b}}(f)(x, y)$ may be viewed as a mapping from $(0, +\infty)$ to H, and it is clear that

$$\mu_{s,\delta}^{\vec{b}}(f)(x) = \left\|\chi_{\Gamma(x)} F_t^{\vec{b}}(f)(x, y)\right\|, \quad \mu_{s,\delta}(f)(x) = \left\|\chi_{\Gamma(x)} F_t(f)(y)\right\|.$$

Note that when $b_1 = \cdots = b_m$, $\mu_{s,\delta}^{\tilde{b}}$ and $\mu_\lambda^{\tilde{b}}$ are just the m order commutators. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-8] and [10]). Our main purpose is to establish the sharp inequalities for the multilinear commutators.

Now we state our main results as following:

Theorem 1. *Let* $0 < \delta < n$ *and* $b_j \in BMO(R^n)$ *for* $j = 1, \cdots, m$. *Then for any* $1 < r < \infty$, *there exists a constant* $C > 0$ *such that for any* $f \in C_0^\infty(R^n)$ *and any* $\tilde{x} \in R^n$,

$$(\mu_{s,\delta}^{\vec{b}}(f))^{\#}(\tilde{x}) \le C\left(\|\vec{b}\|_{BMO} M_{r,\delta}(f)(\tilde{x}) + \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{BMO} M_r(\mu_{s,\delta}^{\vec{b}_{\sigma^c}}(f))(\tilde{x})\right).$$

Theorem 2. *Let* $0 < \delta < n$ *and* $b_j \in BMO(R^n)$ *for* $j = 1, \cdots, m$. *Then* $\mu_{s,\delta}^{\vec{b}}$ *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$, *where* $1 < p < n/\delta$, $1/q = 1/p - \delta/n$.

Remark. Theorems 1 and 2 also hold for $\mu_{\lambda}^{\tilde{b}}$, we omit the details.

3. Proofs of Theorems

To prove the theorems, we need the following lemmas:

Lemma 1 (See [5]). *Let* $0 < \delta < n, 1 < p < n/\delta, 1/q = 1/p - \delta/n$. *Then* $\mu_{s,\delta}$ *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$, *that is,*

$$\|\mu_{s,\delta}(f)\|_{L^q} \leq C\|f\|_{L^p}.$$

Lemma 2 (See [2]). *Let* $0 < \delta < n$, $0 < r < p < n/\delta$, $1/q = 1/p - \delta/n$. *Then* $M_{r,\delta}$ *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$.

Lemma 3. *Let* $1 < r < \infty$, $b_j \in BMO(R^n)$ *for* $j = 1, \cdots, k$. *Then*

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^{k}|b_j(y) - (b_j)_Q|dy \leq C\prod_{j=1}^{k}\|b_j\|_{BMO},$$

and

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^{k}|b_j(y) - (b_j)_Q|^r dy\right)^{1/r} \leq C\prod_{j=1}^{k}\|b_j\|_{BMO}.$$

Proof. Choose $1 < p_j < \infty\, j = 1, \cdots, m$ such that $1/p_1 + \cdots + 1/p_m = 1$, we obtain, by the Hölder's inequality,

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^{k}|b_j(y) - (b_j)_Q|dy \leq \prod_{j=1}^{k}\left(\frac{1}{|Q|}\int_Q |b_j(y) - (b_j)_Q|^{p_j} dy\right)^{1/p_j}$$

$$\leq C\prod_{j=1}^{k}\|b_j\|_{BMO},$$

and

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^{k} |b_j(y) - (b_j)_Q|^r dy\right)^{1/r} \le \prod_{j=1}^{k}\left(\frac{1}{|Q|}\int_Q |b_j(y) - (b_j)_Q|^{p_j r} dy\right)^{1/p_j r}$$

$$\le C \prod_{j=1}^{k} \|b_j\|_{BMO}.$$

The lemma follows.

