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ABOUT ASYMPTOTIC NONSENSITIVITY PROPERTY IN ABSTRACT PROGRAMMED PROBLEM OF THE GAME CONTROL

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Abstract

Abstract maximin problem is considered. The statement variant with constraints of asymptotic character is considered. In main part, the case of «moment» constraints is investigated. In this case, the comparison of different variants of constraint weakening are realized. The conditions of asymptotic nonsensitive under weakening of constraints part are obtained.

Introduction

In the following, some abstract variant of the next informative control problem is considered. Now, we discuss this informative problem in the simplest form. In addition, we fix two linear control systems on the same time interval [0, 1]:

$$\dot{y} = u(t)b_1(t) + \widetilde{b}_1(t), \quad \dot{z} = v(t)b_2(t) + \widetilde{b}_2(t). \tag{0.1}$$

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We suppose that the phase space of the above-mentioned control system is n-dimensional. So, in (0.1), y and z are n-dimensional vectors. We suppose that $u = u(\cdot)$ and $v = v(\cdot)$ are programmed controls defined on the «pointer» $[0, 1[$. Moreover, in (0.1), $b_1 = b_1(\cdot)$, $\tilde{b}_1 = \tilde{b}_1(\cdot)$, $b_2 = b_2(\cdot)$, and $\tilde{b}_2 = \tilde{b}_2(\cdot)$ are n-vector-functions (the functions with n-dimensional values). We have a continuous real-valued function $f_0 = f_0(\cdot, \cdot)$ of two variables of n-dimensional space. Suppose that the cost is defined as $f_0(y(1), z(1))$. In addition, we suppose that the first player I strives to minimize this cost and the second player II strives to maximize this cost. We consider the maximin problem for criterion $f_0(y(1), z(1))$. In addition, the concrete choice of controls $u = u(\cdot)$ and $v = v(\cdot)$ must realize with validity of constraint. We consider the case when these constraints include impulse and «moment» components. We admit a weakening of «moment» constraints both for player I and for player II. As a result, the cost maximin can be changed. We consider the corresponding asymptotic of the cost maximin under employment of more precise constraints.

In addition, we consider different types of the weakening of the «moment» constraints. We will establish some variant of asymptotic non-sensitivity under weakening of constraints part. In the following investigation, approach connected with extension of the initial problem is used. We note constructions of [1]-[4]. For constructing «asymptotic» solutions, the corresponding generalized controls are used (see [1]-[4]). In class of generalized controls, we can consider the game problem with precise constraints (standard constraints). In addition, in many cases, the corresponding generalized maximin defines asymptotics of realizable values of maximin under weakened constraints.

We note that, in [1]-[3], for constructing of generalized control problems measure-valued mappings (controls-measures) are used; in addition, this approach is applied for control problems with geometric constraints. In impulse control, we note original approach of Krasovskii connected with employment of distributions. This approach was basis of many following investigations of control problems with impulse constraints.

In our investigation, for generalized elements in maximin problem, vector finitely additive measures (FAM) are used. The corresponding applications can be connected with game control in linear systems under constraints of impulse character; in addition, linear systems with discontinuity in coefficients under controls can be considered.

1. General Definitions and Designations

As usual, we apply quantors and propositional connectives; $\triangleq$ is equality by definition. We use the term a family for a set all elements of which are sets too. By $\mathcal{P}(H)$ (by $\mathcal{P}'(H)$), we denote the family of all (of all non-empty) subsets of a set H. By B^A, we denote the set of all mappings from a set A into a set B; if $f \in B^A$ and $C \in \mathcal{P}(A)$, then $f^1(C) \triangleq \{f(x) : x \in C\} \in \mathcal{P}(B)$ is the image of the set C under operation of the mapping f (if $C \in \mathcal{P}'(A)$, then $f^1(C) \in \mathcal{P}'(B)$).

By $\mathbb{R}$, we denote real line and suppose that $\mathbb{N} \triangleq \{1; 2; \ldots\}$; if $m \in \mathbb{N}$, then $\overline{1, m} \triangleq \{i \in \mathbb{N} \mid i \leq m\} \in \mathcal{P}'(\mathbb{N})$. Let $\tau_{\mathbb{R}}$ be the usual topology of $\mathbb{R}$ generated by the metric-modulus. As usual, we suppose that elements of $\mathbb{N}$ not are sets. Using this stipulation, for any set T and number $m \in \mathbb{N}$, we apply T^m instead of $T^{\overline{1, m}}$; so, T^m is the set of all processions

$$(t_i)_{i \in \overline{1, m}} : \overline{1, m} \to T.$$

Then, under $m \in \mathbb{N}$, $\mathbb{R}^m$ is m-dimensional arithmetic space; in addition, strongly speaking, we consider elements of $\mathbb{R}^m$ as mapping from $\overline{1, m}$ into $\mathbb{R}$. In general, in space of real-valued functions, we consider linear operations, product, and order as pointwise. Of course, this stipulation extend on spaces $\mathbb{R}^m$, where $m \in \mathbb{N}$. Here, we consider m-dimensional vector as a function from $\overline{1, m}$ into $\mathbb{R}$. For the definiteness, always, we equip finite-dimensional arithmetic space with the norm of the following type: If $m \in \mathbb{N}$ and $x = (x_i)_{i \in \overline{1, m}} \in \mathbb{R}^m$, then

$$\|x\|^{(m)} \triangleq \max_{i \in \overline{1, m}} |x_i| \in [0, \infty[.$$

Of course, under $m \in \mathbb{N}$, we use $\| \cdot \|^{(m)}$ for the mapping

$$x \mapsto \|x\|^{(m)} : \mathbb{R}^m \to [0, \infty[. \tag{1.1}$$

By $\tau_{\mathbb{R}}^{(m)}$, we denote the usual topology of coordinate-wise convergence in $\mathbb{R}^m$. Of course, $\tau_{\mathbb{R}}^{(m)}$ is generated by $\| \cdot \|^{(m)}$. Moreover, we introduce the set $\mathbb{R}_+^m$ of all vectors $(x_i)_{i \in \overline{1, m}} \in \mathbb{R}^m$ with the property $0 \leq x_j \; \forall j \in \overline{1, m}$. In the following, the employment of the norm (1.1) is more suitable. We introduce two types of neighbourhoods in $(\mathbb{R}^m, \tau_{\mathbb{R}}^{(m)})$, where $m \in \mathbb{N}$. Namely, for any $m \in \mathbb{N}$, $S \in \mathcal{P}(\mathbb{R}^m)$, and $\zeta \in \,]0, \infty[$, we suppose that

$$O_\zeta^{(m)}[S] \triangleq \{x \in \mathbb{R}^m \mid \exists\, s \in S : \|x - s\|^{(m)} < \zeta\} \in \tau_{\mathbb{R}}^{(m)}, \tag{1.2}$$

and under any $M \in \mathcal{P}(\overline{1, m})$,

$$\widehat{O}_\zeta^{(m)}[S \mid M] \triangleq \{ (x_i)_{i \in \overline{1, m}} \in \mathbb{R}^m \mid \exists (s_i)_{i \in \overline{1, m}} \in S : (x_j = s_j \quad \forall j \in M) \;\&$$

$$(|x_j - s_j| < \zeta \quad \forall j \in \overline{1, m} \setminus M)\}; \tag{1.3}$$

of course, from (1.2) and (1.3), we obtain that

$$\widehat{O}_\zeta^{(m)}[S \mid M] \subset O_\zeta^{(m)}[S], \tag{1.4}$$

(if $M = \varnothing$, then $\widehat{O}_\zeta^{(m)}[S \mid \varnothing] = O_\zeta^{(m)}[S]$).

