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THE DEMYANOV CONTINUOUS AND CESARI'S PROPERTY

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Abstract

We investigate in this paper the Demyanov metric for classes of unbounded closed, convex sets in $\mathbb{R}^d$, the Cesari's property (Q) for multifunctions is discussed.

1. Introduction and Preliminaries

The concept Cesari's property was first introduced by Cesari in [2] as a useful variant of Kuratowski notion of upper semicontinuity of set-valued maps (multifunctions) and since then it has found important applications in calculus of variations and optimal control. We compare the Cesari's property with the D-continuous set-valued maps. We introduce the following family subsets of $\mathbb{R}^d$:

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$$\mathcal{C} = \left\{A \in \mathbb{R}^d \,:\, A \neq \emptyset, \text{ convex, closed}\right\}; \quad \mathcal{K}^d = \{A \in \mathcal{C} \,:\, A \text{ is compact}\}.$$

Let $A \in \mathbb{R}^d$, $u \in \mathbb{R}^d$.

The support function of a set A we define as

$$p_A(u) = \sup_{a \in A} < a, u >,$$

where $< . >$ is the scalar product.

By $A(u) = \{a \in A :< a, u >= p_A(u)\}$, we denote the face of a set A.

Let $A, B \in \mathcal{K}^d$ and by S^{d-1}, we denote the unit sphere in the space $\mathbb{R}^d$. The Hausdorff metric define as

$$\rho_H(A, B) = \sup_{v \in S^{d-1}} |p_A(v) - p_B(v)|,$$

and the Demyanov metric is defined as

$$\rho_D(A, B) = \sup_{v \in S^{d-1}} \rho_H(A(v), B(v)).$$

We refer to [3] for detailed discussion.

By $0^+ A = \{u \in \mathbb{R}^d \,:\, \forall_{a \in A} \forall_{t \geq 0} a + tu \in A\}$, we denote the recession cone of a set $A \in \mathcal{C}$ and the polar set to A we define as

$$A^0 = \{v \in \mathbb{R}^d \,:\, \forall_{a \in A} < a, v >\leq 0\}.$$

2. The Space $\overline{\mathcal{C}}_K$

We introduce the following equivalence relation on $\mathcal{C}$:

$$A \equiv B \Leftrightarrow 0^+ A = 0^+ B.$$

For the nonempty, closed, convex cone K, we denote by $\mathcal{C}_K$ the equivalence class of all sets in $\mathcal{C}$ having a recession cone K. In particular, the class $\mathcal{C}_0$ having the recession cone $\{0\}$ is the class of sets convex and compact $(\mathcal{C}_0 = \mathcal{K}^d)$.

Now we introduce the following metrics for $A, B \in \mathcal{C}_K$:

$$\rho_1(A, B) = \sup_{v \in riK^0 \cap S^{d-1}} |p_A(v) - p_B(v)|,$$

and

$$\rho_2(A, B) = \sup_{v \in riK^0 \cap S^{d-1}} \rho_H(A(v), B(v)),$$

where *ri* denote the relative interior.

We remark that if $K = \{0\}$ then $\rho_1(A, B) = \rho_H(A, B)$ and $\rho_2(A, B) = \rho_D(A, B)$.

The following example showed that if $K \neq \{0\}$, then $\mathcal{C}_K$ contains elements for which $\rho_1(A, K) = \rho_2(A, K) = \infty$.

Example 2.1. Let $K = \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \geq x_1^2\}$ and $K = \{(0, x_2) : x_2 \geq 0\}$. Then $\rho_1(A, K) = \infty$ so also $\rho_2(A, K) = \infty$.

Now we introduce a subclass $\overline{\mathcal{C}}_K$ consisting of all a sets $A \in \mathcal{C}_K$ such that

$$\rho_2(A, K) < \infty.$$

Observe that $\mathcal{C}_0 = \overline{\mathcal{C}}_0 = \mathcal{K}^d$.

The metric ρ_2 has the following properties:

Lemma 2.1. *Let* $A, B, C, D \in \overline{\mathcal{C}}_K$, $\alpha \geq 0$ *and* $\beta \in [0, 1]$. *Then*

(1) $\rho_2(A + C, B + C) = \rho_2(A, B)$.

(2) $\rho_2(\alpha A, \alpha B) = \alpha\rho_2(A, B)$.

(3) $\rho_2(\beta A + (1 - \beta)C, \beta B + (1 - \beta)D) \leq \beta\rho_2(A, B) + (1 - \beta)\rho_2(C, D)$.

This lemma is easy to prove using definition of a metric and the properties of a support function.

