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WEYL-TYPE INEQUALITY FOR OPERATORS IN BANACH SPACES

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Abstract

Let (x_n) be a Weyl number sequence. We show that for any $0 < \delta \leq 1$, there is a positive number $C = C(\delta)$ such that for arbitrary Riesz operator $T \in L(E)$ and any $n = 1, 2, \cdots$, the inequality

$$\left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{\frac{1}{n}} \leq \left(\prod_{i=1}^{m} \left(\frac{2[\frac{n}{2}]}{[\delta i]}\right)^{\frac{1}{2}}\right)^{\frac{1}{m}} \left(\prod_{i=1}^{m} x_i(T)\right)^{\frac{1}{m}} \leq C\left(\prod_{i=1}^{m} x_i(T)\right)^{\frac{1}{m}}$$

holds, where $C \leq \left(4e\left(1 + \frac{1}{\delta}\right)\right)^{\frac{1}{2}}$ is a constant and $m = \left[\frac{2[\frac{n}{2}]}{[1+\delta]}\right]$. The proof relies mainly on the relationship between absolutely 2-summing norm and multiplicity and injectivity of Weyl number.

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1. Introduction

Since Weyl developed the classical Weyl inequality between eigenvalue of compact operators T acting on a Hilbert space and s-number of operator ([1]), i.e., if H is a Hilbert space and $T \in K(H)$, then

$$\prod_{j=1}^{n} |\lambda_j(T)| \le \prod_{j=1}^{n} a_j(T), \tag{1.1}$$

there have been an extensive literature dealing with inequality between eigenvalue and s-numbers of bounded linear operators acting on general Banach space ([2, 3, 4, 5, 6, 7]). Applying s-numbers of operators to estimate the eigenvalue distribution of operators is a very useful tool ([8, 9, 10]).

Following the basic results on eigenvalues distribution and s-numbers of operators ([1, 2, 3]), more attention has been paid to the various inequality by related authors in the recent years, espacially, for example, Bernd Carl and Aicke Hinrichs ([4, 5, 6, 7]). In [4, 5], the optimal Weyl inequalities in Banach spaces related to arbitrary s-numbers are given, i.e.,

$$\prod_{j=1}^{n} |\lambda_j(T)| \le n^{\frac{n}{2}} \prod_{j=1}^{n} a_j(T), \tag{1.2}$$

where $a_j(T)$ denotes any s-number of operator T. Subsequently in [6], he has given the inequality between geometric means of eigenvalue and an injective and surjective s-numbers sequence in the sense of Pietsch ([13]). At the almost same time, Weyl type inequality related to injective or surjective s-numbers sequence and Banach space of weak type 2 have been given ([7]). In the recent paper [5], the authors proved that Weyl numbers form a minimal multiplicative s-numbers in the sense of Bernd Carl and Aicke Hinrichs. However, a well-known inequality between geometric means of eigenvalues and Weyl numbers is concerned with double Weyl number sequence, which is different from general Weyl inequalities (cf. [2], Lemma 13), i.e.,

$$(\prod_{i=1}^{n}|\lambda_i(T)|)^{\frac{1}{n}} \leq e(\prod_{i=1}^{n}x_i^*(T))^{\frac{1}{n}}, \quad n = 1, 2, \cdots, \tag{1.3}$$

where the double sequence $x_i^*(T)$ is defined by $x_{2i}^*(T) = x_i(T)$ and $x_{2i-1}^*(T) = x_i(T)$. In [11], the constant e is displaced by $\sqrt{2e}$. In the present paper, we will give a Weyl inequality between geometric means of eigenvalues and single Weyl number sequence (cf. Theorem 2.2) instead of double Weyl number sequence. In the sense of minimal multiplicity (cf. [4]), this inequality can be improved. Its type is similar to the results of [6, 7], where all the constants c depend on a given positive number δ. Compared our result with the previous results, We can easily observe that $m = [\frac{n+1}{2}]$ in [2, 11] and $m = n$ in [4], while in this paper, $[\frac{n}{2}] \leq m < n$. Speaking in a certain sense, our result can be seen as a supplement and generalization of the previous results [2, 4, 11]. And they are very good quantities for estimating the asymptotic behaviour of eigenvalues.