Some elements of topology

For any topological space (TS) (X, τ) and a set $A \in \mathcal{P}(X)$, we suppose that $\mathrm{cl}(A, \tau)$ is closure of A in (X, τ) and $\tau|_A$ is the topology of A considered as subspace of (X, τ) (of course, $\tau_A \triangleq \{A \cap G : G \in \tau\}$). We use the neighbourhoods definition of [5, Chapter I]: If (X, τ) is a TS and $x \in X$, then by filter base [5, Chapter I]

$$N_\tau^0(x) \triangleq \{G \in \tau \mid x \in G\} \in \mathcal{P}'(\tau),$$

is defined filter of neighbourhoods

$$N_\tau(x) \triangleq \{Y \in \mathcal{P}(X) \mid \exists G \in N_\tau^0(x) : G \subset Y\}.$$

We consider nets as the triplets of the following type: $(D, \preceq, h)$ is a net in a set Y in the case when $(D, \preceq)$ is a nonempty directed set [6, Chapter II] and $f \in Y^D$. We apply the usual Moore-Smith convergence: If (X, τ) is a TS, $(D, \preceq, f)$ is a net, and $x \in X$, then

$$\left((D, \preceq, f) \xrightarrow{\tau} x\right) \overset{\text{def}}{\Leftrightarrow} (\forall H \in N_\tau(x)\ \exists d_1 \in D \quad \forall d_2 \in D\ (d_1 \preceq d_2) \Rightarrow (f(d_2) \in H)). \tag{1.5}$$

For any net, the corresponding associated filter is defined. Namely, for any net $(D, \preceq, f)$ in a set Z,

$$(Z - \text{ass})[D; \preceq; f] \triangleq \{A \in \mathcal{P}(Z) \mid \exists d \in D \quad \forall \delta \in D \quad (d \preceq \delta) \Rightarrow (f(\delta) \in A)\}. \tag{1.6}$$

Of course, in (1.5) and (1.6), we have the standard definitions of general topology; see [5]-[7].

For any TS $(\mathbf{T}, \mathbf{t})$, $\mathbf{T} \neq \varnothing$, and $k \in \mathbb{N}$, we suppose that $\otimes^k [\mathbf{t}]$ is a natural topology of the set $\mathbf{T}^k$ corresponding to standard product of k samples of TS $(\mathbf{T}, \mathbf{t})$; see [7].

Attraction sets

In the following, we use the notion of attraction set (AS) corresponding to [8]-[12]. Namely, for any nonempty set X, a TS (Y, τ), $Y \neq \varnothing$, a mapping $g \in Y^X$, and a family $\mathcal{X} \in \mathcal{P}'(\mathcal{P}(X))$, we denote by (**as**) $[X;\, Y;\, \tau;\, g;\, \mathcal{X}]$ the set of all $y \in Y$ for which there exists a net $(D, \preceq, h)$ in X with the properties

$$(\mathcal{X} \subset (X - \text{ass})[D; \preceq; f]) \quad \& \quad ((D, \preceq, g \circ h) \xrightarrow{\tau} y). \tag{1.7}$$

In connection with given general definition, we recall the known statement of [9, Proposition 3.3.1] concerning to a sequential realization of AS. Now, we note only one particular case sufficient for almost all following constructions: If (in (1.7)) $\mathcal{X}$ is a directed family with a countable base (see [9, (3.3.17)]) and (Y, τ) is TS with the first axiom of countability, then, for exhausting realization of AS (**as**) $[X;\, Y;\, \tau;\, g;\, \mathcal{X}]$, it is sufficient to use sequences in X, see (1.7).

For any set X, we define the family

$$\beta[X] \triangleq \{\beta \in \mathcal{P}'(\mathcal{P}(X)) \mid \forall B_1 \in \mathcal{B} \quad \forall B_2 \in \mathcal{B} \quad \exists\, B_3 \in \mathcal{B} : B_3 \subset B_1 \cap B_2\},$$

of all directed subfamilies of $\mathcal{P}(X)$; then $\beta_0[X] \triangleq \{\mathfrak{B} \in \beta[X] \mid \varnothing \notin \mathfrak{B}\}$ is the family of all filter bases of the set X.

We note the following useful representation of AS in the case of directed family defining constraints of asymptotic character: If X is a nonempty set, (Y, τ) is a TS, $g \in Y^X$, and $\mathcal{B} \in \beta[X]$, then

$$(\mathbf{as})[X;\, Y;\, \tau;\, g;\, \mathcal{B}] = \bigcap_{B \in \mathcal{B}} \mathrm{cl}(g^1(B), \tau). \tag{1.8}$$

In the following constructions, (1.8) is sufficient for all our goals. If $(U, \tau_1), U \neq \varnothing,$ and $(V, \tau_2), V \neq \varnothing,$ are TS, then by definition $C(U, \tau_1, V, \tau_2)$ is the set of all (τ_1, τ_2)-continuous mappings from U into V. For any TS (T, τ), $T \neq \varnothing$, we suppose $\mathbb{C}(T, \tau) \triangleq C(T, \tau, \mathbb{R}, \tau_{\mathbb{R}})$.

2. Finitely Additive Measures as Generalized Elements

We follows to approach of [8]-[12] connected with employment of finitely additive measures (FAM) in extension constructions for abstract control problems. In given section, we recall some required notions of FAM theory. And what is more, vector FAM will be required.

For simplicity, in designation, now we fix a nonempty set E and a semialgebra $\mathcal{L}$ of subsets of E (in following sections, we will use the corresponding symbols instead of E and $\mathcal{L}$). So, now we have the measurable space $(E, \mathcal{L})$ with a semialgebra of sets. In the following, we use designations of [9], [11], and [13].

The cone $(\text{add})_+[\mathcal{L}]$ of (all) real-valued nonnegative FAM on $\mathcal{L}$ generates the space $\mathbb{A}(\mathcal{L})$ of real-valued FAM on $\mathcal{L}$ with the bounded variation; see [9, p. 39].

In addition, the linear space $\mathbb{A}(\mathcal{L})$ is equipped with the strong norm defined (for any FAM of $\mathbb{A}(\mathcal{L})$) as total variation. Of course, $(\text{add})_+[\mathcal{L}]$ $\subset \mathbb{A}(\mathcal{L})$.

In the following, we use step function and stratum functions. These functions are elements of the Banach space $\mathbb{B}(E)$ of bounded real-valued functions on E with the sup-norm $\|\cdot\|$ (see [14, Chapter IV]). The linear manifold $B_0(E, \mathcal{L})$ is defined as linear span of the set $\{\chi_L : L \in \mathcal{L}\}$ of all indicators of sets of $\mathcal{L}$ (in addition, for $\Lambda \in \mathcal{L}$ $\chi_\Lambda \in \mathbb{R}^{\mathcal{L}}$ is defined by the rule

$$(\chi_\Lambda(x) \triangleq 1 \;\forall\, x \in \Lambda) \quad \& \quad (\chi_\Lambda(y) \triangleq 0 \;\forall y \in E \setminus \Lambda);$$

subsets of E and its indicators are identified). The closure of $B_0(E, \mathcal{L}) \in$ $\mathcal{P}'(\mathbb{B}(E))$ in the topology generated by sup-norm $\|\cdot\|$ is denoted by $B(E, \mathcal{L})$; of course, $B_0(E, \mathcal{L}) \subset B(E, \mathcal{L})$. Then, $B(E, \mathcal{L})$ considered as a subspace of $(\mathbb{B}(E), \|\cdot\|)$ is a Banach space for with the topological conjugate space $B^*(E, \mathcal{L})$ with the traditional (for the Banach space theory) norm is isometrically isomorphic to $\mathbb{A}(\mathcal{L})$ with strong norm-variation (this property is analogous to similar supposition of [14, Chapter IV] for measurable spaces with algebra of sets). In terms of the simplest integration construction in [8, Subsection 3.4], the corresponding isometric isomorphism is defined by natural rule

$$\mu \mapsto \left(\int_E f d\mu\right)_{f \in B(E, \mathcal{L})} : \mathbb{A}(\mathcal{L}) \to B^*(E, \mathcal{L}).$$

Of course, for duality $(B(E, \mathcal{L}), \mathbb{A}(\mathcal{L}))$, the «usual» *-weak topology $\tau_*(\mathcal{L})$ of $\mathbb{A}(\mathcal{L})$ is defined (see [8, Subsection 3.4]). Then $(\mathbb{A}(\mathcal{L}), \tau_*(\mathcal{L}))$ is a locally convex σ-compactum. Along with $\tau_*(\mathcal{L})$, we use topology $\tau_0(\mathcal{L})$ [8, (4.2.9)] of subspace of Tichonoff power of real line $\mathbb{R}$ with discrete topology under employment of $\mathcal{L}$ as the index set. So, $(\mathbb{A}(\mathcal{L}), \tau_0(\mathcal{L}))$ is the subspace of the above-mentioned Tichonoff power. As in [8, Subsection 4.2], we introduce the topologies

$$\left(\tau_*^+(\mathcal{L}) \triangleq \tau_*(\mathcal{L})|_{(\mathrm{add})_+[\mathcal{L}]}\right) \quad \& \quad \left(\tau_0^+(\mathcal{L}) \triangleq \tau_0(\mathcal{L})|_{(\mathrm{add})_+[\mathcal{L}]}\right), \tag{2.1}$$

of the cone $(\mathrm{add})_+[\mathcal{L}]$ for which [8, (4.2.12)] $\tau_*^+(\mathcal{L}) \subset \tau_0^+(L)$. The nonnegative cone of $B_0(E, \mathcal{L})$ (of $B(E, \mathcal{L})$) is denoted as $B_0^+(E, \mathcal{L})$ (as $B^+(E, \mathcal{L})$); see [8, p. 66].

