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SEPARABILISATION PROCEDURE AND MAXIMAL REGULARITY APPLIED TO NON AUTONOMOUS CAUCHY PROBLEM

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Abstract

We introduce a different method to deal with some partial stochastic and ordinary differential equations otherwise, mainly in financial mathematics, including the notions of "maximal regularity" and "random process separabilisation" using composition with appropriate random variables or Brownian motion. We have chosen as model, presented at the Introduction, the problem of Asian options, we then connect it to the Cauchy problem in autonomous and non autonomous cases.

1. Introduction

A lot of stochastic differential equations (SDE) that treat replication problems by a filing (self or not) financing portfolio strategy leads to, in deterministic approach, to partial differential equation (PDE), but in interesting cases, results are restricted to the autonomous case (i.e., time independence of operators).

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Our model is the analysis of the Asian options as presented in [1]. After convenient applications of Itô's formula and following the same way of Black-Scholes [2] for portfolio replication, Prüss et al. [1, 7] proved that it suffices to resolve the equation:

$$(\partial_t - A - x.\Delta y)u = 0, \tag{E}$$

where A is the d-dimensional Black-Scholes operator and $B = x.\nabla y$ is the path dependence of the price averaging in Asian options. The principal argument was that the operator $\mathbb{A} = A + x.\Delta y$ defined on $X_p = L_p(\mathbb{R}^d \times \mathbb{R}_+^d)$ for $1 \leq p < \infty$ endowed with its normal norm, is a generator of a C_0-semi group of contractions with domain

$$D(A) = \begin{cases} W^{2,p}(\mathbb{R}^d; L_p(\mathbb{R}_+^d)), & \text{for all} \quad 1 < p < \infty, \\ \{u \in C_0(\mathbb{R}^d) \bigcap \{ \cap_{q>1} W_{\text{loc}}^{2,q}(\mathbb{R}^d); \quad Au \in C_0(\mathbb{R}^d)\}\}, & \text{if } p = \infty. \end{cases}$$

For more details, see (Lemma 3.2; [1]) and (Proposition 3.1; [1]).

Our purpose is not to restitute the ideas of the authors, but to give an appropriate reason to generalize their approach for non autonomous case. Several authors have already studied the non autonomous Cauchy problem, mainly Aquistapachi, Terreni, Hubert, Arendt and others, see [4], [5], [6], and [7].

The purpose is to contribute modestly to those masterpieces with a new notion: "separabilisation of the range". The main tools are "maximal regularity" and "Monte Carlo method of integral calculus", concepts which are developed in the sequel. The maximal pointwise regularity that must possess the operator family $AoX(t)$ should be susceptible of being simulated numerically in order to give meaning to the approximation stated above. We will study the impact of the support of the random variable X on the choice of model and then we will develop the practical case of a usual law simulated by the Monte Carlo method.

2. The Maximal Regularity

The summarized results in this section are extracted essentially from [8], [9], and [10]. Notations adopted are those of [8].

Let D and X be two Banach spaces, where D is continuously and densely embedded into X. We can see D as a subspace of X with the injection $i : D \to X$ continuous and we will note, up to now, $D \hookrightarrow X$.

2.1. Autonomous case

Let $p > 1$ and let A denote a single operator from $\mathcal{L}(D, X)$ (i.e., A is time-independent). We write $A \in MR_p$ and say that A has L^p-maximal regularity, if the problem

$$\begin{cases} \dot{u} + Au = f & \text{a.e. on} \quad [a, b], \\ u(a) = 0, \end{cases}$$

has a unique solution u in $W^{1,p}(a, b, X) \bigcap L^p(a, b, D)$ for all $f \in L^p(a, b, D)$ and some interval $[a, b]$.

The notation $A \in MR_p$ or just $A \in MR$ is immediate consequence of the most important properties of maximal regularity summarized as follows:

(i) The L^p-maximal regularity is independent of the bounded interval $[a, b]$.

(ii) If $A \in MR_p$ for some $p > 1$, then $A \in MR_p$ for all $p > 1$.

(iii) If $A \in MR$, then $-A$ is a generator of a holomorphic C_0-semigroup on X.

(vi) The converse in (iii) is not true in general Banach spaces with in conditional basis [12].

(v) If $A \in MR$, then for all x in the trace space

$$Tr = \{u(a) : u \in MR(a, b) = W^{1,p}(a, b, X) \bigcap L^p(a, b, D)\},$$

the homogeneous Cauchy problem

$$(HCP) \qquad \begin{cases} \dot{u} + Au = 0 & \text{a.e. on} \quad [a, b], \\ u(a) = x, \end{cases}$$

is well posed.

