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NEW RESULTS OF STEFFENSEN TYPE INEQUALITIES FOR STIELTJES INTEGRAL

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Abstract

Several generalizations and refinements of some Steffensen type inequalities for Stieltjes integral are established.

1. Introduction

The well-known classical Steffensen's inequality [6] states:

Theorem 1. *Let f and g be integrable functions defined on* $[a, b]$ *with f decreasing, and for each* $t \in [a, b]$, $0 \leq g(t) \leq 1$. *Then*

$$\int_{b-\lambda}^{b} f(t)dt \leq \int_{a}^{b} f(t)g(t)dt \leq \int_{a}^{a+\lambda} f(t)dt, \tag{1}$$

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where $\lambda = \int_a^b g(t)dt.$

In [1], the Steffensen's inequality have been generalized in Stieltjes integral type.

Recall that if any one of the two Stieltjes integrals

$$\int_a^b f(t)dg(t) \quad \text{and} \quad \int_a^b g(t)df(t)$$

exists, then the other one also exists and one has

$$\int_a^b f(t)dg(t) + \int_a^b g(t)df(t) = f(b)g(b) - f(a)g(a).$$

In what follows, we always assume that μ is a finite continuous strictly increasing function defined on $[a, b]$. This assures that the inverse μ^{-1} exists and is also a finite continuous strictly increasing function defined on $[\mu(a), \mu(b)]$.

If f is a monotonic function defined on $[a, b]$, then the Stieltjes integral of f with respect to μ, i.e.,

$$\int_a^b f(t)d\mu(t),$$

clearly exists. For brevity, we would like to agree on saying that f is μ-integrable if and only if $\int_a^b f(t)d\mu(t)$ exists.

In [1], the following Stieltjes type inequality for Stieltjes integral was proved:

Theorem 2. *Let f and g be μ-integrable functions defined on $[a, b]$ with f decreasing, and for each $t \in [a, b]$, $0 \le g(t) \le 1$. Then*

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^b f(t)d\mu(t) \le \int_a^b f(t)g(t)d\mu(t) \le \int_a^{\mu^{-1}(\mu(a)+\lambda)} f(t)d\mu(t), \qquad (2)$$

where $\lambda = \int_a^b g(t)d\mu(t).$

If we take $\mu(t) = t$ in Theorem 2, then the inequality (2) reduced to the inequality (1). In [2], a generalization of Theorem 2 was obtained as follows:

Theorem 3. *Let* f, g, *and* h *be* μ-*integrable functions defined on* $[a, b]$ *with f decreasing, and for each* $t \in [a, b]$, $0 \leq g(t) \leq h(t)$. *Then*

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} f(t)h(t)d\mu(t) \leq \int_{a}^{b} f(t)g(t)d\mu(t) \leq \int_{a}^{\mu^{-1}(\mu(a)+\lambda)} f(t)h(t)d\mu(t), \quad (3)$$

provided that there exists $\lambda \in [0, \mu(b) - \mu(a)]$ *such that*

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} h(t)d\mu(t) = \int_{a}^{b} g(t)d\mu(t) = \int_{a}^{\mu^{-1}(\mu(a)+\lambda)} h(t)d\mu(t).$$

If we take $\mu(t) = t$ in Theorem 3, then the inequality (3) reduced to the corrected version of a Mercer's result in [4] which has been proved in [3] and [7] in different ways as follows:

Theorem 4. *Let* f, g, *and h be integrable functions defined on* $[a, b]$ *with f decreasing, and for each* $t \in [a, b]$, $0 \leq g(t) \leq h(t)$. *Then*

$$\int_{b-\lambda}^{b} f(t)h(t)dt \leq \int_{a}^{b} f(t)g(t)dt \leq \int_{a}^{a+\lambda} f(t)h(t)dt,$$

provided that there exists $\lambda \in [0, b-a]$ *such that*

$$\int_{a}^{a+\lambda} h(t)dt = \int_{a}^{b} g(t)dt = \int_{b-\lambda}^{b} h(t)dt.$$

Theorem 5. *Let* f, g, h, *and* k *be a positive* μ-*integrable functions defined on* $[a, b]$ *with* f / k *decreasing, and* $0 \leq g \leq h$. *Then*

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} f(t)h(t)d\mu(t) \leq \int_{a}^{b} f(t)g(t)d\mu(t) \leq \int_{a}^{\mu^{-1}(\mu(a)+\lambda)} f(t)h(t)d\mu(t), \quad (4)$$

provided that there exists $\lambda \in [0, \mu(b) - \mu(a)]$ *such that*

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} h(t)k(t)d\mu(t) = \int_a^b g(t)k(t)d\mu(t) = \int_a^{\mu^{-1}(\mu(a)+\lambda)} h(t)k(t)d\mu(t).$$

Motivated by [7] and [5], in this paper, we will give a further discussion on Steffensen type inequalities for Stieltjes integral. Several generalizations and refinements of some above mentioned inequalities are obtained.