In this section, we fix $r \in \mathbb{N}$. By $B_{0,r}^+[E; \mathcal{L}]$ and $B_r^+[E; \mathcal{L}]$, we denote the sets of all processions

$$(f_i)_{i \in \overline{1, r}} : \overline{1, r} \to B_0^+(E, \mathcal{L}) \text{ and } (\tilde{f}_i)_{i \in \overline{1, r}} : \overline{1, r} \to B^+(E, \mathcal{L}),$$

respectively; of course, $B_{0,r}^+[E; \mathcal{L}] \subset B_r^+[E; \mathcal{L}]$. As a corollary, $B_{0,r}^+[E; \mathcal{L}] = B_0^+(E; \mathcal{L})^r$ and $B_r^+[E; \mathcal{L}] = B^+(E, \mathcal{L})^r$. Moreover, $(\mathrm{add})_r^+[\mathcal{L}] \triangleq (\mathrm{add})_+[\mathcal{L}]^r$ is the set of all processions

$$(\mu_i)_{i \in \overline{1, r}} : \overline{1, r} \to (\text{add})_+[\mathcal{L}].$$

We use the standard variant of the equipment of the (nonempty) set $(\text{add})_r^+[\mathcal{L}]$ with a topology. For this, we apply (2.1). So, $\otimes^r [\tau_*^+(\mathcal{L})]$ and $\otimes^r [\tau_0^+(\mathcal{L})]$ are the required topologies of $(\text{add})_r^+[\mathcal{L}]$ for which

$$\otimes^r [\tau_*^+(\mathcal{L})] \subset \otimes^r [\tau_0^+(\mathcal{L})]. \tag{2.2}$$

Then, triplet $((\text{add})_r^+[\mathcal{L}], \otimes^r [\tau_*^+(\mathcal{L})], \otimes^r [\tau_0^+(\mathcal{L})])$ is a bitopological space in the sense of [16]. Of course, both topologies in (2.2) realize Hausdorff spaces.

We fix a nonzero FAM $\eta \in (\text{add})_+[\mathcal{L}]$ until the end of present section. In addition, $\eta(E) \neq 0$. Using definition of [9, Chapter 3], we introduce the cone

$$(\text{add})^+[\mathcal{L}; \eta] \triangleq \{\mu \in (\text{add})_+[\mathcal{L}] \mid \forall L \in \mathcal{L} \quad (\eta(L) = 0) \Rightarrow (\mu(L) = 0)\}, \tag{2.3}$$

of all weakly absolutely continuous (with respect to η) nonnegative FAM on $\mathcal{L}$. In connection with (2.3), see [17] also. For our goals, the nonempty set

$$(\text{add})_r^+[\mathcal{L}; \eta] \triangleq (\text{add})^+[\mathcal{L}; \eta]^r, \tag{2.4}$$

of all weakly absolutely continuous nonnegative vector FAM is very essential. Of course, (2.4) is the set of all processions

$$(\mu_i)_{i \in \overline{1, r}} : \overline{1, r} \to (\text{add})^+[\mathcal{L}; \eta].$$

The density property

For any $f \in B(E, \mathcal{L})$, we define by $f * \eta$ indefinite integral of f with respect to η; $f * \eta \in \mathbb{A}(\mathcal{L})$. So, $f * \eta$ is η-integral as a set function. Of course, for $f \in B^+(E, \mathcal{L})$, $f * \eta \in (\text{add})^+[\mathcal{L}; \eta]$. And what is more, from statements of [8] and [9], we obtain that

$$(\mathrm{add})^+[\mathcal{L};\ \eta] = \mathrm{cl}(\{f * \eta : f \in B_0^+(E,\ \mathcal{L})\},\ \tau_*(\mathcal{L}))$$
$$= \mathrm{cl}(\{f * \eta : f \in B_0^+(E,\ \mathcal{L})\},\ \tau_0(\mathcal{L}))$$
$$= \mathrm{cl}(\{f * \eta : f \in B_0^+(E,\ \mathcal{L})\},\ \tau_*^+(\mathcal{L}))$$
$$= \mathrm{cl}(\{f * \eta : f \in B_0^+(E,\ \mathcal{L})\},\ \tau_0^+(\mathcal{L})). \tag{2.5}$$

We note that (2.5) has a vector analogs. In this connection, we recall that $\otimes^r\ [\tau_*^+(\mathcal{L})]$ and $\otimes^r\ [\tau_0^+(\mathcal{L})]$ are topologies of the set $(\mathrm{add})_r^+[\mathcal{L}]$, for which (2.2) is valid. Using (2.5), we obtain that

$$(\mathrm{add})_r^+[\mathcal{L};\ \eta] = \mathrm{cl}(\{(f_i * \eta)_{i \in \overline{1,\ r}} : (f_i)_{i \in \overline{1,\ r}} \in B_{0,\ r}^+[E;\ \mathcal{L}]\},\ \otimes^r\ [\tau_*^+(\mathcal{L})])$$
$$= \mathrm{cl}(\{(f_i * \eta)_{i \in \overline{1,\ r}} : (f_i)_{i \in \overline{1,\ r}} \in B_{0,\ r}^+[E;\ \mathcal{L}]\},\ \otimes^r\ [\tau_0^+(\mathcal{L})]). \tag{2.6}$$

3. Integral Constraints and their Relaxations (General Statements)

In this section, we use the space $(E,\ \mathcal{L},\ \eta)$ of the previous section. Now, we recall statements of [13]. Fix a number $\mathbf{n} \in \mathbb{N}$ and a mapping

$$S : \overline{1,\ \mathbf{n}} \times \overline{1,\ r} \to B(E,\ \mathcal{L}). \tag{3.1}$$

So, in this section, we fix (stratum) matriciant S with components $S_{i,\ j}$, $i \in \overline{1,\ \mathbf{n}},\ j \in \overline{1,\ r}.$ Strongly speaking, $\forall\, i \in \overline{1,\ \mathbf{n}},\ \forall\, j \in \overline{1,\ r}$

$$S_{i,\ j} \triangleq S(i,\ j) \in B(E,\ \mathcal{L}).$$

Moreover, in this section, we fix a closed (with respect to topology $\tau_{\mathbb{R}}^{(\mathbf{n})}$) set $\mathbb{Y} \in \mathcal{P}'(\mathbb{R}^{\mathbf{n}})$. Finally, we fix a closed (in $\tau_{\mathbb{R}}^{(r)}$) bounded set $\mathbb{F} \in \mathcal{P}'(\mathbb{R}_+^r)$; we recall that $\mathbb{R}_+^r$ is the nonnegative cone of the space $\mathbb{R}^r$. From the compactness property of $\mathbb{F}$, we obtain that the number

$$\mathbf{c}_{\mathbb{F}} \triangleq \max_{(x_i)_{i \in \overline{1, r}} \in \mathbb{F}} \sum_{i=1}^{r} x_i \in [0, \infty[, \tag{3.2}$$

is defined correctly. We consider the set

$$(r - \text{adm})[\mathbb{F} \mid E; \mathcal{L}; \eta] \triangleq \left\{ (f_i)_{i \in \overline{1, r}} \in B_{0, r}^{+}[E; \mathcal{L}] \mid \left(\int_E f_i d\eta \right)_{i \in \overline{1, r}} \in \mathbb{F} \right\} \in \mathcal{P}'(B_{0, r}^{+}[E; \mathcal{L}]). \tag{3.3}$$