2.2. Non autonomous case

D and X as below, let $p > 1$ and $[0, T]$ be a bounded and fixed interval. We consider the family $(A(t))_{t \in [0, T]}$ and assume that $t \in [0, T] \to A(t) \in \mathcal{L}(D, X)$ is continuous. We will say that A has L^p-maximal regularity if for every $f \in L^p(0, T, X)$, the problem

$$\begin{cases} \dot{u}(t) + A(t)u(t) = f(t) & \text{a.e. on} \quad [0, T], \\ u(0) = 0, \end{cases}$$

has a unique solution $u \in MR[0, T]$.

A great and important result in this case is the possibility to construct an evolution family, which enables us to resolve the non autonomous Cauchy problem with initial value *x* in the trace space *Tr* as defined above. We recall the result and one can see fructuously Lemma 2.2 and Proposition 2.3 in [8].

Proposition 2.1. *Let* $A \in MR_p(0, T)$ *for* $0 < T_0 \leq T$. *Then for every position x in Tr and every time* $s \in [0, T_0]$, *the* $(NCP)_s$

$$\begin{cases} \dot{u} + A(t)u = 0 & \textit{a.e. on} \quad [s; T_0], \\ u(s) = x, \end{cases}$$

is well posed in the sense of [13].

Moreover, if $u(t, s) = u(t)$ *for* $(t, s) \in [0, T_0]^2$ *and* $t \geq s$, *then* $\{u(t, s)\}_{T_0 > t \geq s}$ *is a bounded and strongly continuous evolution family on Tr.*

The continuity of $t \mapsto A(t)$ is not necessary. Arendt and his collaborators have proved that boundness, measurability of $(A(t))_{t \in [0, T]}$ and its relative continuity, as defined in [8] are sufficient and less restrictive conditions for well-posedness under the L^p-maximal regularity. Our main result is to establish that the well-posedness remains under less expansive hypothesis, which obliged the family $(A(t))_{t \in [0, T]}$ to have the punctual L^p-maximal regularity (at least in the continuous case).

3. Random Separabilisation

To better understand a given space, we usually tend to describe its elements with a small number of others so called privileged. In mathematics, mainly in algebraic approaches, the bases are the best candidates. But, in the analytic point of view, one must have at least a dense set of such elements at his disposal. It is the fundamental principle of "separabilisation".

Definition 3.1. Let $A : [0, 1] \to E$ be a family of operators, where E is Banach space. Let X be a random variable from a probabilised space Ω (we choose $\Omega = \mathbb{R}$) into $[0, 1]$. We say X separabilise A, if $(A \circ X(w))_w = (A \circ X_t)_{t \in [0,1]} = A(t)_{t \in H}$ and H is dense in $[0, 1]$.

Proposition 3.2. *Let* $X : \mathbb{R} \to [0, 1]$ *with* $n \in \mathbb{N}$ *and* $\mathbb{Q} = (q_k)_{k \geq 1}$. $u \mapsto X(u) = \frac{1}{2^n} 1_{\mathbb{Q}}(u)$. *Then X separabilise A.*

Proof.

(.) It is trivial because $X(\mathbb{R}) = Q \bigcap [0, 1]$ and obviously $\overline{X(\mathbb{R})} = [0, 1]$.

(.) Moreover, X as defined, is a probability density because

$$\begin{cases} X \geq 0, \\ \int_{\mathbb{R}} X d\sigma = \sum_{n \geq 1} \frac{1}{2^n} = 1, \end{cases}$$

where σ is the usual counting measure.

Theorem 3.3. *Let $(A(t))_{t \in [0, T]}$ be an operator family. Assume that for each $T_0 \in [0, T]$, A has a maximal regularity property in $[O, X(T_0)]$, where X is random variable separabilising A. Then the problem*

$$\begin{cases} \dot{u} + A \circ X(t) u(t) = 0 \quad a.e.\ on \quad [s; T_0], \\ u(s) = x, \end{cases}$$

is well posed for all $s < T_0$, $x \in Tr$, where Tr, in the course of this paper, denotes the trace space as described in [8].

Proof. We begin by noting two main differences with the statement of Proposition 3.2 in [8]. First, when A is not required to have MR for all t, but just for values in countable set. Second, the choice of $X(T_0)$ is arbitrary provided that the operator retains ownership property of MR for all images by X.