First, we introduce s-number sequence in the sense of Pietsch [13]. A non-negative sequence $(s_n)_{n=1}^{\infty}$ is called an s-number sequence if for all operators $T \in L(E, F)$ – – the class of all bounded linear operators between Banach spaces, the sequence satisfies the following:

(1) $\|T\| = s_1(T) \geq s_2(T) \geq \cdots \geq 0$, for $T \in L(E, F)$;

(2) $s_n(T + s) \leq s_n(T) + \|s\|$, for $T, S \in L(E, F)$;

(3) $s_n(RTS) \leq \|R\| s_n(T) \|s\|$, for $S \in L(E_0, E), T \in L(E, F), R \in L(F, F_0)$;

(4) If $\dim(T) < n$, then $s_n(T) = 0$;

(5) $s_n(I_n) = 1$, where $I_n : l_2^n \to l_2^n$ is an identity map from l_2^n to itself.

Now we describe some important examples. For $T \in L(E, F)$ and $n = 1, 2, \cdots$, we define the n-th approximation number

$$a_n(T) = \inf\{\|T - S\|;\ S \in L(E, F),\ \text{rank}\, S < n\},$$

the n-th Gelfand number

$$c_n(T) = \inf\{\|TJ_M^E\|;\ M \subset E,\ \text{codim}\, M < n\},$$

where J_M^E denotes the canonical embedding from M to E, the n-th Kolmogorov number

$$d_n(T) = \inf\{\|Q_N^F T\|;\ N \subset F,\ \dim N < n\},$$

where Q_N^F is the canonical map of F onto the quotient space F / N and the n-th Weyl number:

$$x_n(T) = \sup\{a_n(TA);\ A \in L(l_2, E),\ \|A\| \leq 1\}.$$

An s-number function s is called injective if the following property is satisfied: Let M be a subspace of F, then $s_n(J_M^F T) = s_n(T)$ for all $T \in L(E, M)$. An s-number function s is called surjective if the following property is satisfied : Let E / N be a subspace of E, then $s_n(TQ_N^E) = s_n(T)$ for all $T \in L(E / N, F)$. An s-function is called multiplicative if $s_{m+n-1}(ST) \leq s_m(S)s_n(T)$, for $T \in L(E, F)$, $S \in L(F, G)$ and $m, n = 1, 2, \cdots$. The following properties hold:

(1) the approximation numbers $a_n(T)$ are the largest s-number;

(2) the Gelfand numbers $c_n(T)$ are the largest injective s-number and the Weyl numbers $x_n(T)$ are injective s-number;

(3) the Kolmogorov number $d_n(T)$ are the largest surjective s-number;

(4) the approximation number $a_n(T)$, the Gelfand number $c_n(T)$, and the Weyl number $x_n(T)$ are multiplicative;

(5) the Weyl number are a minimal multiplicative s-number sequence in the sense of Carl and Hinrichs, i.e., let $s_n(T)$ be a multiplicative s-number sequence with the property that $s_n(T) \leq x_n(T)$ for all $T \in L$ and $n = 1, 2, \cdots,$ then

$$x_{2n-1} \leq \left(\prod_{k=1}^{n} s_k(T)\right)^{\frac{1}{n}}. \tag{1.4}$$

2. Our Main Results

We assume that a Riesz operator $T \in L(E)$ acting on a complex Banach space, we assign an eigenvalue sequence $\lambda_n(T)$ as follows: The eigenvalues of T are arranged in an order of non-increasing absolute values and each eigenvalue is counted according to its algebraic multiplicity

$$|\lambda_1(T)| \geq |\lambda_2(T)| \geq \cdots \geq 0.$$

If T possesses less than n eigenvalues λ with $\lambda \neq 0,$ we let $\lambda_n(T) = \lambda_{n+1}(T) = \cdots = 0.$ In the following, we give the main theorem about geometric means of eigenvalues and Weyl numbers of Riesz operators. For this, we first list a useful lemma.