In this section, we consider the following integral constraints:

$$\left(\sum_{j=1}^{r} \int_E S_{i, j} f_j d\eta \right)_{i \in \overline{1, \mathbf{n}}} \in \mathbb{Y}, \tag{3.4}$$

on the choice of $(f_j)_{j \in \overline{1, r}} \in (r - \text{adm})[\mathbb{F} \mid E; \mathcal{L}; \eta]$. In connection with (3.4), we consider the corresponding set of admissible elements; but, it is useful to consider this notion in more general form: If $\mathbf{Y} \in \mathcal{P}(\mathbb{R}^{\mathbf{n}})$, then we suppose that

$$((\mathbf{n}, r) - \text{ADM})[\mathbb{F} \mid E; \mathcal{L}; \eta; \mathbf{Y}; S]$$

$$\triangleq \left\{ (f_i)_{i \in \overline{1, r}} \in (r - \text{adm})[\mathbb{F} \mid E; \mathcal{L}; \eta] \mid \left(\sum_{j=1}^{r} \int_E S_{i, j} f_j d\eta \right)_{i \in \overline{1, \mathbf{n}}} \in \mathbf{Y} \right\}. \tag{3.5}$$

Of course, in (3.5), we can consider the case $\mathbf{Y} = \mathbb{Y}$ supposing the set of admissible (in strong sense) elements; we can consider the variant for which $\mathbf{Y}$ in (3.5) is defined as a neighbourhood of $\mathbb{Y}$ too. In last case, we can use different variants of neighbourhoods (see (1.2), (1.3)).

For the set of generalized elements, we use

$$\Sigma_r[E; \mathcal{L}; \eta; \mathbb{F}] \triangleq \{(\mu_j)_{j \in \overline{1, r}} \in (\text{add})_r^{+}[\mathcal{L}; \eta] \mid (\mu_j(E))_{j \in \overline{1, r}} \in \mathbb{F}\}$$

$$\in (\otimes^r [\tau_*^{+}(\mathcal{L})] - \text{comp})[(\text{add})_r^{+}[\mathcal{L}]], \tag{3.6}$$

(see [13, (4.24),(4.26)]). From (3.2) and (3.3), we obtain that

$$\sum_{i=1}^{r}(f_i * \eta)(E) = \sum_{i=1}^{r}\int_E f_i d\eta \leqslant \mathbf{c}_{\mathbb{F}} \ \ \forall (f_i)_{i\in\overline{1,r}} \in (r - \mathrm{adm})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta]. \tag{3.7}$$

In connection with (3.7), we note some addition connected with (2.6). Namely, from (2.6), (3.3), and (3.6), we obtain that (see [8, (3.4.10)])

$$(f_i * \eta)_{i\in\overline{1,r}} \in \Sigma_r[E;\ \mathcal{L};\ \eta;\ \mathbb{F}] \ \ \forall (f_i)_{i\in\overline{1,r}} \in (r - \mathrm{adm})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta]. \tag{3.8}$$

From (3.3) and (3.8), we obtain that (3.6) is a nonempty set. With regard to (3.8), we introduce the mapping $\mathbb{I}$ in the form of

$$(f_i)_{i\in\overline{1,r}} \mapsto (f_i * \eta)_{i\in\overline{1,r}} : (r - \mathrm{adm})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta] \to \Sigma_r[E;\ \mathcal{L};\ \eta;\ \mathbb{F}]. \tag{3.9}$$

Using (3.6) and (3.9), we obtain under $\tau = \otimes^r[\tau_*^+(\mathcal{L})]$ and $\tau = \otimes^r[\tau_0^+(\mathcal{L})]$ that set $\mathrm{cl}((r - \mathrm{adm})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta],\ \tau)$ is defined correctly. With employment of reasoning similar [13, Proposition 4.2], we obtain that

$$\Sigma_r[E;\ \mathcal{L};\ \eta;\ \mathbb{F}] = \mathrm{cl}(\mathbb{I}^1((r - \mathrm{adm})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta]),\ \otimes^r\ [\tau_*^+(\mathcal{L})])$$

$$= \mathrm{cl}(\mathbb{I}^1((r - \mathrm{adm})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta]),\ \otimes^r\ [\tau_0^+(\mathcal{L})]). \tag{3.10}$$

In connection with (3.10), we realize the passage to subspace of $(\mathrm{add})_r^+[\mathcal{L}]$: We introduce the topologies

$$\left.\begin{array}{l} (\tau_{\Sigma}^*[E;\ \mathcal{L};\ \eta;\ \mathbb{F} \mid r] \triangleq \otimes^r\ [\tau_*^+(\mathcal{L})]\,|_{\Sigma_r[E;\,\mathcal{L};\,\eta;\,\mathbb{F}]}), \\ \text{and} \\ (\tau_{\Sigma}^0[E;\ \mathcal{L};\ \eta;\ \mathbb{F} \mid r] \triangleq \otimes^r\ [\tau_0^+(\mathcal{L})]\,|_{\Sigma_r[E;\,\mathcal{L};\,\eta;\,\mathbb{F}]}), \end{array}\right\} \tag{3.11}$$

of the set (3.6). Of course, from (3.6) and (3.11), we obtain that

$$(\ \Sigma_r[E;\ \mathcal{L};\ \eta;\ \mathbb{F}],\ \tau_{\Sigma}^*[E;\ \mathcal{L};\ \eta;\ \mathbb{F} \mid r]), \tag{3.12}$$

is a nonempty compactum. By (2.2), the following inclusion:

$$\tau_{\Sigma}^{*}[E;\ \mathcal{L};\ \eta;\ \mathbb{F}\,|\,r] \subset \tau_{\Sigma}^{0}[E;\ \mathcal{L};\ \eta;\ \mathbb{F}\,|\,r], \tag{3.13}$$

is realized. From (3.13), we have the obvious comparability property of AS;

$$\begin{aligned}&(\mathbf{as})[(r-\mathrm{adm})[\mathbb{F}\,|\,E;\ \mathcal{L};\ \eta];\ \Sigma_r[E;\ \mathcal{L};\ \eta;\ \mathbb{F}];\ \tau_{\Sigma}^{0}[E;\ \mathcal{L};\ \eta;\ \mathbb{F}\,|\,r];\ \mathbb{I};\ \mathcal{X}]\\ &\quad\subset (\mathbf{as})[(r-\mathrm{adm})[\mathbb{F}\,|\,E;\ \mathcal{L};\ \eta];\ \Sigma_r[E;\ \mathcal{L};\ \eta;\ \mathbb{F}];\ \tau_{\Sigma}^{*}[E;\ \mathcal{L};\ \eta;\ \mathbb{F}\,|\,r];\ \mathbb{I};\ \mathcal{X}]\\ &\qquad\forall\,\mathcal{X}\in\mathcal{P}'(\mathcal{P}((r-\mathrm{adm})[\mathbb{F}\,|\,E;\ \mathcal{L};\ \eta])).\end{aligned} \tag{3.14}$$

Moreover, from (3.10) and (3.11), the following universal density property are realized:

$$\begin{aligned}&\mathrm{cl}(\mathbb{I}^1((r-\mathrm{adm})[\mathbb{F}\,|\,E;\ \mathcal{L};\ \eta]),\ \tau_{\Sigma}^{*}[E;\ \mathcal{L};\ \eta;\ \mathbb{F}\,|\,r])\\ &\qquad=\mathrm{cl}(\mathbb{I}^1((r-\mathrm{adm})[\mathbb{F}\,|\,E;\ \mathcal{L};\ \eta]),\ \tau_{\Sigma}^{0}[E;\ \mathcal{L};\ \eta;\ \mathbb{F}\,|\,r])\\ &\qquad=\Sigma_r[E;\ \mathcal{L};\ \eta;\ \mathbb{F}].\end{aligned} \tag{3.15}$$

Now, we recall (3.5). For this definition, we introduce the corresponding generalized analogue of (3.5). Let $\mathcal{S}$ be the following mapping:

$$(\mu_j)_{j\in\overline{1,r}} \mapsto \left(\sum_{j=1}^{r}\int_E S_{i,j}\,d\mu_j\right)_{i\in\overline{1,\mathbf{n}}} :\ \Sigma_r[E;\ \mathcal{L};\ \eta;\ \mathbb{F}] \to \mathbb{R}^{\mathbf{n}}. \tag{3.16}$$