2. Main Results

We first provide a useful lemma.

Lemma. *Let* f, g, *and* h *be* μ-*integrable functions defined on* $[a, b]$. *Suppose also that* $\lambda \in [0, \mu(b) - \mu(a)]$ *is a real number such that*

$$\int_a^{\mu^{-1}(\mu(a)+\lambda)} h(t)d\mu(t) = \int_a^b g(t)d\mu(t) = \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} h(t)d\mu(t).$$

Then

$$\int_a^b f(t)g(t)d\mu(t) = \int_a^{\mu^{-1}(\mu(a)+\lambda)} (f(t)h(t) - [f(t) - f(\mu^{-1}(\mu(a)+\lambda))][h(t) - g(t)])d\mu(t)$$

$$+ \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} [f(t) - f(\mu^{-1}(\mu(a)+\lambda))]g(t)d\mu(t), \tag{5}$$

and

$$\int_a^b f(t)g(t)d\mu(t) = \int_a^{\mu^{-1}(\mu(b)-\lambda)} [f(t) - f(\mu^{-1}(\mu(b)-\lambda))]g(t)d\mu(t)$$

$$+ \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} (f(t)h(t) - [f(t) - f(\mu^{-1}(\mu(b)-\lambda))]$$

$$\times [h(t) - g(t)])d\mu(t). \tag{6}$$

Proof. Observe that $\lambda \in [0, \mu(b) - \mu(a)]$, we get $\mu(a) \le \mu(a) + \lambda \le \mu(b)$ and $\mu(a) \le \mu(b) - \lambda \le \mu(b)$, which also imply that $a \le \mu^{-1}(\mu(a) + \lambda) \le b$ and $a \le \mu^{-1}(\mu(b) - \lambda) \le b$.

By assumption

$$\int_a^{\mu^{-1}(\mu(a)+\lambda)} h(t)d\mu(t) = \int_a^b g(t)d\mu(t),$$

it is not difficult to deduce that

$$\int_a^{\mu^{-1}(\mu(a)+\lambda)} (f(t)h(t) - [f(t) - f(\mu^{-1}(\mu(a)+\lambda))][h(t) - g(t)])d\mu(t)$$

$$- \int_a^b f(t)g(t)d\mu(t)$$

$$= \int_a^{\mu^{-1}(\mu(a)+\lambda)} (f(t)h(t) - f(t)g(t) - [f(t) - f(\mu^{-1}(\mu(a)+\lambda))][h(t) - g(t)])d\mu(t)$$

$$+ \int_a^{\mu^{-1}(\mu(a)+\lambda)} f(t)g(t)d\mu(t) - \int_a^b f(t)g(t)d\mu(t)$$

$$= \int_a^{\mu^{-1}(\mu(a)+\lambda)} f(\mu^{-1}(\mu(a)+\lambda))[h(t) - g(t)])d\mu(t) - \int_{\mu^{-1}(\mu(a)+\lambda)}^b f(t)g(t)d\mu(t)$$

$$= f(\mu^{-1}(\mu(a)+\lambda))(\int_a^{\mu^{-1}(\mu(a)+\lambda)} h(t)d\mu(t) - \int_a^{\mu^{-1}(\mu(a)+\lambda)} g(t)d\mu(t))$$

$$- \int_{\mu^{-1}(\mu(a)+\lambda)}^b f(t)g(t)d\mu(t)$$

$$= f(\mu^{-1}(\mu(a)+\lambda))(\int_a^b g(t)d\mu(t) - \int_a^{\mu^{-1}(\mu(a)+\lambda)} g(t)d\mu(t))$$

$$- \int_{\mu^{-1}(\mu(a)+\lambda)}^b f(t)g(t)d\mu(t)$$

$$= f(\mu^{-1}(\mu(a)+\lambda))\int_{\mu^{-1}(\mu(a)+\lambda)}^b g(t)d\mu(t) - \int_{\mu^{-1}(\mu(a)+\lambda)}^b f(t)g(t)d\mu(t)$$

$$= \int_{\mu^{-1}(\mu(a)+\lambda)}^b [f(\mu^{-1}(\mu(a)+\lambda)) - f(t)]g(t)d\mu(t),$$

which is just the desired identity (5) asserted by the Lemma.

