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DECOMPOSITION OF SYMPLECTIC STRUCTURES

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Abstract

We consider the decomposition of symplectic structures by Poisson structures or presymplectic structures and give an example which admits no non-trivial compact decompositions.

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1. Introduction

Symplectic structures on differentiable manifolds are interesting in both physics and mathematics (e.g., Abraham and Marsden [1], Guillemin and Sternberg [13]). In physics, it gives the framework of classical mechanics and is studied at the present time (e.g., Cantrijn et al. [9], Norris [18], Paufler and Roemer [19], Sardanashvily [20]). On the other hand, in mathematics, symplectic structures appear in not only differential geometry but also topology (e.g., Atiyah [2], Gromov [12]), complex geometry (e.g., Atiyah and Bott [3], Donaldson [10], Mabuchi [15]), and statistics (e.g., Barndorff-Nielsen and Jupp [4], Friedrich [11], Nakamura [17]).

A symplectic structure is a structure on a manifold possessing the integrability and non-degenerateness. As generalizations of this structure there are Poisson and presymplectic structures. These are considered as "degenerate" symplectic structures and have resemblance and different points. For example, these two structures define foliations of the manifold but the directions of leaves are different.

In this paper, we discuss how to construct a symplectic structure on a manifold by degenerate structures, i.e., Poisson structures and presymplectic structures. This is considered as a decomposition of a symplectic structure and is described by states of foliations defined by these structures. In general, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be foliations of a manifold M. Denote by $D_1(x)$ and $D_2(x)$ tangent spaces of leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$ passing through $x \in M$, respectively. We define $\mathcal{F}_1 = \mathcal{F}_2$ if $D_1(x) = D_2(x)$ for any $x \in M$ and $\mathcal{F}_1 \oplus \mathcal{F}_2 = M$ if $D_1(x) \oplus D_2(x) = T_xM$ for any $x \in M$.

Theorem 1.1. (i) *Let* $\prod_i$, $i = 1, 2,$ *be regular Poisson structures on M and let* $\mathcal{F}_{\prod_i}$ *be the foliation of M defined by* $\prod_i$. *Assume that* $\operatorname{rank}(\prod_1) + \operatorname{rank}(\prod_2) = \dim M$. *Then* $\prod_i$, $i = 1, 2,$ *define a symplectic structure on M if and only if* $\mathcal{F}_{\prod_1} \oplus \mathcal{F}_{\prod_2} = M$.

(ii) *Let* ω_i, $i = 1, 2,$ *be presymplectic structures with constant rank on M and let* $\mathcal{F}_{\omega_i}$ *be foliations defined by* ω_i. *Assume that*

$\mathrm{rank}(\omega_1) + \mathrm{rank}(\omega_2) = \dim M$. *Then* ω_i, $i = 1, 2$, *define a symplectic structure on M if and only if* $\mathcal{F}_{\omega_1} \oplus \mathcal{F}_{\omega_2} = M$.

(iii) *Let* $\prod$ *and* ω *be a regular Poisson structure and a presymplectic structure with constant rank on M and let* $\mathcal{F}_{\prod}$ *and* $\mathcal{F}_{\omega}$ *be foliations of M defined by* $\prod$ *and* ω, *respectively. Assume that* $\mathrm{rank}(\prod) + \mathrm{rank}(\omega) = \dim M$. *Then* $\prod$ *and* ω *define a symplectic structure on M if and only if* $\mathcal{F}_{\prod} = \mathcal{F}_{\omega}$ *and there exists an integrable distribution* $\mathfrak{D} = \{D(x)\}_{x \in M}$ *on M such that the restriction of* ω *to* $\mathfrak{D}$ *is nondegenerate at each point in* $x \in M$.

Conversely, if there exists a foliation $\mathcal{F}_1$ *of a symplectic manifold* (M, Ω) *such that the restriction of* Ω *to every leaf of* $\mathcal{F}_1$ *is non-degenerate, then there is a regular Poisson structure on M. In addition, if there exists a foliation* $\mathcal{F}_2$ *of M such that* $\mathcal{F}_1 \oplus \mathcal{F}_2 = M$, *then there exist regular Poisson structures* $\prod_1, \prod_2$ *and presymplectic structures* ω_1, ω_2 *with constant rank on M such that* $\prod_1$ *is compatible with* ω_1 *and* $\prod_2$ *is compatible with* ω_2.