Then, the generalized variant of (3.5) defined for precise $\mathbb{Y}$-constraint is realized in the following form:

$$\begin{aligned}&((\mathbf{n},\ r)-\mathcal{ADM})[\mathbb{F}\,|\,E;\ \mathcal{L};\ \eta;\ \mathbb{Y};\ S] \triangleq \mathcal{S}^{-1}(\mathbb{Y})\\ &\qquad=\left\{(\mu_j)_{j\in\overline{1,r}}\in\Sigma_r[E;\ \mathcal{L};\ \eta;\ \mathbb{F}]\,\middle|\,\left(\sum_{j=1}^{r}\int_E S_{i,j}\,d\mu_j\right)_{i\in\overline{1,\mathbf{n}}}\in\mathbb{Y}\right\},\end{aligned} \tag{3.17}$$

is the set of $\mathbb{Y}$-admissible generalized elements. In connection with (3.5), we note that

$$((\mathbf{n},\ r) - \mathbb{ADM})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta;\ \mathbb{Y};\ S]$$

$$\triangleq \Big\{((\mathbf{n},\ r) - \mathrm{ADM})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta;\ \mathcal{O}_\zeta^{(\mathbf{n})}[\mathbb{Y}];\ S] : \zeta \in]0,\ \infty[\Big\}$$

$$\in \beta[(r - \mathrm{adm})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta]]. \qquad (3.18)$$

In (3.18), we have the natural variant of «asymptotic constraints», connected with «uniform» weakening of standard $\mathbb{Y}$-constraints. We will consider another variants of weakening of the $\mathbb{Y}$-constraint. Namely, we will consider «nonuniform» weakening. For this, again we introduce the set

$$((\mathbf{n},\ r) - \mathrm{step})[E;\ \mathcal{L};\ S] \triangleq \{M \in \mathcal{P}(\overline{1,\ \mathbf{n}}) \mid S_{i,\ j} \in B_0(E,\ \mathcal{L}) \quad \forall\, i \in M\ \forall j \in \overline{1,\ r}\}. \qquad (3.19)$$

In (3.19), the graduatedness sets of our matriciant are considered. For any $M \in ((\mathbf{n},\ r) - \mathrm{step})[E;\ \mathcal{L};\ S]$ and $\zeta \in]0, \infty[$, the set $\widehat{O}_\zeta^{(\mathbf{n})}[Y \mid M]$ is defined. Then, along with (3.18), we consider the families

$$((\mathbf{n},\ r) - \mathbf{ADM})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta;\ \mathbb{Y};\ M;\ S]$$

$$\triangleq \Big\{((\mathbf{n},\ r) - \mathrm{ADM})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta;\ \widehat{\mathcal{O}}_\zeta^{(\mathbf{n})}[\mathbb{Y} \mid M];\ S] : \zeta \in]0,\ \infty[\Big\}$$

$$\in \beta[(r - \mathrm{adm})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta]] \quad \forall M \in ((\mathbf{n},\ r) - \mathrm{step})[E;\ \mathcal{L};\ S]. \qquad (3.20)$$

So, we have two types of «asymptotic» constraints. Namely, we weaken $\mathbb{Y}$-constraint with respect to «all directions» and with respect to a part of «directions». In the following, we will establish that, for the both above-mentioned variants, the corresponding asymptotic is defined by (3.17). In addition, we require that, in the last case, the choice *M* from the family (3.19) is assumed (in this connection, we keep in mind the variant (3.20)).

Remark 3.1. In connection with (3.19) and (3.20), we note that $\varnothing \in ((\mathbf{n}, r) - \text{step})[E; \mathcal{L}; S]$; therefore, the family

$$((\mathbf{n}, r) - \mathbf{ADM})[\mathbb{F} \mid E; \mathcal{L}; \eta; \mathbb{Y}; \varnothing; S] \in \beta[(r - \text{adm})[\mathbb{F} \mid E; \mathcal{L}; \eta]],$$

is defined correctly. In addition,

$$((\mathbf{n}, r) - \mathbb{ADM})[\mathbb{F} \mid E; \mathcal{L}; \eta; \mathbb{Y}; S] = ((\mathbf{n}, r) - \mathbf{ADM})[\mathbb{F} \mid E; \mathcal{L}; \eta; \mathbb{Y}; \varnothing; S]. \tag{3.21}$$

So, for the case $M = \varnothing$ (we keep in mind the choice of M from the family (3.19)), we obtain unique constraints of asymptotic character.

For the end of the present section, we fix the set

$$M \in ((\mathbf{n}, r) - \text{step})[E; \mathcal{L}; S]. \tag{3.22}$$

Proposition 3.1. *The set* (3.17) *of all generalized admissible elements realizes the* «*unique*» *asymptotic of admissible* (*in the usual sense*) *sets under a weakening of* $\mathbb{Y}$*-constraint*

$$\begin{aligned}
&((\mathbf{n}, r) - \mathcal{ADM})[\mathbb{F} \mid E; \mathcal{L}; \eta; \mathbb{Y}; S] \\
&\quad = (\mathbf{as})[(r - \text{adm})[\mathbb{F} \mid E; \mathcal{L}; \eta]; \textstyle\sum_r[E; \mathcal{L}; \eta; \mathbb{F}]; \tau_{\Sigma}^{*}[E; \mathcal{L}; \eta; \mathbb{F} \mid r]; \\
&\qquad\qquad \mathbb{I}; ((\mathbf{n}, r) - \mathbb{ADM})[\mathbb{F} \mid E; \mathcal{L}; \eta; \mathbb{Y}; S]] \\
&\quad = (\mathbf{as})[(r - \text{adm})[\mathbb{F} \mid E; \mathcal{L}; \eta]; \textstyle\sum_r[E; \mathcal{L}; \eta; \mathbb{F}]; \tau_{\Sigma}^{*}[E; \mathcal{L}; \eta; \mathbb{F} \mid r]; \\
&\qquad\qquad \mathbb{I}; ((\mathbf{n}, r) - \mathbf{ADM})[\mathbb{F} \mid E; \mathcal{L}; \eta; \mathbb{Y}; M; S]] \\
&\quad = (\mathbf{as})[(r - \text{adm})[\mathbb{F} \mid E; \mathcal{L}; \eta]; \textstyle\sum_r[E; \mathcal{L}; \eta; \mathbb{F}]; \tau_{\Sigma}^{0}[E; \mathcal{L}; \eta; \mathbb{F} \mid r]; \\
&\qquad\qquad \mathbb{I}; ((\mathbf{n}, r) - \mathbb{ADM})[\mathbb{F} \mid E; \mathcal{L}; \eta; \mathbb{Y}; S]] \\
&\quad = (\mathbf{as})[(r - \text{adm})[\mathbb{F} \mid E; \mathcal{L}; \eta]; \textstyle\sum_r[E; \mathcal{L}; \eta; \mathbb{F}]; \tau_{\Sigma}^{0}[E; \mathcal{L}; \eta; \mathbb{F} \mid r]; \\
&\qquad\qquad \mathbb{I}; ((\mathbf{n}, r) - \mathbf{ADM})[\mathbb{F} \mid E; \mathcal{L}; \eta; \mathbb{Y}; M; S]].
\end{aligned} \tag{3.23}$$

The corresponding proof is the simple corollary of statements of [13, Section 4] (in particular, we keep in mind Theorems 4.1 and 4.2 of [13]). The base property is the equalities chain (2.6). This proof differ from analogous statements of [9, Chapter 4] only technical details. Therefore, we omit this proof.