Now by assumption

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} h(t)d\mu(t) = \int_{a}^{b} g(t)d\mu(t),$$

we can also deduce that

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} (f(t)h(t) - [f(t) - f(\mu^{-1}(\mu(b) - \lambda))][h(t) - g(t)])d\mu(t)$$

$$- \int_{a}^{b} f(t)g(t)d\mu(t)$$

$$= \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} (f(t)h(t) - f(t)g(t) - [f(t) - f(\mu^{-1}(\mu(b) - \lambda))][h(t) - g(t)])d\mu(t)$$

$$+ \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} f(t)g(t)d\mu(t) - \int_{a}^{b} f(t)g(t)d\mu(t)$$

$$= \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} f(\mu^{-1}(\mu(b) - \lambda))[h(t) - g(t)])d\mu(t) - \int_{a}^{\mu^{-1}(\mu(b)-\lambda)} f(t)g(t)d\mu(t)$$

$$= f(\mu^{-1}(\mu(b) - \lambda))(\int_{\mu^{-1}(\mu(a)+\lambda)}^{b} h(t)d\mu(t) - \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} g(t)d\mu(t))$$

$$- \int_{a}^{\mu^{-1}(\mu(b)-\lambda)} f(t)g(t)d\mu(t)$$

$$= f(\mu^{-1}(\mu(b) - \lambda))(\int_{a}^{b} g(t)d\mu(t) - \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} g(t)d\mu(t))$$

$$- \int_{a}^{\mu^{-1}(\mu(b)-\lambda)} f(t)g(t)d\mu(t)$$

$$= f(\mu^{-1}(\mu(b) - \lambda))\int_{a}^{\mu^{-1}(\mu(b)-\lambda)} g(t)d\mu(t) - \int_{a}^{\mu^{-1}(\mu(b)-\lambda)} f(t)g(t)d\mu(t)$$

$$= \int_a^{\mu^{-1}(\mu(b)-\lambda)} [f(\mu^{-1}(\mu(b)-\lambda)) - f(t)]g(t)d\mu(t),$$

which is just the desired identity (6) asserted by the Lemma.

Theorem 6. *Let f, g, and h be μ-integrable functions defined on $[a, b]$ with f decreasing, and $0 \le g \le h$. Suppose also that $\lambda \in [0, \mu(b) - \mu(a)]$ is a real number such that*

$$\int_a^{\mu^{-1}(\mu(a)+\lambda)} h(t)d\mu(t) = \int_a^b g(t)d\mu(t) = \int_{\mu^{-1}(\mu(b)-\lambda)}^b h(t)d\mu(t).$$

Then we have the following inequalities:

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^b f(t)h(t)d\mu(t)$$

$$\le \int_{\mu^{-1}(\mu(b)-\lambda)}^b (f(t)h(t) - [f(t) - f(\mu^{-1}(\mu(b)-\lambda))][h(t) - g(t)])d\mu(t)$$

$$\le \int_a^b f(t)g(t)d\mu(t)$$

$$\le \int_a^{\mu^{-1}(\mu(a)+\lambda)} (f(t)h(t) - [f(t) - f(\mu^{-1}(\mu(a)+\lambda))][h(t) - g(t)])d\mu(t)$$

$$\le \int_a^{\mu^{-1}(\mu(a)+\lambda)} f(t)h(t)d\mu(t). \tag{7}$$

Proof. Since f is decreasing on $[a, b]$ and $0 \le g \le h$, we can conclude that

$$\int_a^{\mu^{-1}(\mu(a)+\lambda)} [f(t) - f(\mu^{-1}(\mu(a)+\lambda))][h(t) - g(t)]d\mu(t) \ge 0, \tag{8}$$

$$\int_{\mu^{-1}(\mu(a)+\lambda)}^b [f(t) - f(\mu^{-1}(\mu(a)+\lambda))]g(t)d\mu(t) \le 0, \tag{9}$$