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q |\mu_{s,\delta}^{\vec{b}}(f)(x) - C_0| dx$$

$$\le C\left(\|\vec{b}\|_{BMO} M_{r,\delta}(f)(\tilde{x}) + \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{BMO} M_r(\mu_{s,\delta}^{\vec{b}}(f)(\tilde{x})\right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. We first consider the case $m = 1$. We write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{R^n \setminus 2Q}$,

$$F_t^{b_1}(f)(x, y) = (b_1(x) - (b_1)_{2Q})F_t(f)(y) - F_t((b_1 - (b_1)_{2Q})f_1)(y)$$

$$- F_t((b_1 - (b_1)_{2Q})f_2)(y),$$

then

$$|\mu_{s,\delta}^{b_1}(f)(x) - \mu_{s,\delta}(((b_1)_{2Q} - b_1)f_2)(x_0)|$$

$$= \left|\|\chi_{\Gamma(x)} F_t^{b_1}(f)(x, y)\| - \|\chi_{\Gamma(x_0)} F_t(((b_1)_{2Q} - b_1)f_2)(y)\|\right|$$

$$\le \|\chi_{\Gamma(x)} F_t^{b_1}(f)(x, y) - \chi_{\Gamma(x_0)} F_t(((b_1)_{2Q} - b_1)f_2)(y)\|$$

$$\leq \|\chi_{\Gamma(x)}(b_1(x)-(b_1)_{2Q})F_t(f)(y)\| + \|\chi_{\Gamma(x)}F_t((b_1-(b_1)_{2Q})f_1)(y)\|$$

$$+ \|\chi_{\Gamma(x)}F_t((b_1-(b_1)_{2Q})f_2)(y) - \chi_{\Gamma(x_0)}F_t((b_1-(b_1)_{2Q})f_2)(y)\|$$

$$= A(x) + B(x) + C(x).$$

For $A(x)$, by Hölder's inequality with exponent $1/r + 1/r' = 1$, we get

$$\frac{1}{|Q|}\int_Q A(x)dx = \frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}||\mu_{s,\delta}(f)(x)|dx$$

$$\leq \left(\frac{C}{|2Q|}\int_{2Q}|b_1(x)-(b_1)_{2Q}|^{r'}dx\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q |\mu_{s,\delta}(f)(x)|^r dx\right)^{1/r}$$

$$\leq C\|b_1\|_{BMO}M_r(\mu_{s,\delta}(f))(\tilde{x}).$$

For $B(x)$, taking $1 < r < p < q < n/\delta, 1/q = 1/p - \delta/n, r = pt$, by the boundness of $\mu_{s,\delta}$ from $L^p(R^n)$ to $L^q(R^n)$ and Hölder's inequality with exponent $1/t + 1/t' = 1$, we have

$$\frac{1}{|Q|}\int_Q B(x)dx = \frac{1}{|Q|}\int_Q [\mu_{s,\delta}((b_1-(b_1)_{2Q})f_1)(x)]dx$$

$$\leq \left(\frac{1}{|Q|}\int_{R^n}[\mu_{s,\delta}((b_1-(b_1)_{2Q})f\chi_{2Q})(x)]^q dx\right)^{1/q}$$

$$\leq C\frac{1}{|Q|^q}\left(\int_{R^n}|b_1(x)-(b_1)_{2Q}|^p|f(x)\chi_{2Q}(x)|^p dx\right)^{1/p}$$

$$\leq C|Q|^{(-1/q)+(1/pt')+(1-\delta pt/n)/pt}\left(\frac{1}{|2Q|}\int_{2Q}|b_1-(b_1)_{2Q}|^{pt'}dx\right)^{1/pt'}$$

$$\times\left(\frac{1}{|2Q|^{1-\delta pt/n}}\int_{2Q}|f(x)|^{pt}dx\right)^{1/pt}$$

$$= C|Q|^{(-1/q)+(1/pt')+(1-\delta r/n)/r}\left(\frac{1}{|2Q|}\int_{2Q}|b_1-(b_1)_{2Q}|^{pt'}dx\right)^{1/pt'}$$

$$\times\left(\frac{1}{|2Q|^{1-\delta r/n}}|f(x)|^r dx\right)^{1/r}$$

$$\leq C\|b_1\|_{BMO}M_{r,\delta}(f)(\tilde{x}).$$

For $C(x)$, by the Minkowski's inequality, we obtain

$$C(x)\leq\left(\int\int_{R_+^{n+1}}\|(\chi_{\Gamma(x)}-\chi_{\Gamma(x_0)})F_t((b_1-(b_1)_{2Q})f_2(y)\|)^2\frac{dydt}{t^{n+3}}\right)^{1/2}$$