Attainability property

In this section, we fix $\mathbf{k} \in \mathbb{N}$ and mapping

$$(i,\ j) \mapsto A_{i,\,j} : \overline{1,\ \mathbf{k}} \times \overline{1,\ r} \to B(E,\ \mathcal{L}). \tag{3.24}$$

Of course, for any $(f_j)_{j \in \overline{1,\,r}} \in B^+_{0,\,r}[E;\ \mathcal{L}],$ the vector

$$\left(\sum_{j=1}^{r} \int_E A_{i,\,j} f_j d\eta \right)_{i \in \overline{1,\,\mathbf{k}}} \in \mathbb{R}^{\mathbf{k}},$$

is defined. As a corollary, we define the mapping

$$\widehat{\mathcal{A}} : (r - \mathrm{adm})[\,\mathbb{F} \,|\, E;\ \mathcal{L};\ \eta\,] \to \mathbb{R}^{\mathbf{k}},$$

by the following rule:

$$\widehat{\mathcal{A}}\,(\,(f_i)_{i \in \overline{1,\,r}}\,) \triangleq \left(\sum_{j=1}^{r} \int_E A_{i,\,j} f_j d\eta \right)_{i \in \overline{1,\,\mathbf{k}}} \quad \forall (\,f_j\,)_{j \in \overline{1,\,r}} \in (r - \mathrm{adm})[\,\mathbb{F} \,|\, E;\ \mathcal{L};\ \eta\,]. \tag{3.25}$$

We consider AS on the values of $\widehat{A}$: For any $\mathcal{Z} \in \mathcal{P}'(\mathcal{P}((r - \mathrm{adm})[\mathbb{F} | E; \mathcal{L}; \eta]))$

$$(\mathbf{as})[\,(r - \mathrm{adm})[\,\mathbb{F} \,|\, E;\ \mathcal{L};\ \eta\,];\ \mathbb{R}^{\mathbf{k}};\ \tau_{\mathbb{R}}^{(\mathbf{k})};\ \widehat{\mathcal{A}} : \mathcal{Z}\,] \in \mathcal{P}(\,\mathbb{R}^{\mathbf{k}}).$$

Moreover, along with $\widehat{A}$, we consider the mapping

$$\widetilde{\mathcal{A}} : \Sigma_r\,[\,E;\ \mathcal{L};\ \eta;\ \mathbb{F}\,] \to \mathbb{R}^{\mathbf{k}},$$

by the following rule:

$$\widetilde{\mathcal{A}}((\mu_j)_{j\in\overline{1,r}}) \triangleq \left(\sum_{j=1}^{r}\int_E A_{i,j}d\mu_j\right)_{i\in\overline{1,\mathbf{k}}} \quad \forall (\mu_j)_{j\in\overline{1,r}} \in \Sigma_r[E; \mathcal{L}; \eta; \mathbb{F}]. \tag{3.26}$$

By (3.9), the mapping $\widetilde{\mathcal{A}} \circ \mathbb{I}$ is defined. By [8, (3.4.11)], from (3.25) and (3.26), the equality

$$\widehat{\mathcal{A}} = \widetilde{\mathcal{A}} \circ \mathbb{I}, \tag{3.27}$$

follows. From definition of the $*$-weak topology, the obvious continuity property

$$\widetilde{\mathcal{A}} \in C\left(\Sigma_r[E; \mathcal{L}; \eta; \mathbb{F}], \tau_{\Sigma}^{*}[E; \mathcal{L}; \eta; \mathbb{F} \mid r], \mathbb{R}^{\mathbf{k}}, \tau_{\mathbb{R}}^{(\mathbf{k})}\right), \tag{3.28}$$

is realized. By (3.27) and (3.28), we obtain the very important property: Using [9, Proposition 5.2.1], we obtain that

$$(\mathbf{as})\left[(r-\mathrm{adm})[\mathbb{F} \mid E; \mathcal{L}; \eta]; \mathbb{R}^{\mathbf{k}}, \tau_{\mathbb{R}}^{(\mathbf{k})}; \widehat{\mathcal{A}}; \mathfrak{R}\right]$$

$$= \widetilde{\mathcal{A}}^1((\mathbf{as})[(r-\mathrm{adm})[\mathbb{F} \mid E; \mathcal{L}; \eta]; \Sigma_r[E; \mathcal{L}; \eta; \mathbb{F}]; \tau_{\Sigma}^{*}[E; \mathcal{L}; \eta; \mathbb{F} \mid r]; \mathbb{I}; \mathfrak{R}])$$

$$\forall \mathfrak{R} \in \mathcal{P}'(\mathcal{P}((r-\mathrm{adm})[\mathbb{F} \mid E; \mathcal{L}; \eta])); \tag{3.29}$$

in this connection, see [12, (3.3), Proposition 3.2]. From (3.29) and Proposition 3.1, the important result follows:

Proposition 3.2. *The next equality chain takes place*

$$\widetilde{\mathcal{A}}^1(((\mathbf{n}, r)-\mathcal{ADM})[\mathbb{F} \mid E; \mathcal{L}; \eta; \mathbb{Y}; S])$$

$$= (\mathbf{as})\left[(r-\mathrm{adm})[\mathbb{F} \mid E; \mathcal{L}; \eta]; \mathbb{R}^{\mathbf{k}}, \tau_{\mathbb{R}}^{(\mathbf{k})}; \widehat{\mathcal{A}}; ((\mathbf{n}, r)-\mathbb{ADM})[\mathbb{F} \mid E; \mathcal{L}; \eta; \mathbb{Y}; S]\right]$$

$$= (\mathbf{as})\left[(r-\mathrm{adm})[\mathbb{F} \mid E; \mathcal{L}; \eta]; \mathbb{R}^{\mathbf{k}}, \tau_{\mathbb{R}}^{(\mathbf{k})}; \widehat{\mathcal{A}}; ((\mathbf{n}, r)-\mathbf{ADM})[\mathbb{F} \mid E; \mathcal{L}; \eta; \mathbb{Y}; M; S]\right]. \tag{3.30}$$

We recall (3.18) and (3.20); in these families, we have two different (generally speaking) variants of constraints of asymptotic character. Now, we introduce the known π-refinement relation: If $\mathfrak{X}$ and $\mathfrak{Y}$ are families, then

$$(\mathfrak{X} \dashv \mathfrak{Y}) \overset{\text{def}}{\Leftrightarrow} (\forall\, A \in \mathfrak{X} \quad \exists\, B \in \mathfrak{Y} : B \subset A). \tag{3.31}$$

In the following, we use (3.31) without additional clarifications. Of course, in the capacity of $\mathfrak{X}$ and $\mathfrak{Y}$, we can use (see (3.31)) families of subsets of $(r - \text{adm})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta]$. In addition,

$$\forall \mathfrak{X} \in \mathcal{P}'(\mathcal{P}((r - \text{adm})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta]))\ \forall\, \mathfrak{Y} \in \mathcal{P}'(\mathcal{P}((r - \text{adm})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta]))$$

$$(\mathfrak{X} \dashv \mathfrak{Y}) \Rightarrow (\,(\mathbf{as})[\,(r - \text{adm})[\,\mathbb{F} \mid E;\ \mathcal{L};\ \eta\,];\ \mathbb{R}^{\mathbf{k}};\ \tau_{\mathbb{R}}^{(\mathbf{k})};\ \widehat{\mathcal{A}};\ \mathfrak{Y}\,]$$

$$\subset (\mathbf{as})[\,(r - \text{adm})[\,\mathbb{F} \mid E;\ \mathcal{L};\ \eta\,];\ \mathbb{R}^{\mathbf{k}};\ \tau_{\mathbb{R}}^{(\mathbf{k})};\ \widehat{\mathcal{A}};\ \mathfrak{X}\,]\,). \tag{3.32}$$

From (1.4), (3.5), (3.18), (3.20), and (3.31), we obtain that

$$((\mathbf{n},\ r) - \mathbb{ADM})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta;\ \mathbb{Y};\ S] \dashv ((\mathbf{n},\ r) - \mathbf{ADM})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta;\ \mathbb{Y};\ M;\ S]. \tag{3.33}$$

In connection with (3.33), we note that the natural construction of $\dashv$ onto the product

$$\mathcal{P}'(\mathcal{P}((r - \text{adm})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta])) \times \mathcal{P}'(\mathcal{P}((r - \text{adm})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta])),$$

is a reflexive relation on $\mathcal{P}'(\mathcal{P}((r - \text{adm})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta]))$.