Though we may consider inexpensively the case of

(iv) $\mathcal{F}_{\prod_1} \oplus \mathcal{F}_{\omega_1} = M$; (v) $\mathcal{F}_{\prod_1} = \mathcal{F}_{\prod_2}$; (vi) $\mathcal{F}_{\omega_1} = \mathcal{F}_{\omega_2}$,

by means of conditions of foliations, we lack information to define a symplectic structure on *M*. The case (iv) was studied of Vaisman [22] and applied to the geometric quantization (in this case, we have a horsymplectic structure as in De Barros [5]. Cf. dual P-p structures in Błaszak and Marciniak [6]). In our work, a result in Vaisman [22] is crucial. For (v) and (vi), we only note that by the equivariant Darboux theorem the local theory is completely determined (e.g., Guillemin and Sternberg [13]).

By Theorem 1.1, there exists a decomposition of a symplectic structure Ω on M by Poisson structures or presymplectic structures, if and only if there exist foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of M such that $\mathcal{F}_1 \oplus \mathcal{F}_2 = M$ and the restriction of Ω to no fewer than leaves of one of foliations (and then all leaves of both foliations) is nondegenerate. The pair $(\mathcal{F}_1, \mathcal{F}_2)$ of these foliations is called a *decomposition* of Ω. If all leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$ are compact, then the decomposition $(\mathcal{F}_1, \mathcal{F}_2)$ is called *compact*. By taking a direct product of compact symplectic manifolds, we obtain an example of compact decompositions. We easily have an example of non compact decompositions on 4-dimensional torus T^4 by 2-dimensional planes with irrational slant. It is well know that every compact homogeneous symplectic manifold with Kirillov-Kostant-Souriau form is expressed as a coadjoint orbit of suitable coadjoint action. In view of Theorem 1.1, we have the following:

Theorem 1.2. *On compact Kählerian homogeneous space* $(G_2/H, \Omega)$, *there exist no non-trivial compact decompositions of* Ω *by invariant structures.*

Note that H in the theorem above is either $U(2)$ or T^2 and the second Betti number of G_2/T^2 is two.

2. Poisson and Presymplectic Structures

Let M be an n-dimensional differentiable manifold. We donote by $\Lambda^p(TM)$, the space of all p-vectors, i.e., anti-symmetric contravariant tensor fields of type $(p, 0)$, where $\Lambda^0(TM) = C^\infty(M)$. The Lie derivative L_X, $X \in \Lambda^1(TM)$, on M acts on $\Lambda^q(TM)$ by

$$(L_X Q)(x) = \frac{d}{dt_{|t=0}}\{\exp(-tX)_* Q(\exp tX(x))\}, \quad x \in M, \quad Q \in \Lambda^q(TM). \tag{2.1}$$

For $X_1, \dots, X_p \in \Lambda^1(TM)$, we define

$$[X_i, Q] \quad := L_{X_i} Q,$$

$$[X_1 \wedge \cdots \wedge X_p, Q] := \sum_{i=1}^{p} (-1)^{i+1} X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_p \wedge [X_i, Q], \tag{2.2}$$

then we have the operator

$$[\ ,\ ] : \Lambda^p(M) \times \Lambda^q(M) \to \Lambda^{p+q-1}(M), \tag{2.3}$$

satisfying (2.1), which is called the *Schouten-Nijenhuis bracket*. (For detail, see, e.g., Vaisman [23]). A 2-vector $\prod \in \Lambda^2(TM)$ is called a *Poisson structure*, if $[\Pi, \Pi] = 0$. If we locally express $\prod$ as

$$\Pi = \sum_{i, j=1}^{n} \Pi_{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j},$$

then $[\Pi, \Pi] = 0$ is equivalent to

$$\Pi_{hi} \frac{\partial}{\partial x_h} \Pi_{jk} + \Pi_{hj} \frac{\partial}{\partial x_h} \Pi_{ki} + \Pi_{hk} \frac{\partial}{\partial x_h} \Pi_{ij} = 0.$$

We next give an alternative definition of Poisson structures. For $f, g \in C^\infty(M)$, we define the bilinear form $\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ on $C^\infty(M)$ by

$$\{f, g\} := \Pi(df, dg) = \sum_{i, j=1}^{n} \Pi_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j},$$

and call the *Poisson bracket*. This bracket satisfies

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0, \tag{2.4}$$

for any $f, g, h \in C^\infty(M)$ if and only if $\prod$ is a Poisson structure on *M*.