From lemma, we can prove the following theorem:

Theorem 2.1. *Let* $(A_n), (B_n)$ *be a sequence sets contains in* $\overline{\mathcal{C}}_K$ *converges, respectively, in* ρ_2 *metric to* A *and* B *and a sequence* α_n *converges to* α *for all* $\alpha_n, \alpha \geq 0$. *Then*

(1) $\lim_{n\to\infty} \rho_2(A_n + B_n, A + B) = 0$.

(2) $\lim_{n\to\infty} \rho_2(\alpha_n A_n, \alpha A) = 0$.

Proof. Using lemma, we have that

$$\rho_2(A_n + B_n, A + B) \leq \rho_2(A_n + B_n, A + B_n) + \rho_2(A + B_n, A + B)$$

$$= \rho_2(A_n, A) + \rho_2(B_n, B).$$

For scalar multiplication, we get (assume that $\alpha \geq \alpha_n$):

$$\rho_2(\alpha_n A_n, \alpha A) \leq \rho_2(\alpha_n A_n, \alpha_n A) + \rho_2(\alpha_n A, \alpha A)$$

$$= \alpha_n \rho_2(A_n, A) + \rho_2(\alpha_n A, \alpha A).$$

From lemma and assumption, we obtain that

$$\rho_2(\alpha_n A, \alpha A) = \rho_2(\alpha_n A + K, \alpha_n A + (\alpha - \alpha_n)A) = \rho_2(K, (\alpha - \alpha_n)A)$$

$$= \rho_2((\alpha - \alpha_n)K, (\alpha - \alpha_n)A) = (\alpha - \alpha_n)\rho_2(A, K).$$

For $\alpha \leq \alpha_n$, the proof similar.

The following example showing that the space $\overline{\mathcal{C}}_K$ is not separable.

Example 2.2. Let $K = \{(0, x_2) : x_2 \geq 0\}$.

We consider the family sets

$$A_\alpha = \{(x_1, x_2) : 0 \leq x_1 \leq 1, x_2 \geq \alpha x_1, \alpha \in [0, 1]\},$$

where $\rho_2(A_\alpha, A_\beta) = \sqrt{1 + (\max\{\alpha, \beta\})^2} \geq 1$.

Theorem 2.2. *The space* $(\overline{\mathcal{C}}_K, \rho_2)$ *is complete.*

Proof. Let $\{A_n\}$ be a Cauchy sequence in $\overline{\mathcal{C}}_K$. Due to definition ρ_2, the sequence $\{A_n(v)\}$ is a Cauchy sequence in $\{K^d, \rho_H)$ which is known to be complete. So for any $v \in riK^o \cap S^{d-1}$, $\rho_H(A_n(v), A(v)) \to 0$. Hence $\rho_2(A_n, A) \to 0$. We proof that $A \in \overline{\mathcal{C}}_K$. Using the triangle inequality, we obtain

$$\rho_2(A, K) \leq \rho_2(A, A_n) + \rho_2(A_n, K).$$

So $\rho_2(A, K) < \infty$.

3. On *D*-Continuity of Multifunction and Cesari's Property

Consider the multifunction $F : \mathbb{R}^d \to \overline{\mathcal{C}}_K$. We say that multifunction F is D-continuous at $x_0 \in \mathbb{R}^d$ if $\lim_{x \to x_0} \rho_2(F(x), F(x_0)) = 0$.

Now recall the Cesari's property. We say that a multifunction $F : \mathbb{R}^d \to \overline{\mathcal{C}}_K$ satisfies the Cesari's property at x_0 if

$$F(x_0) = \bigcap_{\delta > 0} clco \bigcup_{x \in B(x_0, \delta)} F(x),$$

where $B(x_0, \delta) = \{x \in \mathbb{R}^d : |x - x_0| < \delta\}$.

Lohne in [5] give definition upper C-limits multifunction F by

$$\limsup_{x \to x_0} F(x) = \bigcup_{x_n \to x_0} \limsup_{n \to \infty} F(x_n) = \bigcap_{n \in \mathbb{N}} clco \bigcup_{k \geq n} F(x_k).$$

Now we give the following result:

Theorem 3.1. *Let* $F : \mathbb{R}^d \to \overline{C}_{0^+F(x_0)}$ *be a D-continuous at* $x_0 \in \mathbb{R}^d$. *Then for all* $v \in ri(0^+F(x_0))^o \cap S^{d-1}$

$$(F(x_0))(v) = \bigcap_{\delta>0} clco \bigcup_{x\in B(x_0,\delta)} (F(x))(v).$$