Lemma 2.1 ([12], p. 234)**.** *Assuming E_{2n} a $2n$-dimension Banach space, then there is an isomorphism $u \in L(E_{2n}, l_2^{2n})$ such that*

$$\pi_2(u) = (2n)^{\frac{1}{2}}, \|u^{-1}\| = 1, \tag{2.1}$$

where $\pi_2(u)$ denotes absolutely 2-summing norm of u (cf. [2], Section 5).

In the sequel, $[x]$ is the integer part of x for $1 < x < \infty$ and $[x] = 1$ if $0 < x \leq 1$.

Theorem 2.2. *If T and $|\lambda_n(T)|$ are the above, then for any $0 < \delta \leq 1$, there is a positive number $C = C(\delta)$ such that for arbitrary Riesz operator $T \in L(E)$ and any $n = 1, 2, \cdots,$ the inequality*

$$(\prod_{i=1}^{n} |\lambda_i(T)|)^{\frac{1}{n}} \leq (\prod_{i=1}^{m} (\frac{2[\frac{n}{2}]}{[\delta i]})^{\frac{1}{2}})^{\frac{1}{m}} (\prod_{i=1}^{m} x_i(T))^{\frac{1}{m}} \leq C(\prod_{i=1}^{m} x_i(T))^{\frac{1}{m}} \qquad (2.2)$$

holds, where $x_i(T)$ denotes the Weyl number of T and $m = [\frac{2[\frac{n}{2}]}{[1+\delta]}]$. For the constant C, we get $C \leq (4e(1 + \frac{1}{\delta}))^{\frac{1}{2}}$.

Remark 2.3. Using the formula (1.3), we can obtain Weyl inequality between eigenvalue and minimal multiplicity s-number in the sense of Carl and Hinrichs (cf. [4]).

Proof. If $|\lambda_n(T)| = 0,$ then there is nothing to prove. So we assume that $|\lambda_n(T)| \neq 0.$ When $n = 1,$ the result is obvious. If we have proved when $n = 2p (p \in N),$ the result is true, then the inequality still holds when $n = 2p + 1,$ which follows from the inequality:

$$(\prod_{i=1}^{2p+1} |\lambda_i(T)|)^{\frac{1}{2p+1}} \leq (\prod_{i=1}^{2p} |\lambda_i(T)|)^{\frac{1}{2p}}.$$

Therefore, it is sufficient to prove that the result holds for all even natural numbers.

Without loss of generation, we replace n by $2n$. In the following, we prove

$$(\prod_{i=1}^{2n} |\lambda_i(T)|)^{\frac{1}{2n}} \leq C(\prod_{i=1}^{m} x_i(T))^{\frac{1}{m}},$$

where $m = [\frac{2n}{1+\delta}]$. From Riesz operator $T \in L(E)$, we can find a $2n$-dimensional subspace E_{2n} of E invariant under T such that the restriction of T to E_{2n} has precisely $\lambda_1(T), \lambda_2(T), \cdots, \lambda_{2n}(T)$ as its eigenvalues (cf. [8], 3.2.23; [12]). Following the above lemma, we can obtain that there is an isomorphism map $u \in L(E_{2n}, l_2^{2n})$ such that $\pi_2(u) = (2n)^{\frac{1}{2}}, \|u^{-1}\| = 1$. With the principle of related operators (cf. [8], 3.3.4; [4]), we draw a conclusion

$$(\prod_{i=1}^{2n} |\lambda_i(T)|)^{\frac{1}{2n}} = (\prod_{i=1}^{2n} |\lambda_i(uT_{2n}u^{-1})|)^{\frac{1}{2n}}. \tag{2.3}$$