For $f \in C^{\infty}(M)$, the vector field $X_f \in \mathfrak{X}(M)$ defined by

$$X_f g = \{g, f\},$$

is called the *Hamiltonian vector field* of *f*. Define a map

$$\prod^{\sharp} : T^* M \to TM,$$

by $\prod^{\sharp}(df) = X_f$. If a Poisson structure is defined from a symplectic structure on *M*, then this map is non-degenerate (linear isomorphic at each point in *M*), while this $\prod^{\sharp}$ is degenerate in general. The additive structures to define the inverse of $\prod^{\sharp}$ is given in the next section (Lemma 3.1). We define the rank of $\prod$ at $x \in M$ as the rank of the linear map $\prod^{\sharp}(x)$ and denote by $\mathrm{rank}(\prod(x))$. Note that $\mathrm{rank}(\prod(x))$ is always even. A Poisson structure $\prod$ whose rank is constant on *M* is especially called *regular*. For regular Poisson manifolds, we have the following:

Lemma 2.1. *For every point x in a regular Poisson manifold* $(M, \{\,,\})$, *there exists a local chart* $(U, x^1, \dots, x^r, y^1, \dots, y^r, z^1, \dots, z^{n-2r})$ *of x such that*

$$\{x^i, y^j\} = \delta_{ij}, \quad \{x^i, z^j\} = \{y^i, z^j\} = 0,$$

$$\{x^i, x^j\} = \{y^i, y^j\} = \{z^i, z^j\} = 0,$$

where $2r := \mathrm{rank} \prod$.

In this lemma, $\{z^i = \text{const.}\}$ defines a global foliation $\mathcal{F}$ of *M*. Indeed, for $x \in M$, if we set

$$D(x) := \{X \in T_x M;\ {}^{\exists} f \in C^{\infty}(M) \text{ s.t. } X_f(x) = X\},$$

then $\mathrm{rank}(\prod(x)) = \dim D(x)$ and $\bigcup_{x \in M} D(x)$ defines the foliation $\mathcal{F}$. Here the integrability of $\bigcup_{x \in M} D(x)$ follows from $[\prod, \prod] = 0$ or the

Jacobi identity (2.4). Moreover, Π induces a symplectic structure on each leaf of $\mathcal{F}$. Note that on general Poisson manifold, the distribution defined above is integrable and define a foliation called the *symplectic foliation*.

Lemma 2.2. *Let M be a* $2n$*-dimensional differentiable manifold*, Ω *be a nondegenerate* 2*-form on M, and* $\mathcal{F}$ *be a foliation of M. Assume that all leaves of* $\mathcal{F}$ *are of dimension* $2k$ *and the restriction of* Ω *to every leaf defines a symplectic structure on the leaf. Then, there exists a Poisson structure on M induced from the symplectic structures on the leaves of* $\mathcal{F}$.

For the proof, see Vaisman [23]. Note that this lemma holds for more general situation and the Poisson structure constructed in the lemma is called a *Dirac structure*.

We next review the definition and properties of presymplectic structures. The pair (M, ω) of a differentiable manifold M and 2-form ω on M is a *presymplectic* manifold, if ω is closed 2-form on M. Because ω is a 2-form, at each $x \in M$, we have the linear map

$$\omega^\flat : T_xM \to T_x^*M$$
$$X \mapsto i(X)\omega,$$

where i denotes the interior product. In the case of symplectic manifolds, this map is isomorphic. By the closedness of ω, we have a foliation on M. Indeed, let $V(x) := \{X \in T_xM;\ i(X)\omega = 0\}$, $x \in M$. We define the rank of ω at $x \in M$ by the codimension of $V(x)$ and denote by $\mathrm{rank}(\omega(x))$. Note that $\mathrm{rank}(\omega(x))$ is always even. From now on, we assume that the rank of ω is constant on M. It follows that $\mathfrak{V} := \bigcup_{x\in M} V(x)$ defines a distribution on M by the Frobenius theorem. Denote the maximal connected integral submanifold of $\mathfrak{V}$ through $x \in M$ by $N(x)$. The foliation defined by $N(x)$, $x \in M$, is called the *null foliation* (in Vaisman [22], it is called the *vertical foliation*). We denote by M/N the leaf space of the foliation. In general, M/N admits no manifold structures.