Proof. Let $x_0 \in \mathbb{R}^d$ and assume that F is D-continuous at x_0. Then for all $v \in ri(0^+F(x_0))^o \cap S^{d-1}$

$$\lim_{x\to x_0} \rho_H((F(x))(v), (F(x_0))(v)) = 0.$$

Using definition Hausdorff metric, we have that for any $w \in S^{d-1}$

$$\lim_{x\to x_0} p_{(F(x))(v)}(w) = p_{(F(x_0))(v)}(w).$$

With the aid of ([5], Proposition 2.1), we obtain

$$p_{(F(x_0))(v)}(w) \geq \limsup_{x\to x_0} p_{(F(x))(v)}(w) \geq \limsup_{n\to\infty} p_{(F(x_n))(v)}(w)$$
$$\geq p_{\limsup_{n\to\infty}(F(x_n))(v)}(w).$$

Hence $\limsup_{n\to\infty}(F(x_n))(v) \subset (F(x_0))(v)$ for the some sequence $x_n \to x_0$. So

$$\limsup_{x\to x_0}(F(x))(v) \subset (F(x_0))(v),$$

for all $v \in ri(0^+F(x_0)^o \cap S^{d-1}$. The following equality ([5], Proposition 3.1)

$$\bigcap_{\delta>0} clco \bigcup_{x\in B(x_0,\delta)} F(x) = \limsup_{x\to x_0} F(x),$$

implies the Cesari's property.

The next example shows the Cesari's property not implies D-continuous.

Example 3.1. Let $K \subset \mathbb{R}^2$ where $K = \{(0, x_2) : x_2 \geq 0\}$.

$$F(t) = \begin{cases} \{(x_1, x_2) : 0 \leq x_1 \leq 1, x_2 \geq \frac{1}{|t|} x_1\}, & \text{for } t \neq 0, \\ \{(0, x_2) : x_2 \geq 0\}, & \text{for } t = 0. \end{cases}$$

Observe that $\limsup_{t \to 0} F(t) \subset F(0)$ but $\lim_{t \to 0} \rho_1(F(t), F(0)) = 1$ and $\lim_{t \to 0} \rho_2(F(t), F(0)) = \infty$.

We will close this section with stability result which tells us that set-valued D-converges is preserved by set-valued integration.

Let $T = [a, b]$ be an interval in $\mathbb{R}$ and let $F : T \to \mathcal{K}^d$. Then we define

$$\int_T F(t)dt = \{\int_T f(t)dt : f \in L^1, f(t) \in F(t) \text{ a.e. in } T\}.$$

This is called the Aumann integral.

First proof the following lemma.

Lemma 3.1. *Let $F : T \to \mathcal{K}^d$ be a measurable and*

$$\sup\{|f| : f(t) \in F(t)\} \leq \varphi(t), \varphi \in L^1, \textit{ then for } v \in \mathbb{R}^d$$

$$p_{\int_T F(t)dt}(v) = \int_T p_{F(t)}(v)dt.$$

Proof. Remark that for all $v \in \mathbb{R}^d$

$$p_{\int_T F(t)dt}(v) = \sup_{f(t) \in F(t)} < v, \int_T f(t)dt >= \int_T \sup_{f(t) \in F(t)} < v, f(t) > dt$$

$$= \int_T p_{F(t)}(v)dt.$$

Theorem 3.2. *Suppose* $F_n : T \to \mathcal{K}^d$ *for* $n = 0, 1 \ldots$ *are measurable,* $\sup\{|f| : f(t) \in F_n(t)\} \leq \varphi(t)$ *for each* n, *where* $\varphi \in L^1(T)$ *and* $\lim_{n\to\infty} \rho_D(F_n(t), F_0(t)) = 0$ *for each* $t \in T$.

Then $\lim_{n\to\infty} \rho_D(\int_T F_n(t)dt, \int_T F_0(t)dt) = 0.$

Proof. Using lemma and the definition Demyanov metric, we get

$$\rho_D(\int_T F_n(t)dt, \int_T F_0(t)dt) = \sup_{v\in S^{d-1}} \rho_H((\int_T F_n(t)dt)(v), (\int_T F_0(t)dt)(v))$$

$$= \sup_{v\in S^{d-1}} \rho_H(\int_T F_n(t)(v)dt, \int_T F_0(t)(v)dt) = \sup_{v\in S^{d-1}} \sup_{u\in S^{d-1}} |p_{\int_T F_n(t)(v)dt}(u)$$

$$- p_{\int_T F_0(t)(v)dt}(u)| \leq \sup_{v\in S^{d-1}} \sup_{u\in S^{d-1}} \int_T |p_{F_n(t)(v)}(u) - p_{F_0(t)(v)}(u)|$$

$$= \int_T \rho_D(F_n(t), F_0(t))dt.$$

It remains to use the assumption of theorem.

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