To prove Theorem 3.3, we need two lemmas.

Lemma 3.4. *Let δ be a jauge (see* [13]*) (i.e., $\delta : [0, 1] \to \mathbb{R}_+$) and $t_i \in [0, T]$, $i = 0, 1, \ldots, n$. Then, there exists a subdivision $(\Lambda_i)_{i=1}^n$ such that $\forall i \in \{0, 1, \ldots, n-1\}$ $t_i \in [\Lambda_i, \Lambda_{i+1}] \subset [t_i - \delta_{t_i}, t_i + \delta_{t_i}]$.*

Proof. Suppose by contradiction that the conclusion does not hold, i.e., there does not exist a subdivision $(\tau_i)_{i=0}^n$ such that

$$[\Lambda_i, \Lambda_{i+1}] \subset [t_i - \delta_{t_i}, t_i + \delta_{t_i}]. \tag{1}$$

Consider two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ with $a_0 = a, b_0 = b$ and if a_n and b_n are constructed, then one of $[a_n, \frac{a_n + b_n}{2}]$ or $[\frac{a_n + b_n}{2}, b_n]$ verifies the assumption below, otherwise, connecting the two subdivisions, we will recover $[a_n, b_n]$ in which, to find the property (1) is impossible. Therefore, we construct a decreasing net of nested intervals, whose intersection, since $[a, b]$ is compact, is reduced to $\{c_0\}$. Thus, as $\delta_{c_0} > 0, \exists k_0 > 0,$

$$c_0 \in [a_{k_0}, b_{k_0}] \subset]c_0 - \frac{1}{2}\delta_{c_0}, c_0 + \frac{1}{2}\delta_{c_0}[.$$

It holds that, $[a_{k_0}, b_{k_0}]$ satisfies (1), this is a contradiction.

□

Remark. The previous lemma is a classical result for Henstock-Kurzweil integral theory. It can be proved in many different ways but one must always use a compactness argument.

Now, let A be a single operator and $B : [0, 1] \to L(D, X)$ be a family of operator such that for random variable presented in Lemma 3.4; $B \circ X$ is measurable and satisfies a regularity inequality

$$\forall x \in X; \quad \forall t \in \mathbb{R}, \quad \|B \circ X(t)\|_X \leq \alpha_t \|x\|_D + \eta\|x\|_X, \tag{2}$$

for some constants α_t such that the norm of the operator

$$L : L^p(a, b, D) \to L^p(a, b, X),$$

$$u \to (L_A : t \to \dot{u} + Au),$$

with the appropriate domain verifies $\|R(\lambda, L_A)\| \leq \frac{1}{2\alpha_t}$.

We can prove the following lemma:

Lemma 3.5. *Assume that* $A \in MR$ *and* B *verifies the inequality* (2). *Then, for every* $\lambda > 0$, *the following problem*:

$$\begin{cases} \dot{u} + \lambda u + Au + B \circ X(t)u(t) = f, \\ u(0) = x, \end{cases}$$

has a unique solution for every $x \in Tr$ *and* $f \in L^p(a, b, X)$.

Proof. It suffices to reshuffling the functional argument in [8]. The adaptation is obvious because $B(t)$ and $B \circ X(t)$ verify simultaneously the inequality (2). □

The first main result in this paper is the following theorem:

Theorem 3.6. *Let* $A : [0, T] \to L(D, X)$ *be relatively continuous* (*see* [8] *for precise definition of relative continuity*) *such that the mapping* $t \mapsto \|A(t)\|$ *is measurable, for all* $X(t)$. *Let* X *be a random variable separabilising* A. *Assume that* $A \circ X(t) \in MR$ *for all* $t \in [0, 1]$ *and that for every* φ *in the dual set* X^* *relative to* X, *the mapping* $\varphi \circ A \circ X$ *is measurable*.

Then $A \in MR_p[0, \tau]$ *for all* $\tau \in [0, T]$ *and* $p > 1$.

Proof. Let $f \in L^p(0, T, X)$ and A relatively continuous means that, for each $t \in [0, T]$, there exists $\delta_t > 0$ such that

$$|s - t| < \delta_t \Rightarrow \|A(t)x - A(s)x\| \leq \frac{1}{2\alpha_t} \|x\|_D + \eta_t \|x\|_X.$$

By Lemma 3.4, we can find a subdivision $(\Lambda_i)_{i=0}^n$ and a sequence $t_0 = 0 < t_1 < \ldots < t_n = T$ such that $[\tau_i, \tau_{i+1}] \subset [t_i - \delta_t, t_i + \delta_t]$.