Let (S, ω_S) be a symplectic manifold and N be a manifold, and set $M := S \times N$. Then the pull-back $\mathrm{pr}_1^* \omega_S$ of ω on S with respect to the projection onto the first factor $\mathrm{pr}_1 : M \to S$ defines a presymplectic structure on M. Conversely, every presymplectic manifold with constant rank locally has this form. Indeed, the following Darboux type theorem holds for presymplectic manifolds.

Lemma 2.3. *For every point x in a presymplectic manifold (M, ω), there exists a local chart $(U; x^1, \ldots, x^r, y^1, \ldots, y^r, z^1, \ldots, z^{n-2r})$ such that*

$$\omega = \sum_{i=1}^{r} a_i(x^1, \ldots, x^r, y^1, \ldots, y^r) dx^i \wedge dy^i.$$

Here r is the half of the rank of ω.

3. Proof of Theorem 1.1

We first collect properties about Poisson and presymplectic structures before giving the proof of Theorem 1.1. Poisson and presymplectic structures may be considered as "degenerate" symplectic structures. We first see differences between these structures. If ω is a presymplectic structure with constant rank on a manifold M, then there is a map

$$\omega^\flat : TM \to T^*M.$$

On the other hand, if we have a Poisson structure $\prod$ on M, then we have

$$\prod^\sharp : T^*M \to TM.$$

Although in both cases, we have maps between TM and T^*M, directions of maps are different. Furthermore, Poisson and presymplectic structures define foliations of M. However, the directions of leaves are different (for Poisson structures, the resulting foliations are symplectic foliations, whereas for presymplectic structures these are null foliations).

For a regular Poisson structure $\prod$ and a presymplectic structure ω with constant rank (ranks are both $2r$), if

$$\prod^{\sharp} \circ \omega^{\flat}|_{\mathrm{Im}(\prod^{\sharp})} = \mathrm{id} \quad \text{and} \quad \omega^{\flat} \circ \prod^{\sharp}|_{\mathrm{Im}(\omega^{\flat})} = \mathrm{id}, \tag{3.1}$$

hold, then $\prod$ is *compatible* with ω or ω is *compatible* with $\prod$. The existence of compatible structures is given by

Lemma 3.1 (Vaisman [22]). (i) *Let* (M, ω) *be a presymplectic manifold and* $\mathfrak{V}$ *be the distribution on TM defined by* ω. *If there exists an integrable distribution* $\mathfrak{D}$ *on M satisfying* $\mathfrak{D} \oplus \mathfrak{V} = TM$, *then there is a* ω-*compatible Poisson structure* $\prod$ *on M.*

(ii) *Let* $(M, \prod)$ *be a Poisson manifold and set* $\mathfrak{V} := \prod(T^*M)$. *If there exists an integrable distribution* $\mathfrak{D}$ *on M satisfying* $\mathfrak{D} \oplus \mathfrak{V} = TM$, *then there is a* $\prod$-*compatible presymplectic structure* ω *on M.*

Proof. (i) By $TM = \mathfrak{D} \oplus \mathfrak{V}$, we have $T^*M = \mathfrak{D}^* \oplus \mathfrak{V}^*$, where $\mathfrak{D}^* = \{\alpha \in T^*M;\ \alpha(v) = 0 \text{ for all } v \in \mathfrak{V}\}$. For any $\alpha \in T^*M$, we set

$$\alpha = \alpha^P + \alpha_p \in \mathfrak{D}^* \oplus \mathfrak{V}^*,$$

and define $\prod^{\sharp} : T^*M \to TM$ and 2-vector $\prod$ by

$$\prod^{\sharp}(\alpha) := (\omega^{\flat})^{-1}(\alpha^P), \quad \prod(\alpha, \beta) := \omega((\omega^{\flat})^{-1}(\alpha^P), (\omega^{\flat})^{-1}(\beta^P)).$$

Since $\mathfrak{D}$ is integrable, $\prod$ satisfies (2.4) and then define a Poisson structure on M (cf. Lichnerowicz [14]). Obviously $\prod$ is ω-compatible. We also obtain (ii) the same as (i). $\square$

For a presymplectic manifold M, a Poisson bracket is defined on not hole $C^{\infty}(M)$ but the subalgebra $C^{\infty}(M/N) := \{f \in C^{\infty}(M);\ f$ is constant on any leaf of the null foliation$\}$. The reason why a Poisson bracket cannot be defined is that ω is degenerate and lack of information about null directions. A sufficient condition to define a Poisson bracket on $C^{\infty}(M)$ is (i) of Lemma 3.1, and (ii) of Lemma 3.1 is a condition that a Poisson structure determines a closed 2-form.