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b}[f(t)-f(\mu^{-1}(\mu(b)-\lambda))][h(t)-g(t)])d\mu(t) \le 0, \tag{10}$$

and

$$\int_{a}^{\mu^{-1}(\mu(b)-\lambda)}[f(t)-f(\mu^{-1}(\mu(b)-\lambda))]g(t)d\mu(t) \ge 0. \tag{11}$$

By (5), (8), and (9), we find that

$$\int_a^b f(t)g(t)d\mu(t)$$

$$\le \int_a^{\mu^{-1}(\mu(a)+\lambda)}(f(t)h(t)-[f(t)-f(\mu^{-1}(\mu(a)+\lambda))][h(t)-g(t)])d\mu(t)$$

$$\le \int_a^{\mu^{-1}(\mu(a)+\lambda)} f(t)h(t)d\mu(t), \tag{12}$$

and by (6), (10), and (11), we get

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} f(t)h(t)d\mu(t)$$

$$\le \int_{\mu^{-1}(\mu(b)-\lambda)}^{b}(f(t)h(t)-[f(t)-f(\mu^{-1}(\mu(b)-\lambda))][h(t)-g(t)])d\mu(t)$$

$$\le \int_a^b f(t)g(t)d\mu(t). \tag{13}$$

Consequently, inequalities (7) follow by combining the inequalities (12) and (13). The proof is completed.

In particular, if we take $h(t) \equiv 1$, then we obtain the following refinement of Steffensen type inequality (2).

Corollary 1. *Let f and g be* μ*-integrable functions defined on* $[a, b]$ *with f decreasing, and for each* $t \in [a, b]$, $0 \leq g(t) \leq 1$. *Then*

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} f(t)d\mu(t)$$

$$\leq \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} (f(t) - [f(t) - f(\mu^{-1}(\mu(b) - \lambda))][1 - g(t)])d\mu(t)$$

$$\leq \int_{a}^{b} f(t)g(t)d\mu(t)$$

$$\leq \int_{a}^{\mu^{-1}(\mu(a)+\lambda)} (f(t) - [f(t) - f(\mu^{-1}(\mu(a) + \lambda))][1 - g(t)])d\mu(t)$$

$$\leq \int_{a}^{\mu^{-1}(\mu(a)+\lambda)} f(t)d\mu(t), \tag{14}$$

where $\lambda = \int_{a}^{b} g(t)d\mu(t)$.

Theorem 7. *Let* f, g, h, *and* ψ *be* μ*-integrable functions defined on* $[a, b]$ *with* f *decreasing, and* $0 \leq \psi(t) \leq g(t) \leq h(t) - \psi(t)$ *for* $t \in [a, b]$. *Suppose also that* $\lambda \in [0, \mu(b) - \mu(a)]$ *is a real number such that*

$$\int_{a}^{\mu^{-1}(\mu(a)+\lambda)} h(t)d\mu(t) = \int_{a}^{b} g(t)d\mu(t) = \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} h(t)d\mu(t).$$

Then we have the following inequalities:

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} f(t)h(t)d\mu(t) + \int_{a}^{b} |[f(t) - f(\mu^{-1}(\mu(b) - \lambda))]\psi(t)| d\mu(t)$$

$$\leq \int_{a}^{b} f(t)g(t)d\mu(t)$$

$$\leq \int_{a}^{\mu^{-1}(\mu(a)+\lambda)} f(t)h(t)d\mu(t) - \int_{a}^{b} |[f(t) - f(\mu^{-1}(\mu(a) + \lambda)]\psi(t)| d\mu(t). \tag{15}$$

Proof. Since f is decreasing on $[a, b]$ and $0 \le \psi(t) \le g(t) \le h(t) - \psi(t)$ for $t \in [a, b]$, we have

$$\int_a^{\mu^{-1}(\mu(a)+\lambda)} [f(t) - f(\mu^{-1}(\mu(a) + \lambda))][h(t) - g(t)]d\mu(t)$$

$$+ \int_{\mu^{-1}(\mu(a)+\lambda)}^b [f(\mu^{-1}(\mu(a) + \lambda)) - f(t)]g(t)d\mu(t)$$

$$= \int_a^{\mu^{-1}(\mu(a)+\lambda)} |f(t) - f(\mu^{-1}(\mu(a) + \lambda))|[h(t) - g(t)]d\mu(t)$$