For all $i \in \{0, 1, \ldots, n\}$, $A \circ X(t_i)$, by assumption, is in MR, there exists $j_i \in \{0, 1, \ldots, n\}$ such that $X(t_i) \in [\tau_{j_i}, \tau_{j_{i+1}}]$. Consider for all $s \in [\tau_{j_i}, \tau_{j_{i+1}}]$, the perturbation

$$B_{j_i} : [\tau_{j_i}, \tau_{j_{i+1}}] \to L(D, X),$$

$$s \mapsto A(s) - A \circ X(t_i).$$

The operator B_{j_i} satisfies Proposition 2.1 and the problem

$$\begin{cases} A \circ X(t_i)u + B_{j_i}(s)u(s) = f(s) \quad \text{a.e. on} \quad [\tau_{j_i}, \tau_{j_{i+1}}], \\ u(\tau_{j_i}) = 0, \end{cases}$$

has a unique solution.

To achieve the proof, it suffices to prove the strong measurability of $t \mapsto A \circ X(t)$ on $[0, T]$. Fortunately, X and $\|A\|$ are measurable, by composition $t \mapsto \|A \circ X(t)x\|$ is measurable too. On the other hand, by the separabilisation tool, $A \circ X$ has a separable rank. One can deduce easily using Pettis theorem (see [14]). □

4. Separabilisation with Brownian Motion

Our second main result concern how to separabilise operators by Brownian motion. The main tools are:

Lemma 4.1. *The Brownian motion in one dimension space is recurrent. In the other words*, $\forall x \in [0, T]$, $P(\exists t > 0, B_t = x) > 0$.

Note that in $\mathbb{R}^3$, this property does not hold and one can easily prove that $\inf\{t; B_t = x\} = +\infty$ for x in some U open subset of $\mathbb{R}^3$.

Lemma 4.2. *One-point sets in* $\mathbb{R}$, *are not polar, i.e.,* $P_x(T_A < \infty) > 0$, $\forall x \in \mathbb{R}$ *(i.e., for all* $x \geq 0$, *the hitting time* T_A *are stopping times and events* $(T_A < \infty)$ *are measurable for each* $A \subset [0, T]$*).*

The proofs of Lemmas 4.1 and 4.2 are very classical (see [12]).

Theorem 4.3. *Let* $(B_t)_{t\geq 0}$ *be a Brownian motion, A is a family of operator such that* $A \in MR_p$ *for every* $p > 1$ *and relatively continuous. Let* $(t_n)_{n\geq 0}$ *be a dense sequence in* $[0, T]$. *Consider* $E_i = \{t > 0, B_t = t_i\}$.

Then $A \circ B(t)$ *is* MR_p *and the problem*

$$\begin{cases} \dot{u} + A \circ B(t) = f \quad \text{a.e. on} \quad [0, T], \\ u(0) = x, \end{cases}$$

is well posed for every $f \in L^p(0, T, D)$ *and x in Tr.*

Proof. For each $i > 0$, the set E_i is trivially measurable and by Lemma 4.2 is not empty. By Lemma 4.1, the probability to reach each t_i, by Brownian motion, is not null.

Let $\varepsilon > 0$, by relative continuity of A, for all $t > 0$, there exist η, $\delta_t > 0$ such that

$$|t - s| < \delta \; \|A(t)x - A(s)x\| \leq \frac{1}{2\alpha_t} \|x\|_D + \eta \|x\|_X,$$

where α_t denote the constant in relation (2). By Lemma 3.4, there exists a finite subsequence $(t_{n_j})_{j=0}^{n}$ and a subdivision $(\tau_j)_j^n$ such that

$$t_{n_j} \in [\tau_j, \tau_{j+1}] \subset [t_{n_j} - \delta_{t_{n_j}}, t_{n_j} + \delta_{t_{n_j}}],$$

for every $s \in E_i \cap [\tau_j, \tau_{j+1}]$, the small perturbation $B(s) = A(t_{n_j}) - A(s)$ remains relatively continuous. So, the problem

$$\begin{cases} \dot{u}(s) + A \circ B(s) = f(s) \quad \text{a.e. on} \quad s \in [\tau_j, \tau_{j+1}], \\ u(\tau_j) = x, \end{cases}$$

is well posed. By connecting as in [8], the proof is complete. □

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