$$\leq C\int_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)|$$

$$\times\left|\int\int_{|x-y|\leq t}\frac{\chi_{\Gamma(z)}(y,t)dydt}{|y-z|^{2n-2-2\delta}t^{n+3}}-\int\int_{|x_0-y|\leq t}\frac{\chi_{\Gamma(z)}(y,t)dydt}{|y-z|^{2n-2-2\delta}t^{n+3}}\right|^{1/2}dz$$

$$\leq\int_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)|$$

$$\times\left(\int\int_{|y|\leq t,|x+y-z|\leq t}\left|\frac{1}{|x+y-z|^{2n-2-2\delta}}-\frac{1}{|x_0+y-z|^{2n-2-2\delta}}\right|\frac{dydt}{t^{n+3}}\right)^{1/2}dz$$

$$\leq\int_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)|$$

$$\times\left(\int\int_{|y|\leq t,|x+y-z|\leq t}\frac{|x-x_0|}{|x+y-z|^{2n-1-2\delta}}t^{-n-3}dydt\right)^{1/2}dz,$$

note that $|x-z|\leq 2t$, $|x+y-z|\geq|x-z|-t\geq|x-z|-3t$ when $|y|\leq t$, $|x+y-z|\leq t$, then, for $x\in Q$,

$$C(x) \le C\int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}|\,|f(z)|\,|x - x_0|^{1/2}$$

$$\times \left(\iint_{|y|\le t,\,|x+y-z|\le t} \frac{t^{-n}\,dydt}{|x + y - z|^{2n+2-2\delta}} \right)^{1/2} dz$$

$$\le C\int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}\|f(z)\|x - x_0|^{1/2}$$

$$\times \left(\iint_{|y|\le t,\,|x+y-z|\le t} \frac{t^{-n}\,dydt}{(|x - z| - 3t)^{2n+2-2\delta}} \right)^{1/2} dz$$

$$\le C\int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}\|f(z)\|x - x_0|^{1/2}$$

$$\times \left(\int_{|x-z|/2}^{\infty} \frac{dt}{(|x - z| - 3t)^{2n+2-2\delta}} \right)^{1/2} dz$$

$$\le C\int_{(2Q)^c} |b_1(z) - (b_1)_{2Q}|\,|f(z)|\, \frac{|x_0 - x|^{1/2}}{|x_0 - z|^{n+1/2-\delta}}\, dz$$

$$\le C\sum_{k=1}^{\infty} \int_{2^{k+1}Q\setminus 2^kQ} |x_0 - x|^{1/2}|x_0 - z|^{-(n+1/2-\delta)}|b_1(z) - (b_1)_{2Q}|\,|f(z)|dz$$

$$\le C\sum_{k=1}^{\infty} 2^{-k}|2^{k+1}Q|^{\delta/n-1} \int_{2^{k+1}Q} |b_1(z) - (b_1)_{2Q}|\,|f(z)|dz$$

$$\le C\sum_{k=1}^{\infty} 2^{-k/2}\left(\frac{1}{|2^{k+1}Q|^{1-\delta/n}} \int_{2^{k+1}Q} |(b_1(z) - (b_1)_{2Q})|^{r'} dz \right)^{1/r'}$$

$$\times \left(\frac{1}{|2^{k+1}Q|^{1-\delta/n}} \int_{2^{k+1}Q} |f(z)|^{r} dz \right)^{1/r}$$

$$= C\sum_{k=1}^{\infty} 2^{-k/2}|2^{k+1}Q|^{(\delta/r'n)+\delta(1-r)/rn}$$

$$\times \left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|(b_1(z)-(b_1)_{2Q}|^{r'}dz\right)^{1/r'}$$

$$\times \left(\frac{1}{|2^{k+1}Q|^{1-\delta r/n}}\int_{2^{k+1}Q}|f(z)|^{r}dz\right)^{1/r}$$

$$\leq C\sum_{k=1}^{\infty} 2^{-k/2}k\|b_1\|_{BMO}M_{r,\delta}(f)(\tilde{x})$$

$$\leq C\|b_1\|_{BMO}M_{r,\delta}(f)(\tilde{x}),$$

thus

$$\frac{1}{|Q|}\int_Q C(x)dx \leq C\|b_1\|_{BMO}M_{r,\delta}(f)(\tilde{x}).$$

Now, we consider the case $m \geq 2$, we have known that, for $b = (b_1, \cdots, b_m)$,

$$F_t^{\vec{b}}(f)(x, y) = \int_{|y-z|\leq t}\frac{\Omega(y-z)}{|y-z|^{n-1-\delta}}\left[\prod_{j=1}^{m}(b_j(x)-b_j(z))\right]f(z)dz$$