Theorem 3.1. *If* $\mathcal{Z} \in \mathcal{P}'(\mathcal{P}((r - \text{adm})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta]))$, *then*

$$(((\mathbf{n},\ r) - \mathbb{ADM})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta;\ \mathbb{Y};\ S] \dashv \mathcal{Z}) \,\&$$

$$(\mathcal{Z} \dashv ((\mathbf{n},\ r) - \mathbf{ADM})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta;\ \mathbb{Y};\ M;\ S])$$

$$\Rightarrow \left((\mathbf{as})\left[(r - \text{adm})[\,\mathbb{F} \mid E;\ \mathcal{L};\ \eta\,];\ \mathbb{R}^{\mathbf{k}};\ \tau_{\mathbb{R}}^{(\mathbf{k})};\ \widehat{\mathcal{A}};\ \mathcal{Z}\right]\right)$$

$$= \widetilde{\mathcal{A}}^{1}(((\mathbf{n},\ r) - \mathcal{ADM})[\mathbb{F} \mid E;\ \mathcal{L};\ \eta;\ \mathbb{Y};\ S]). \tag{3.34}$$

Proof of Theorem 3.1 is reduced to the immediate combination of (3.32) and Proposition 3.2. In Theorem 3.1, we have the statement about asymptotic nonsensitivity in the range of constraints of asymptotic character. The corresponding range is defined in terms of π-refinement (see [16]). The essential part of the used construction is Proposition 3.2.

In the following, we use constructions of this section in two variants. Namely, as in Introduction, we consider the above-mentioned extension procedures for first and second players (player I and player II) separately. Of course, for this, some designation changes will be necessary.

4. Extension of a Game Problem

We use notions of Section 3. Fix the following two measurable spaces with semialgebras of sets: $(E_1, \mathcal{L}_1)$ and $(E_2, \mathcal{L}_2)$, where $E_1 \neq \varnothing$, $E_2 \neq \varnothing$, $\mathcal{L}_1$ is a semialgebra of subsets of E_1; $\mathcal{L}_2$ is a semialgebra of subsets of E_2. Therefore, we use constructions of Section 3 under suppositions $(E, \mathcal{L}) = (E_1, \mathcal{L}_1)$ or $(E, \mathcal{L}) = (E_2, \mathcal{L}_2)$.

We fix $r_1 \in \mathbb{N}$ and $r_2 \in \mathbb{N}$ in the capacity of dimensions of instantaneous controls of players I and II, respectively. We obtain the following sets of step vector-functions:

$$B^+_{0, r_1}[E_1; \mathcal{L}_1], \quad B^+_{0, r_2}[E_2; \mathcal{L}_2]. \tag{4.1}$$

In the following, vector-functions of the sets (4.1) are used for usual controls of players I and II. For nonempty sets $(\text{add})^+_{r_1}[\mathcal{L}_1]$ and $(\text{add})^+_{r_2}[\mathcal{L}_2]$, we use topologies $\otimes^{r_1}[\tau^+_*(\mathcal{L}_1)], \otimes^{r_1}[\tau^+_0(\mathcal{L}_1)]$ and $\otimes^{r_2}[\tau^+_*(\mathcal{L}_2)]$, $\otimes^{r_2}[\tau^+_0(\mathcal{L}_2)]$, respectively.

Fix $\eta_1 \in (\text{add})_+[\mathcal{L}_1]$ and $\eta_2 \in (\text{add})_+[\mathcal{L}_2]$; suppose that $\eta_1(E_1) \neq 0$ and $\eta_2(E_2) \neq 0$. In the terms of η_1 and η_2, we form the nonempty sets

$$(\text{add})^+_{r_1}[\mathcal{L}_1; \eta_1], \quad (\text{add})^+_{r_2}[\mathcal{L}_2; \eta_2].$$

Fix $\mathbf{n}_1 \in \mathbb{N}$ and $\mathbf{n}_2 \in \mathbb{N}$. We consider $\mathbf{n}_1$ and $\mathbf{n}_2$ as two concrete variants of $\mathbf{n}$ of Section 3. Moreover, fix mappings

$$S^{(1)} : \overline{1,\, \mathbf{n}_1} \times \overline{1,\, r_1} \to B(E_1,\, \mathcal{L}_1), \quad S^{(2)} : \overline{1,\, \mathbf{n}_2} \times \overline{1,\, r_2} \to B(E_2,\, \mathcal{L}_2);$$

in the following, $S^{(1)}$ and $S^{(2)}$ are considered in the capacity of matriciant for players I and II, respectively. As in Section 3, we suppose that

$$\left(S^{(1)}_{i,\,j} \triangleq S^{(1)}(i,\, j) \quad \forall\, i \in \overline{1,\, \mathbf{n}_1} \quad \forall\, j \in \overline{1,\, r_1}\right),$$

and

$$\left(S^{(2)}_{i,\,j} \triangleq S^{(2)}(i,\, j) \quad \forall\, i \in \overline{1,\, \mathbf{n}_2} \quad \forall\, j \in \overline{1,\, r_2}\right).$$

Let $\mathbb{Y}_1 \in \mathcal{P}'(\mathbb{R}^{\mathbf{n}_1})$ and $\mathbb{Y}_2 \in \mathcal{P}'(\mathbb{R}^{\mathbf{n}_2})$. We suppose that $\mathbb{Y}_1$ is closed in $\mathrm{TS}(\mathbb{R}^{\mathbf{n}_1},\, \tau_{\mathbb{R}}^{(\mathbf{n}_1)})$ and $\mathbb{Y}_2$ is closed in $\mathrm{TS}(\mathbb{R}^{\mathbf{n}_2},\, \tau_{\mathbb{R}}^{(\mathbf{n}_2)})$. So, $\mathbb{Y}_1$ and $\mathbb{Y}_2$ are nonempty closed sets in the corresponding finite-dimensional spaces. By $\mathbb{Y}_1$ and $\mathbb{Y}_2$, the constraints

$$\left(\sum_{j=1}^{r_1} \int_{E_1} S^{(1)}_{i,\,j} \widetilde{f}_j d\eta_1\right)_{i \in \overline{1,\, \mathbf{n}_1}} \in \mathbb{Y}_1 \quad \text{and} \quad \left(\sum_{j=1}^{r_2} \int_{E_2} S^{(2)}_{i,\,j} \widehat{f}_j d\eta_2\right)_{i \in \overline{1,\, \mathbf{n}_2}} \in \mathbb{Y}_2, \tag{4.2}$$

on the choice of $(\widetilde{f}_j)_{j \in \overline{1,\, r_1}} \in B^+_{0,\, r_1}[E_1;\, \mathcal{L}_1]$ and $(\widehat{f}_j)_{j \in \overline{1,\, r_2}} \in B^+_{0,\, r_2}[E_2;\, \mathcal{L}_2]$, respectively, are considered. Along with (4.1) and (4.2), constraints of impulse character will use too. We call (4.2) moment constraints.

Abstract constraints of impulse character

Fix bounded sets $\mathbb{F}_1 \in \mathcal{P}'(\mathbb{R}^{r_1}_+)$ and $\mathbb{F}_2 \in \mathcal{P}'(\mathbb{R}^{r_2}_+)$. We suppose that $\mathbb{F}_1$ is closed in $(\mathbb{R}^{r_1},\, \tau_{\mathbb{R}}^{(r_1)})$ and $\mathbb{F}_2$ is closed in $(\mathbb{R}^{r_2},\, \tau_{\mathbb{R}}^{(r_2)})$. Moreover, we suppose that $\mathbb{F}_1$ and $\mathbb{F}_2$ are bounded sets. We consider

$$(r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1]$$

$$\triangleq \left\{ (f_i)_{i \in \overline{1, r_1}} \in B^{+}_{0, r_1}[E_1;\ \mathcal{L}_1] \mid \left(\int_{E_1} f_i d\eta_1 \right)_{i \in \overline{1, r_1}} \in \mathbb{F}_1 \right\} \in \mathcal{P}'(B^{+}_{0, r_1}[E_1;\ \mathcal{L}_1]);$$

$$(r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2]$$

$$\triangleq \left\{ (f_i)_{i \in \overline{1, r_2}} \in B^{+}_{0, r_2}[E_2;\ \mathcal{L}_2] \mid \left(\int_{E_2} f_i d\eta_2 \right)_{i \in \overline{1, r_2}} \in \mathbb{F}_2 \right\} \in \mathcal{P}'(B^{+}_{0, r_2}[E_2;\ \mathcal{L}_2]).$$