$$+ \int_{\mu^{-1}(\mu(a)+\lambda)}^b |f(\mu^{-1}(\mu(a) + \lambda)) - f(t)|g(t)d\mu(t)$$

$$\ge \int_a^{\mu^{-1}(\mu(a)+\lambda)} |f(t) - f(\mu^{-1}(\mu(a) + \lambda))|\psi(t)d\mu(t)$$

$$+ \int_{\mu^{-1}(\mu(a)+\lambda)}^b |f(\mu^{-1}(\mu(a) + \lambda)) - f(t)|\psi(t)d\mu(t)d\mu(t)$$

$$= \int_a^b |[f(t) - f(\mu^{-1}(\mu(b) - \lambda))]\psi(t)|d\mu(t), \tag{16}$$

and

$$\int_a^{\mu^{-1}(\mu(b)-\lambda)} [f(t) - f(\mu^{-1}(\mu(b) - \lambda))]g(t)d\mu(t)$$

$$+ \int_{\mu^{-1}(\mu(b)-\lambda)}^b [f(\mu^{-1}(\mu(b) - \lambda)) - f(t)][h(t) - g(t)]d\mu(t)$$

$$= \int_a^{\mu^{-1}(\mu(b)-\lambda)} |f(t) - f(\mu^{-1}(\mu(b) - \lambda))|g(t)d\mu(t)$$

$$+ \int_{\mu^{-1}(\mu(b)-\lambda)}^b |f(\mu^{-1}(\mu(b) - \lambda)) - f(t)|[h(t) - g(t)]d\mu(t)$$

$$\geq \int_a^{\mu^{-1}(\mu(b)-\lambda)} |f(t) - f(\mu^{-1}(\mu(b) - \lambda))| \psi(t) d\mu(t)$$

$$+ \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} |f(\mu^{-1}(\mu(b) - \lambda)) - f(t)| \psi(t) d\mu(t)$$

$$= \int_a^b |[f(t) - f(\mu^{-1}(\mu(b) - \lambda))] \psi(t)| d\mu(t). \tag{17}$$

By combining the identities (5), (6) and inequalities (16), (17), we get the inequality (15) asserted by Theorem 7. Thus the proof is completed.

Corollary 2. *Let f and g be* μ*-integrable functions defined on* $[a, b]$ *with f decreasing, and* $0 \leq M \leq g(t) \leq 1 - M$ *for* $t \in [a, b]$*. Then*

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} f(t) d\mu(t) + M \int_a^b |[f(t) - f(\mu^{-1}(\mu(b) - \lambda))]| d\mu(t)$$

$$\leq \int_a^b f(t) g(t) d\mu(t)$$

$$\leq \int_a^{\mu^{-1}(\mu(a)+\lambda)} f(t) d\mu(t) - M \int_a^b |[f(t) - f(\mu^{-1}(\mu(a) + \lambda)]| d\mu(t), \tag{18}$$

where $\lambda = \int_a^b g(t) d\mu(t).$

Remark 1. Clearly, the inequality (18) is a refinement and generalization of Steffensen type inequality (2). Indeed, in its special case when $M = 0$, the inequality (18) would reduce to Steffensen type inequality (2).

Now, we would like to give a general result on a considerably improved version of Steffensen type inequality (2) by introducing the additional parameters λ_1 and λ_2.

Theorem 8. *Let f and g be μ-integrable functions defined on $[a, b]$ with f decreasing, and $0 \le M \le g(t) \le 1 - M$ for $t \in [a, b]$. Also let $0 \le \lambda_1 \le \int_a^b g(t)d\mu(t) \le \lambda_2 \le \mu(b) - \mu(a)$. Then we have the following inequalities*:

$$\int_{\mu^{-1}(\mu(b)-\lambda_1)}^{b} f(t)d\mu(t) + f(b)\left(\int_a^b g(t)d\mu(t) - \lambda_1\right)$$

$$+ M\int_a^b \left|f(t) - f\left(\mu^{-1}\left(\mu(b) - \int_a^b g(t)d\mu(t)\right)\right)\right|d\mu(t)$$

$$\le \int_a^b f(t)g(t)d\mu(t)$$

$$\le \int_a^{\mu^{-1}(\mu(a)+\lambda_2)} f(t)d\mu(t) - f(b)\left(\lambda_2 - \int_a^b g(t)d\mu(t)\right)$$

$$- M\int_a^b \left|f(t) - f\left(\mu^{-1}\left(\mu(a) + \int_a^b g(t)d\mu(t)\right)\right)\right|d\mu(t). \tag{19}$$