$$= \int_{|y-z|\leq t}[((b_1(x)-(b_1)_{2Q})-(b_1(z)-(b_1)_{2Q}))\cdots((b_m(x)-(b_m)_{2Q})$$

$$-(b_m(z)-(b_m)_{2Q}))]\frac{\Omega(y-z)}{|y-z|^{n-1-\delta}}f(z)dz$$

$$= \sum_{j=0}^{m}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int_{|y-z|\leq t}(b(z)$$

$$-(b)_{2Q})_{\sigma^c}\frac{\Omega(y-z)}{|y-z|^{n-1-\delta}}f(z)dz$$

$$= (b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q}) F_t(f)(y)$$

$$+ (-1)^m F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f)(y)$$

$$+ \sum_{j=0}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_{2Q})_\sigma \int_{|y-z| \le t} (b(z) - b(x))_{\sigma^c} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z) dz$$

$$= (b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q}) F_t(f)(y)$$

$$+ (-1)^m F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f)(y)$$

$$+ \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} c_{m,j} (b(x) - (b)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x, y),$$

thus,

$$|\mu_{s,\delta}^{\vec{b}}(f)(x) - \mu_{s,\delta}((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q})) f_2)(x_0)|$$

$$\le \| \chi_{\Gamma(x)} F_t^{\vec{b}}(f)(x, y) - \chi_{\Gamma(x_0)} F_t(((b_1)_{2Q} - b_1) \cdots ((b_m)_{2Q} - b_m) f_2)(y) \|$$

$$\le \| \chi_{\Gamma(x)} (b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q}) F_t(f)(y) \|$$

$$+ \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \| \chi_{\Gamma(x)} (\tilde{b}(x) - (b_m)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x, y) \|$$

$$+ \| \chi_{\Gamma(x)} F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f_1)(y) \|$$

$$+ \| \chi_{\Gamma(x)} F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f_2)(y) - \chi_{\Gamma(x_0)}$$

$$\times F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f_2)(y) \|$$

$$= S_1(x) + S_2(x) + S_3(x) + S_4(x).$$

For $S_1(x)$, by Hölder's inequality with exponent $1/r' + 1/r = 1$ and Lemma 2, we get

$$\frac{1}{|Q|}\int_Q S_1(x)dx$$

$$\leq C\frac{1}{|Q|}\int_Q\left|\prod_{j=1}^{m}(b_j(x)-(b_j)_{2Q})\right|\left|\mu_{s,\delta}(f)(x)\right|dx$$

$$\leq C\left(\frac{1}{|2Q|}\int_{2Q}\left|\prod_{j=1}^{m}(b_j(x)-(b_j)_{2Q})\right|^{r'}dx\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q|\mu_{s,\delta}(f)(x)|^r dx\right)^{1/r}$$

$$\leq C\|\vec{b}\|_{BMO}M_r(\mu_{s,\delta}(f))(\tilde{x}).$$

For $S_2(x)$, by Hölder's inequality with exponent $1/r' + 1/r = 1$ and Lemma 2, we get

$$\frac{1}{|Q|}\int_Q S_2(x)dx$$

$$= \frac{1}{|Q|}\int_Q\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left|(b(x)-(b)_{2Q})_\sigma\mu_{s,\delta}^{\vec{b}_{\sigma^c}}(f)(x)\right|dx$$

$$\leq \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|}\int_Q|(b(x)-(b)_{2Q})_\sigma||\mu_{s,\delta}^{\vec{b}_{\sigma^c}}(f)(x)|dx$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|2Q|}\int_{2Q}|(b(x)-(b)_{2Q})_\sigma|^{r'}dx\right)^{1/r'}$$

$$\times\left(\frac{1}{|Q|}\int_Q|\mu_{s,\delta}^{\vec{b}_{\sigma^c}}(f)(x)|^r dx\right)^{1/r}$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{BMO}M_r(\mu_{s,\delta}^{\vec{b}_{\sigma^c}}(f))(\tilde{x}).$$