By Lemma 3.1, we may prove Theorem 1.1 as follows. (i) and (ii) of Theorem 1.1 is easy to check by the definition of symplectic structures and (iii) is reduced to (i) or (ii) by Lemma 3.1. The latter half of Theorem 1.1 follows from Lemma 2.2.

Remark 3.2. (i) Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be foliations on a symplectic manifold (M, Ω) such that $\mathcal{F}_1 \oplus \mathcal{F}_2 = M$. Assume that $\Omega|_{D_1(x)}$ is nondegenerate for all $x \in M$. In this situation,

$$\Omega(D_1(x), D_2(x)) = \{0\}, \quad \forall x \in M,$$

does not hold in general. For example, let

$$M = T^4 = \mathbb{R}^4/\mathbb{Z}^4, \quad \Omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4,$$

where x^1, x^2, x^3, x^4 is the coordinates on T^2 induced from the canonical coordinates of $\mathbb{R}^4$. If we define

$$D_1 = \operatorname{span}\{\partial_1, \partial_2\}, \quad D_2 = \operatorname{span}\{\partial_1 + \partial_3, \partial_2 + \partial_4\},$$

where $\partial_i := \partial/\partial x^i (i = 1, 2, 3, 4)$, then

- $\Omega|_{D_1}$ and $\Omega|_{D_2}$ are nondegenerate;
- $\Omega(X, Y) = 1$ for $X := \partial_2 \in L_1$ and $Y := \partial_1 + \partial_3 \in L_2$.

(ii) For a decomposition of symplectic structure Ω on M by presymplectic structures ω_1 and ω_2, i.e., $\Omega = \omega_1 + \omega_2 (\mathcal{F}_{\omega_1} \oplus \mathcal{F}_{\omega_2} = M)$, there does not necessarily exist symplectic manifolds $(M_1, \omega_1), (M_2, \omega_2)$ such that $M = M_1 \times M_2$.

4. Proof of Theorem 1.2

Let $\mathfrak{D}_1 = \{(D_1)_p\}_{p\in G_2/H}$ and $\mathfrak{D}_2 = \{(D_2)_p\}_{p\in G_2/H}$ be non-trivial invariant integrable distributions on a compact Kählerian homogeneous space G_2/H on which G_2 acts effectively. To prove Theorem 1.2, we may show the following: Let $L_i(o)$ denote the leaf of $\mathfrak{D}_i$ through the origin $o \in G_2/H$ $(i = 1, 2)$. If both $L_1(o)$ and $L_2(o)$ are compact, then

$$T_o(G_2/H) = (D_1)_o \oplus (D_2)_o \quad \text{(direct sum)},$$

cannot hold. In order to show this, we first see the following:

Proposition 4.1. *Let $\mathfrak{D} = \{D_p\}_{p\in G_2/H}$ be a non-trivial, invariant, integrable distribution on a compact Kählerian homogeneous space G_2/H on which G_2 acts effectively. Suppose that the leaf $L(o)$ of $\mathfrak{D}$ through o is compact. Then, there exists a compact, connected subgroup K of maximal rank of G_2, which satisfies two conditions*:

(i) $H \subsetneq K \subsetneq G_2$;

(ii) $D_p = d\tau_g(T_o(K/H))$ *for any point* $p = \tau_g(o) \in G_2/H$,

where τ_g, $g \in G_2$, *denotes a transformation of* G_2/H *defined by* $\tau_g(aH) := gaH$ *for* $aH \in G_2/H$.