Proof. By assumption, it is clear that

$$\mu(a) \le \mu(a) + \lambda_1 \le \mu(a) + \int_a^b g(t)d\mu(t) \le \mu(a) + \lambda_2 \le \mu(b),$$

and

$$\mu(a) \le \mu(b) - \lambda_2 \le \mu(b) - \int_a^b g(t)d\mu(t) \le \mu(b) - \lambda_1 \le \mu(b),$$

which also imply that

$$a \le \mu^{-1}(\mu(a) + \lambda_1) \le \mu^{-1}\left(\mu(a) + \int_a^b g(t)d\mu(t)\right) \le \mu^{-1}(\mu(a) + \lambda_2) \le b,$$

and

$$a \le \mu^{-1}(\mu(b) - \lambda_2) \le \mu^{-1}\left(\mu(b) - \int_a^b g(t)d\mu(t)\right) \le \mu^{-1}(\mu(b) - \lambda_1) \le b.$$

By direct computation, we get

$$\int_a^b f(t)g(t)d\mu(t) - \int_a^{\mu^{-1}(\mu(a)+\lambda_2)} f(t)d\mu(t) + f(b)\left(\lambda_2 - \int_a^b g(t)d\mu(t)\right)$$

$$= \int_a^b f(t)g(t)d\mu(t) - \int_a^{\mu^{-1}(\mu(a)+\lambda_2)} f(t)d\mu(t) + \int_a^{\mu^{-1}(\mu(a)+\lambda_2)} f(b)d\mu(t)$$

$$- \int_a^b f(b)g(t)d\mu(t)$$

$$= \int_a^b [f(t) - f(b)]g(t)d\mu(t) - \int_a^{\mu^{-1}(\mu(a)+\lambda_2)} [f(t) - f(b)]d\mu(t)$$

$$\leq \int_a^b [f(t) - f(b)]g(t)d\mu(t) - \int_a^{\mu^{-1}(\mu(a)+\int_a^b g(t)d\mu(t))} [f(t) - f(b)]d\mu(t), \quad (20)$$

where the last inequality follows from the assumption that

$$a \leq \mu^{-1}\left(\mu(a) + \int_a^b g(t)d\mu(t)\right) \leq \mu^{-1}(\mu(a) + \lambda_2) \leq b,$$

and

$$f(t) - f(b) \geq 0 \text{ for } t \in [a, b].$$

On the other hand, since the function $f(t) - f(b)$ is μ-integrable and decreasing on $[a, b]$, thus by using Corollary 2 with the following substitution:

$$f(t) \mapsto f(t) - f(b),$$

in (18), we find that

$$\int_a^b [f(t) - f(b)]g(t)d\mu(t) - \int_a^{\mu^{-1}(\mu(a)+\int_a^b g(t)d\mu(t))} [f(t) - f(b)]d\mu(t)$$

$$\leq -M\int_a^b \left|f(t) - f(b) - f\left(\mu^{-1}\left(\mu(a) + \int_a^b g(t)d\mu(t)\right)\right)\right| d\mu(t). \quad (21)$$

By combining the inequalities (20) and (21), we obtain

$$\int_a^b f(t)g(t)d\mu(t) - \int_a^{\mu^{-1}(\mu(a)+\lambda_2)} f(t)d\mu(t) + f(b)(\lambda_2 - \int_a^b g(t)d\mu(t))$$

$$\leq -M\int_a^b |f(t) - f(b) - f(\mu^{-1}(\mu(a) + \int_a^b g(t)d\mu(t)))|d\mu(t),$$

which is the second inequality in the assertion (19) of Theorem 8.

Similarly, we can also prove that

$$\int_a^b f(t)g(t)d\mu(t) - \int_{\mu^{-1}(\mu(b)-\lambda_1)}^b f(t)d\mu(t) - f(b)(\int_a^b g(t)d\mu(t) - \lambda_1)$$

$$\geq \int_a^b [f(t) - f(b)]g(t)d\mu(t) - \int_{\mu^{-1}(\mu(b)-\int_a^b g(t)d\mu(t))}^b [f(t) - f(b)]g(t)d\mu(t)$$

$$\geq M\int_a^b |f(t) - f(\mu^{-1}(\mu(b) - \int_a^b g(t)d\mu(t))|d\mu(t),$$

which implies the first inequality in the assertion (19) of Theorem 8. Thus completes the proof of Theorem 8.