For $S_3(x)$, we choose $1 < r < p < q < n/\delta, 1/q = 1/p - \delta/n, r = pt$, by the boundness of $\mu_{s,\delta}$ from $L^p(R^n)$ to $L^q(R^n)$ and Hölder's inequality with $1/t + 1/t' = 1$, we get

$$\frac{1}{|Q|}\int_Q S_3(x)dx$$

$$= \frac{1}{|Q|}\int_Q \|\mu_{s,\delta}(\prod_{j=1}^m (b_j - (b_j)_{2Q})f_1)(x)\| dx$$

$$\leq \left(\frac{1}{|Q|}\int_{R^n} |\mu_{s,\delta}(\prod_{j=1}^m (b_j - (b_j)_{2Q})f\chi_{2Q})(x)|^q dx\right)^{1/q}$$

$$\leq C\frac{1}{|Q|^{1/q}}\left(\int_{R^n} |\prod_{j=1}^m (b_j(x) - (b_j)_{2Q})|^p |f(x)\chi_{2Q})(x)|^p dx\right)^{1/p}$$

$$\leq C\frac{1}{|Q|^{1/q}}\left(\int_{2Q} |\prod_{j=1}^m (b_j(x) - (b_j)_{2Q})|^{pt'} dx\right)^{1/pt'}\left(\int_{2Q} |f(x)|^{pt} dx\right)^{1/pt}$$

$$\leq C|Q|^{(-1/q)+(1/pt')-(1-(\delta pt/n)/pt)}$$

$$\times\left(\frac{1}{|2Q|}\int_{2Q} |\prod_{j=1}^m (b_j(x) - (b_j)_{2Q})|^{pt'} dx\right)^{1/pt'}$$

$$\times\left(\frac{1}{|2Q|^{1-\delta pt/n}}\int_{2Q} |f(x)|^{pt} dx\right)^{1/pt}$$

$$\leq C\|\vec{b}\|_{BMO} M_{r,\delta}(f)(\tilde{x}).$$

For $S_4(x)$, similar to the proof of $C(x)$ in case $m = 1$, we obtain

$$S_4(x) \leq C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ} |x - x_0|^{1/2}|x_0 - z|^{-(n+1/2-\delta)}\prod_{j=1}^m |b_j(z) - (b_j)_{2Q}||f(z)|dz$$

$$\leq C\sum_{k=1}^{\infty}2^{-k/2}|2^{k+1}Q|^{\delta/n-1}\int_{2^{k+1}Q}\prod_{j=1}^{m}|b_j(z)-(b_j)_{2Q}||f(z)|dz$$

$$\leq C\sum_{k=1}^{\infty}2^{-k/2}\left(\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}\left|\prod_{j=1}^{m}(b_j(z)-(b_j)_{2Q})\right|^{r'}dz\right)^{1/r'}$$

$$\times\left(\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}|f(z)|^{r}dz\right)^{1/r}$$

$$= C\sum_{k=1}^{\infty}2^{-k/2}|2^{k+1}Q|^{(\delta/r'n)+\delta(1-r)/rn}$$

$$\times\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|\prod_{j=1}^{m}(b_j(z)-(b_j)_{2Q})\right|^{r'}dz\right)^{1/r'}$$

$$\times\left(\frac{1}{|2^{k+1}Q|^{1-\delta r/n}}\int_{2^{k+1}Q}|f(z)|^{r}dz\right)^{1/r}$$

$$\leq C\|\vec{b}\|_{BMO}M_{r,\delta}(f)(\tilde{x}),$$

thus,

$$\frac{1}{|Q|}\int_{Q}S_4(x)dx\leq C\|\vec{b}\|_{BMO}M_{r,\delta}(f)(\tilde{x}).$$

This completes the proof of Theorem 1.

Proof of Theorem 2. We first consider the case $m = 1$. Choose $1 < r < p$ in Theorem 1 and by Lemma 2, we have

$$\|\mu_{s,\delta}^{b_1}(f)\|_{L^q}\leq\|M(\mu_{s,\delta}^{b_1})(f)\|_{L^q}\leq C\|(\mu_{s,\delta}^{b_1}(f))^{\#}\|_{L^q}$$

$$\leq C\|M_r(\mu_{s,\delta}(f))\|_{L^q}+C\|M_{r,\delta}(f)\|_{L^q}$$

$$\leq C\|\mu_{s,\delta}(f)\|_{L^q} + C\|M_{r,\delta}(f)\|_{L^q}$$

$$\leq C\|f\|_{L^p} + C\|f\|_{L^p}$$

$$\leq C\|f\|_{L^p}.$$

When $m \geq 2$, we may get the conclusion of Theorem 2 by induction. This finishes the proof.

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