Proof. Let $\mathfrak{g}_2 := \mathrm{Lie}(G_2)$ and $\mathfrak{h} := \mathrm{Lie}(H)$. By Theorem 2 in Matsushima [16, p. 56], H is the centralizer of a torus of G_2 and then

$$\operatorname{rank} G_2 = \operatorname{rank} H. \tag{4.1}$$

It is clear that G_2/H is a reductive homogeneous space and so there exists a vector subspace $\mathfrak{m}$ of $\mathfrak{g}_2$, which satisfies two conditions:

$$\text{(a) } \mathfrak{g}_2 = \mathfrak{h} \oplus \mathfrak{m} \quad \text{and} \quad \text{(b) } \mathrm{Ad}(h)\mathfrak{m} \subset \mathfrak{m} \ \text{ for all } h \in H.$$

Hereafter, we identify $T_o(G_2/H)$ with $\mathfrak{m}$ and assume D_o is a subspace of $\mathfrak{m}$. Note that $\{0\} \subsetneq D_o \subsetneq \mathfrak{m}$ because $\mathfrak{D}$ is non-trivial. Since $\mathfrak{D} = \{D_p\}_{p \in G_2/H}$ is invariant and integrable, we are able to deduce that

$$\mathrm{Ad}(h)D_o \subset D_o \text{ for all } h \in H, \ [D_o, D_o] \subset \mathfrak{h} \oplus D_o. \tag{4.2}$$

Define a subspace $\mathfrak{k}$ of $\mathfrak{g}_2$ by

$$\mathfrak{k} := \mathfrak{h} \oplus D_o.$$

Then (4.2) implies that $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}_2$. Let K be the connected Lie subgroup of G_2 with $\mathrm{Lie}(K) = \mathfrak{k}$. By virtue of $\mathfrak{h} \subsetneq \mathfrak{k} \subsetneq \mathfrak{g}_2$, we see that $H_0 \subsetneq K \subsetneq G_2$. Here H_0 denotes the identity component of H. Therefore, the first condition (i) holds;

$$H \subsetneq K \subsetneq G_2, \tag{4.3}$$

because H is connected (see Matsushima [16, Theorem 2, p. 56]). Since H is closed in G_2 and since the identity mapping of K into G_2 is continuous, H is closed in K. From now on, we are going to verify that $L(o)$ coincides with K/H. In terms of $\mathfrak{k} = \mathfrak{h} \oplus D_o$, we conclude that $T_o(K/H) = D_o$. Accordingly, the second condition (ii) holds;

$$D_p = d\tau_g(T_o(K/H)) \text{ for any point } p = \tau_g(o) \in G_2/H,$$

because $\mathfrak{D} = \{D_p\}_{p \in G_2/H}$ is invariant. Hence both $L(o)$ and K/H are connected integral manifolds of $\mathfrak{D}$ through $o \in G_2/H$. It follows that

$$K/H \subset L(o),$$

because $L(o)$ is the leaf of $\mathfrak{D}$ through o. Note that K/H is an open submanifold of $L(o)$. Let us show that the converse inclusion is also true. Since $L(o)$ is compact, it is a complete Riemannian manifold with respect to the metric induced from G_2/H. Thus, any point $x \in L(o)$ can be jointed to o by a broken geodesic of $L(o)$. Since K/H is a complete totally geodesic submanifold of $L(o)$, we confirm that K/H contains this broken geodesic. Hence $x \in K/H$. This shows $L(o) \subset K/H$. We have verified that

$$L(o) = K/H.$$

Since both $L(o)$ and H are compact (see Matsushima [16, Theorem 2, p. 56] again), K is also compact. It follows from (4.1) and (4.3) that

$$\operatorname{rank} G_2 = \operatorname{rank} K,$$

namely, K is a subgroup of maximal rank of G_2. Consequently, we have proved Proposition 4.1. □

It is known that a compact Kählerian homogeneous space G_2/H is either

$$G_2/U(2) \quad \text{or} \quad G_2/T^2 \tag{4.4}$$

(cf. Bordemann et al. [7, p. 643]), where T^2 is a maximal torus of G_2. It is also known that a connected closed subgroup K of maximal rank of G_2 is isomorphic to one of the following:

$$SU(3), \quad SU(2) \times SU(2), \quad U(2), \quad T^2 \tag{4.5}$$

(cf. Borel and de Siebenthal [8, p. 219]). Taking (4.4) and (4.5) into consideration, we will demonstrate that "$T_o(G_2/H) = (D_1)_o \oplus (D_2)_o$" cannot hold in two cases $G_2/H = G_2/U(2)$ and $G_2/H = G_2/T^2$.