Remark 2. It is clear that Steffensen type inequality (2) would follow as a special case of the inequality (19) when

$$M = 0 \quad \lambda_1 = \lambda_2.$$

Moreover, it is worth noticing that the inquality (19) is stronger than Steffensen type inequality (2) if $f(b) \geq 0$.

Theorem 9. *Let h be a positive μ-integrable functions on $[a, b]$ and f, g be μ-integrable functions on $[a, b]$ such that $\frac{f}{h}$ is decreasing, and $0 \leq g(t) \leq 1, t \in [a, b]$. Suppose also that $\lambda \in [0, \mu(b) - \mu(a)]$ is a real number such that*

$$\int_a^{\mu^{-1}(\mu(a)+\lambda)} h(t)d\mu(t) = \int_a^b h(t)g(t)d\mu(t) = \int_{\mu^{-1}(\mu(b)-\lambda)}^b h(t)d\mu(t).$$

Then we have the following inequalities:

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} f(t)d\mu(t)$$

$$\leq \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} (f(t) - [\frac{f(t)}{h(t)} - \frac{f(\mu^{-1}(\mu(b)-\lambda))}{h(\mu^{-1}(\mu(b)-\lambda))}]h(t)[1-g(t)])d\mu(t)$$

$$\leq \int_{a}^{b} f(t)g(t)d\mu(t)$$

$$\leq \int_{a}^{\mu^{-1}(\mu(a)+\lambda)} (f(t) - [\frac{f(t)}{h(t)} - \frac{f(\mu^{-1}(\mu(a)+\lambda))}{h(\mu^{-1}(\mu(a)+\lambda))}]h(t)[1-g(t)])d\mu(t)$$

$$\leq \int_{a}^{\mu^{-1}(\mu(a)+\lambda)} h(t)d\mu(t).$$

Proof. Take the substitutions $f(t) \mapsto \frac{f(t)}{h(t)}$ and $g(t) \mapsto h(t)g(t)$ in Theorem 6.

Theorem 10. *Let* k *be a positive* μ*-integrable functions on* $[a, b]$ *and* f, g, h *be* μ*-integrable functions on* $[a, b]$ *such that* $\frac{f}{k}$ *is decreasing, and* $0 \leq g(t) \leq h(t)$, $t \in [a, b]$. *Suppose also that* $\lambda \in [0, \mu(b) - \mu(a)]$ *is a real number such that*

$$\int_{a}^{\mu^{-1}(\mu(a)+\lambda)} h(t)k(t)d\mu(t) = \int_{a}^{b} g(t)k(t)d\mu(t) = \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} h(t)k(t)d\mu(t).$$

Then we have the following inequalities:

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} f(t)h(t)d\mu(t)$$

$$\leq \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} (f(t)h(t) - [\frac{f(t)}{k(t)} - \frac{f(\mu^{-1}(\mu(b)-\lambda))}{k(\mu^{-1}(\mu(b)-\lambda))}]k(t)[h(t)-g(t)])d\mu(t)$$

$$\leq \int_a^b f(t)g(t)d\mu(t)$$

$$\leq \int_a^{\mu^{-1}(\mu(a)+\lambda)} (f(t)h(t) - [\frac{f(t)}{k(t)} - \frac{f(\mu^{-1}(\mu(a)+\lambda))}{k(\mu^{-1}(\mu(a)+\lambda))}]k(t)[h(t) - g(t)])d\mu(t)$$

$$\leq \int_a^{\mu^{-1}(\mu(a)+\lambda)} f(t)h(t)d\mu(t).$$

Proof. Take the substitutions $f(t) \mapsto \frac{f(t)}{k(t)}$, $g(t) \mapsto k(t)g(t)$, and $h(t) \mapsto k(t)h(t)$ in Theorem 6.