On finite dimensional Hilbert spaces, all s-numbers of bounded operators coincide, which is just the singular value of operator. Applying classical Weyl inequality to the operator $uT_{2n}u^{-1}$ (cf. [1]), we can get

$$(\prod_{i=1}^{2n} |\lambda_i(uT_{2n}u^{-1})|)^{\frac{1}{2n}} \le (\prod_{i=1}^{2n} x_i(uT_{2n}u^{-1}))^{\frac{1}{2n}}. \tag{2.4}$$

Let $m = [\frac{2n}{1+\delta}]$, for $0 < \delta \le 1$, we may arrive at

$$(\prod_{i=1}^{2n} x_i(uT_{2n}u^{-1}))^{\frac{1}{2n}} \le (\prod_{i=1}^{m} x_{[\delta i]+i-1}(uT_{2n}u^{-1}))^{\frac{1}{m}}. \tag{2.5}$$

For the right hand of the above inequality (2.5), with the multiplicity of Weyl number and Lemma 2.1, we have

$$x_{[\delta i]+i-1}(uT_{2n}u^{-1}) \le x_{[\delta i]}(u)x_i(T_{2n})\|u^{-1}\| = x_{[\delta i]}(u)x_i(T_{2n}).$$

By absolutely 2-summing norm and [2] Lemma 8, with $[\delta i] \le n$, we have

$$([\delta i])^{\frac{1}{2}} x_{[\delta i]}(u) \le \pi_2(u) = (2n)^{\frac{1}{2}}.$$

Let I_{2n} be canonical injection from E_{2n} into E. Obviously $I_{2n}T_{2n} = TI_{2n}$. According to injectivity and the definition of Weyl number, we can obtain

$$x_i(T_{2n}) = x_i(I_{2n}T_{2n}) = x_i(TI_{2n}) \leq x_i(T)\|I_{2n}\| = x_i(T).$$

Therefore,

$$(\prod_{i=1}^{m} x_{[\delta i]+i-1}(uT_{2n}u^{-1}))^{\frac{1}{m}} \leq (\prod_{i=1}^{m} x_i(T))^{\frac{1}{m}} (\prod_{i=1}^{m} (\frac{2n}{[\delta i]})^{\frac{1}{2}})^{\frac{1}{m}}. \tag{2.6}$$

We observe that $[\delta i] \geq \frac{\delta i}{2}$, $m = [\frac{2n}{1+\delta}] \geq \frac{n}{1+\delta}$. Following again from Taylor expanding theorem with $\frac{m^m}{m!} \leq e^m$, we have

$$(\prod_{i=1}^{m} (\frac{2n}{[\delta i]})^{\frac{1}{2}})^{\frac{1}{m}} \leq (4e(1+\frac{1}{\delta}))^{\frac{1}{2}}. \tag{2.7}$$

Combined with the above (2.3)-(2.7), the proof is complete.

Remark 2.4. In a special case, if $\delta = 1$, we can obtain a better inequality

$$(\prod_{i=1}^{n} |\lambda_i(T)|)^{\frac{1}{n}} \leq (\prod_{i=1}^{m} (\frac{2[\frac{n}{2}]}{i})^{\frac{1}{2}})^{\frac{1}{m}} (\prod_{i=1}^{m} x_i(T))^{\frac{1}{m}} \leq \sqrt{2e}(\prod_{i=1}^{m} x_i(T))^{\frac{1}{m}}$$

hold, where $m = 2[\frac{n}{2}]$. Indeed $\prod_{i=1}^{m} \frac{2[\frac{n}{2}]}{i} \leq (2e)^m$ according to the above proof. It follows that $C \leq \sqrt{2e}$.

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