Remark 4.2. Two actions of G_2 on $G_2/U(2)$ and on G_2/T^2 are almost effective but not effective. However, we hereafter assume that two actions of G_2 on $G_2/U(2)$ and on G_2/T^2 are effective. Because, if necessary, one may consider two effective actions of G_2/Z on $(G_2/Z)/(U(2)/Z)$ and on $(G_2/Z)/(T^2/Z)$ instead of them, and consider $SU(3)/Z$, $(SU(2) \times SU(2))/Z$, $U(2)/Z$, T^2/Z instead of (4.5), where Z denotes the center of G_2.

Case $G_2/H = G_2/U(2)$**.** Let $D_1 = \{(D_1)_p\}_{p \in G_2/U(2)}$ and $D_2 = \{(D_2)_p\}_{p \in G_2/U(2)}$ be two, non-trivial, invariant, integrable distributions on $G_2/U(2)$ such that both $L_1(o)$ and $L_2(o)$ are compact, where $L_i(o)$ is the leaf of $\mathfrak{D}_i$ through the origin $o \in G_2/U(2)$. A connected closed subgroup K of maximal rank of G_2 satisfying $U(2) \subsetneq K$ is either $SU(3)$ or $SU(2) \times SU(2)$ by (4.5). Therefore, Proposition 4.1 assures that

$$(D_i)_o = T_o(SU(3)/U(2)) \quad \text{or} \quad T_o((SU(2) \times SU(2))/U(2)).$$

Since $\dim SU(3)/U(2) = 4$, $\dim (SU(2) \times SU(2))/U(2) = 2$ and $\dim G_2/U(2) = 10$, we conclude that "$T_o(G_2/U(2)) = (D_1)_o \oplus (D_2)_o$" cannot hold.

Case $G_2/H = G_2/T^2$**.** Let $\mathfrak{D}_1 = \{(D_1)_p\}_{p \in G_2/T^2}$ and $\mathfrak{D}_2 = \{(D_2)_p\}_{p \in G_2/T^2}$ be two, non-trivial, invariant, integrable distributions on G_2/T^2 such that both $L_1(o)$ and $L_2(o)$ are compact, where $L_i(o)$ is the leaf of $\mathfrak{D}_i$ through the origin $o \in G_2/T^2$. A connected closed subgroup K of maximal rank of G_2 satisfying $T^2 \subsetneq K$ is either of the $SU(3)$, $SU(2) \times SU(2)$ or $U(2)$ by (4.5). By Proposition 4.1, we comprehend that

$$(D_i)_o = T_o(SU(3)/T^2), \quad T_o((SU(2) \times SU(2))/T^2) \text{ or } T_o(U(2)/T^2).$$

It is natural that $\dim SU(3)/T^2 = 6$, $\dim (SU(2) \times SU(2))/T^2 = 4$, $\dim U(2)/T^2 = 2$, and $\dim G_2/T^2 = 12$. If "$T_o(G_2/T^2) = (D_1)_o \oplus (D_2)_o$" holds, then $(D_1)_o = (D_2)_o = T_o(SU(3)/T^2)$. However, such a case cannot occur by Lemma 4.3 below. Therefore, "$T_o(G_2/T^2) = (D_1)_o \oplus (D_2)_o$" cannot hold and Theorem 1.2 holds.

Lemma 4.3. *Let* $\mathfrak{D}' = \{(D')_p\}_{p \in G_2/T^2}$ *be an invariant distribution on* G_2/T^2 *such that* $T_o(G_2/T^2) = (D')_o \oplus D_o$. *Then* $\mathfrak{D}'$ *cannot be integrable. Here* $\mathfrak{D}$ *is the invariant integrable distribution* $\{D_p\}_{p \in G_2/T^2}$ $= \{d\tau_g(T_o(SU(3)/T^2))\}_{\tau_g(o) \in G_2/T^2}$.