Theorem 11. *Let h be a positive μ-integrable functions on $[a, b]$ and f, g, ψ be μ-integrable functions on $[a, b]$ such that $\frac{f}{h}$ is decreasing, and $0 \leq \psi(t) \leq g(t) \leq 1 - \psi(t)$ for $t \in [a, b]$. Suppose also that $\lambda \in [0, \mu(b) - \mu(a)]$ is a real number such that*

$$\int_a^{\mu^{-1}(\mu(a)+\lambda)} h(t)d\mu(t) = \int_a^b h(t)g(t)d\mu(t) = \int_{\mu^{-1}(\mu(b)-\lambda)}^b h(t)d\mu(t).$$

Then we have the following inequalities:

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^b f(t)d\mu(t) + \int_a^b |[\frac{f(t)}{h(t)} - \frac{f(\mu^{-1}(\mu(b)-\lambda))}{h(\mu^{-1}(\mu(b)-\lambda))}]h(t)\psi(t)|d\mu(t)$$

$$\leq \int_a^b f(t)g(t)d\mu(t)$$

$$\leq \int_a^{\mu^{-1}(\mu(a)+\lambda)} f(t)d\mu(t) - \int_a^b |[\frac{f(t)}{h(t)} - \frac{f(\mu^{-1}(\mu(a)+\lambda))}{h(\mu^{-1}(\mu(a)+\lambda))}]h(t)\psi(t)|d\mu(t).$$

Proof. Take the substitutions $f(t) \mapsto \frac{f(t)}{h(t)}$, $g(t) \mapsto h(t)g(t)$ and $\psi(t) \mapsto h(t)\psi(t)$ in Theorem 7.

Theorem 12. *Let* k *be a positive* μ-*integrable functions on* $[a, b]$ *and* f, g, h, ψ *be* μ-*integrable functions on* $[a, b]$ *such that* $\frac{f}{k}$ *is decreasing, and* $0 \le \psi(t) \le g(t) \le h(t) - \psi(t)$ *for* $t \in [a, b]$. *Suppose also that* $\lambda \in [0, \mu(b) - \mu(a)]$ *is a real number such that*

$$\int_a^{\mu^{-1}(\mu(a)+\lambda)} h(t)k(t)d\mu(t) = \int_a^b g(t)k(t)d\mu(t) = \int_{\mu^{-1}(\mu(b)-\lambda)}^b h(t)k(t)d\mu(t).$$

Then we have the following inequalities:

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^b f(t)h(t)d\mu(t) + \int_a^b |[\frac{f(t)}{k(t)} - \frac{f(\mu^{-1}(\mu(b)-\lambda))}{k(\mu^{-1}(\mu(b)-\lambda))}]k(t)\psi(t)|d\mu(t)$$

$$\le \int_a^b f(t)g(t)d\mu(t)$$

$$\le \int_a^{\mu^{-1}(\mu(a)+\lambda)} f(t)h(t)d\mu(t) - \int_a^b |[\frac{f(t)}{k(t)} - \frac{f(\mu^{-1}(\mu(a)+\lambda))}{k(\mu^{-1}(\mu(a)+\lambda))}]k(t)\psi(t)|d\mu(t). \tag{22}$$

Proof. Take the substitutions $f(t) \mapsto \frac{f(t)}{k(t)}$, $g(t) \mapsto k(t)g(t)$, $h(t) \mapsto k(t)h(t)$ and $\psi(t) \mapsto k(t)\psi(t)$ in Theorem 7.

Remark 3. Clearly, the inequality (22) is a refinement and generalization of Steffensen type inequality (4).

Corollary 3. *Let* f, g, h, ψ *be* μ-*integrable functions on* $[a, b]$ *with* f *is decreasing, and* $0 \le \psi(t) \le g(t) \le h(t) - \psi(t)$ *for* $t \in [a, b]$. *Suppose also that* $\lambda \in [0, \mu(b) - \mu(a)]$ *is a real number such that*

$$\int_a^{\mu^{-1}(\mu(a)+\lambda)} h(t)d\mu(t) = \int_a^b g(t)d\mu(t) = \int_{\mu^{-1}(\mu(b)-\lambda)}^b h(t)d\mu(t).$$

Then we have the following inequalities:

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} f(t)h(t)d\mu(t) + \int_a^b |[f(t) - f(\mu^{-1}(\mu(b) - \lambda))]\psi(t)| d\mu(t)$$

$$\leq \int_a^b f(t)g(t)d\mu(t)$$

$$\leq \int_a^{\mu^{-1}(\mu(a)+\lambda)} f(t)h(t)d\mu(t) - \int_a^b |[f(t) - f(\mu^{-1}(\mu(a) + \lambda))]\psi(t)| d\mu(t). \quad (23)$$

Remark 4. Clearly, the inequality (23) is a refinement and generalization of Steffensen type inequality (3).

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