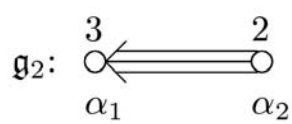
Proof. Let $\mathfrak{t}^2 := \mathrm{Lie}(T^2)$. Then $\mathfrak{t}^2$ is a maximal abelian subalgebra of $\mathfrak{g}_2$, and $\mathfrak{g}_2$ is decomposed into the direct sum

$$\mathfrak{g}_2 = \mathfrak{t}^2 \oplus \bigoplus_{j=1}^{6} V_{\theta_j},$$

where V_{θ_j} is an ad $\mathfrak{t}^2$-invariant subspace of $\mathfrak{g}_2$ and $\dim V_{\theta_j} = 2$ for each $j \in \{1, \ldots, 6\}$; moreover, there exist a basis $\{X_{1,j}, X_{2,j}\}_{j=1}^{6}$ of $\bigoplus_{j=1}^{6} V_{\theta_j}$ and roots $\theta_j : \mathfrak{t}^2 \to \mathbb{R}$, which satisfy

$$\mathrm{ad}(Z)Y = (X_{1,j} \;\; X_{2,j}) \begin{pmatrix} 0 & -\theta_j(Z) \\ \theta_j(Z) & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix},$$

for any $Z \in \mathfrak{t}^2$ and $Y = \lambda X_{1,j} + \mu X_{2,j} \in V_{\theta_j}$ (cf. Toda and Mimura [21, Chapter 5]). Let $\{\alpha_1, \alpha_2\}$ denote the set of simple roots in $\Delta^+ := \{\theta_j\}_{j=1}^{6}$. We assume that the Dynkin diagram of $\{\alpha_1, \alpha_2\}$ is as follows:



Note that in this case Δ^+ is described as

$$\Delta^+ = \{\alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_2\}.$$

In the setting, we are going to prove that the distribution $\mathfrak{D}'$ cannot be integrable. Let us identify $T_o(G_2/T^2)$ with $\bigoplus_{j=1}^{6} V_{\theta_j}$. Since $\mathfrak{D}'$ and $\mathfrak{D}$

are invariant, $(D')_o$ and D_o are given by combinations of V_{θ_j}, $\theta_j \in \Delta^+$. Direct computation enables us to see that the combination of roots in Δ^+ generating $\mathfrak{su}(3) = \mathrm{Lie}(SU(3))$ is one of the following:

(1) $\{\alpha_1, \alpha_1 + \alpha_2, \alpha_2\}$,

(2) $\{\alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$,

(3) $\{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2\}$,

(4) $\{\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$,

(5) $\{3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_2\}$.

Accordingly, one of the following five cases only occurs

(1) $D_o = \mathfrak{su}(3)/\mathfrak{t}^2 = V_{\alpha_1} \oplus V_{\alpha_1+\alpha_2} \oplus V_{\alpha_2}$,

(2) $D_o = \mathfrak{su}(3)/\mathfrak{t}^2 = V_{\alpha_1} \oplus V_{\alpha_1+\alpha_2} \oplus V_{2\alpha_1+\alpha_2}$,

(3) $D_o = \mathfrak{su}(3)/\mathfrak{t}^2 = V_{\alpha_1} \oplus V_{2\alpha_1+\alpha_2} \oplus V_{3\alpha_1+\alpha_2}$,

(4) $D_o = \mathfrak{su}(3)/\mathfrak{t}^2 = V_{\alpha_1+\alpha_2} \oplus V_{2\alpha_1+\alpha_2} \oplus V_{3\alpha_1+2\alpha_2}$,

(5) $D_o = \mathfrak{su}(3)/\mathfrak{t}^2 = V_{3\alpha_1+\alpha_2} \oplus V_{3\alpha_1+2\alpha_2} \oplus V_{\alpha_2}$.

Since $\mathfrak{D} = \{D_p\}_{p\in G_2/T^2}$ is integrable, it must satisfy $[D_o, D_o] \subset \mathfrak{t}^2 \oplus D_o$. Therefore, the last case (5) only occurs. In this case, $(D')_o$ is as follows:

$$(D')_o = V_{\alpha_1} \oplus V_{\alpha_1+\alpha_2} \oplus V_{2\alpha_1+\alpha_2},$$

because $T_o(G_2/T^2) = (D')_o \oplus D_o$ and $(D')_o$ is given by a combination of V_{θ_j}. Thus, it is impossible for $(D')_o$ to satisfy $[(D')_o, (D')_o] \subset \mathfrak{t}^2 \oplus (D')_o$ in this case. Consequently, $\mathfrak{D}' = \{(D')_p\}_{p\in G_2/T^2}$ cannot be integrable. □

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