In connection with definitions of Section 3, we suppose that

$$(((\mathbf{n}_1,\ r_1) - \mathrm{ADM})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1;\ \widetilde{\mathbf{Y}};\ S^{(1)}]$$

$$\triangleq \left\{ (f_i)_{i \in \overline{1, r_1}} \in (r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1] \mid \left(\sum_{j=1}^{r_1} \int_{E_1} S^{(1)}_{i,j} f_j d\eta_1 \right)_{i \in \overline{1, \mathbf{n}_1}} \in \widetilde{\mathbf{Y}} \right\},$$

$$\forall \widetilde{\mathbf{Y}} \in \mathcal{P}(\mathbb{R}^{\mathbf{n}_1}));$$

and

$$(((\mathbf{n}_2,\ r_2) - \mathrm{ADM})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2;\ \widehat{\mathbf{Y}};\ S^{(2)}]$$

$$\triangleq \left\{ (f_i)_{i \in \overline{1, r_2}} \in (r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2] \mid \left(\sum_{j=1}^{r_2} \int_{E_2} S^{(2)}_{i,j} f_j\, d\eta_2 \right)_{i \in \overline{1, \mathbf{n}_2}} \in \widehat{\mathbf{Y}} \right\},$$

$$\forall \widehat{\mathbf{Y}} \in \mathcal{P}(\mathbb{R}^{\mathbf{n}_2})). \tag{4.3}$$

Of course, the sets (4.3) will be used in cases, when questions of approximate validity of $\mathbb{Y}_1$-constraints and $\mathbb{Y}_2$-constraints are considered. In these cases, in (4.3), neighbourhoods of $\mathbb{Y}_1$ and $\mathbb{Y}_2$ are required to use.

Now, we consider concrete variants of construction of previous section connected with generalized elements. Namely,

$$\Sigma_{r_1}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1] \triangleq \{(\mu_j)_{j\in\overline{1,r_1}} \in (\mathrm{add})^+_{r_1}[\mathcal{L}_1;\ \eta_1] \mid (\mu_j(E_1))_{j\in\overline{1,r_1}} \in \mathbb{F}_1\}, \tag{4.4}$$

is the set of generalized elements of the first player. Analogously,

$$\Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2] \triangleq \{(\nu_j)_{j\in\overline{1,r_2}} \in (\mathrm{add})^+_{r_2}[\mathcal{L}_2;\ \eta_2] \mid (\nu_j(E_2))_{j\in\overline{1,r_2}} \in \mathbb{F}_2\}. \tag{4.5}$$

So, in fact, we introduce generalized controls of both players. Of course, usual controls can be represented as generalized controls. For this goal, we introduce the mapping $\mathbf{I}$ by the rule

$$(f_i)_{i\in\overline{1,r_1}} \mapsto (f_i * \eta_1)_{i\in\overline{1,r_1}} : (r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1] \to \Sigma_{r_1}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1]. \tag{4.6}$$

Then, $(r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1]$ can be considered as everywhere dense subset of the set (4.4)

$$\Sigma_{r_1}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1] = \mathrm{cl}(\mathbf{I}^1((r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1]),$$

$$\tau^*_\Sigma[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1 \mid r_1])$$

$$= \mathrm{cl}(\mathbf{I}^1((r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1]),\ \tau^0_\Sigma[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1 \mid r_1]), \tag{4.7}$$

where

$$\tau^*_\Sigma[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1 \mid r_1] \triangleq \otimes^{r_1}\ [\tau^+_*(\mathcal{L}_1)]\mid_{\Sigma_{r_1}[E_1;\mathcal{L}_1;\eta_1;\mathbb{F}_1]},$$

and

$$\tau^0_\Sigma[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1 \mid r_1] \triangleq \otimes^{r_1}\ [\tau^+_0(\mathcal{L}_1)]\mid_{\Sigma_{r_1}[E_1;\mathcal{L}_1;\eta_1;\mathbb{F}_1]}.$$

The more detailed reasonings are reduced in the previous section.

Analogously, $(r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2]$ can be considered in the form of everywhere dense subset of the set (4.5); for this goal, we introduce the following mapping $\mathbf{J}$:

$$(f_i)_{i \in \overline{1, r_2}} \mapsto (f_i * \eta_2)_{i \in \overline{1, r_2}} : (r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2]$$
$$\to \Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2],$$

considered as the immersion operator. In addition,

$$\Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2] = \mathrm{cl}(\mathbf{J}^1((r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2]),$$
$$\tau^*_\Sigma[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2 \mid r_2])$$
$$= \mathrm{cl}(\mathbf{J}^1((r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2]),$$
$$\tau^0_\Sigma[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2 \mid r_2]), \tag{4.8}$$

where

$$\tau^*_\Sigma[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2 \mid r_2] \triangleq \otimes^{r_2}\ [\tau^+_*(\mathcal{L}_2)] \mid_{\Sigma_{r_2}[E_2;\mathcal{L}_2;\eta_2;\mathbb{F}_2]},$$

and

$$\tau^0_\Sigma[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2 \mid r_2] \triangleq \otimes^{r_2}\ [\tau^+_0(\mathcal{L}_2)] \mid_{\Sigma_{r_2}[E_2;\mathcal{L}_2;\eta_2;\mathbb{F}_2]}.$$

Moreover, we introduce two natural generalized operators. The mapping $\mathcal{S}_1$ is defined as

$$(\mu_j)_{j \in \overline{1, r_1}} \mapsto \left(\sum_{j=1}^{r_1} \int_{E_1} S^{(1)}_{i,j}\, d\mu_j \right)_{i \in \overline{1, \mathbf{n}_1}} : \Sigma_{r_1}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1] \to \mathbb{R}^{\mathbf{n}_1}; \tag{4.9}$$

of course, (4.9) is generalized operator of player I. Analogously, the mapping $\mathcal{S}_2$ is defined in the form

$$(\nu_j)_{j\in\overline{1, r_2}} \mapsto \left(\sum_{j=1}^{r_2}\int_{E_2} S_{i,j}^{(2)}\, d\nu_j\right)_{i\in\overline{1, \mathbf{n}_2}} : \Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2] \to \mathbb{R}^{\mathbf{n}_2}; \tag{4.10}$$

(4.10) is generalized operator of player II.

The sets of admissible generalized elements of players I and II are defined as preimages of $\mathbb{Y}_1$ and $\mathbb{Y}_2$ under operation of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively;

$$\mathfrak{M} \triangleq ((\mathbf{n}_1, r_1) - \mathcal{ADM})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{Y}_1;\ S^{(1)}]$$

$$= \{(\mu_j)_{j\in\overline{1, r_1}} \in \Sigma_{r_1}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1] \mid \mathcal{S}_1((\mu_j)_{j\in\overline{1, r_1}}) \in \mathbb{Y}_1\}, \tag{4.11}$$

$$\mathfrak{N} \triangleq ((\mathbf{n}_2, r_2) - \mathcal{ADM})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{Y}_2;\ S^{(2)}]$$

$$= \{(\nu_j)_{j\in\overline{1, r_2}} \in \Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2] \mid \mathcal{S}_2((\nu_j)_{j\in\overline{1, r_2}}) \in \mathbb{Y}_2\}. \tag{4.12}$$

Elements of (4.11) (of (4.12)) and only they are admissible generalize controls of player I (of player II). We note that this admissibility is regarded in the sense of precise constraints corresponding to sets $\mathbb{Y}_1$ and $\mathbb{Y}_2$, respectively.

Now, we introduce two variants of constraints of asymptotic character. Let

$$\mathfrak{A}_1 \triangleq ((\mathbf{n}_1, r_1) - \mathbb{ADM})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{Y}_1;\ S^{(1)}]$$

$$= \left\{((\mathbf{n}_1, r_1) - \mathrm{ADM})\left[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathcal{O}_\zeta^{(\mathbf{n}_1)}[\mathbb{Y}_1];\ S^{(1)}\right] : \zeta \in\ ]0, \infty[\right\}$$

$$\in \beta[(r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1]], \tag{4.13}$$

$$\mathfrak{A}_2 \triangleq ((\mathbf{n}_2, r_2) - \mathbb{ADM})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{Y}_2;\ S^{(2)}]$$

$$= \left\{((\mathbf{n}_2, r_2) - \mathrm{ADM})\left[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathcal{O}_\zeta^{(\mathbf{n}_2)}[\mathbb{Y}_2];\ S^{(2)}\right] : \zeta \in\ ]0, \infty[\right\}$$

$$\in \beta[(r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2]]. \tag{4.14}$$

In (4.13) and (4.14), we use weakening of constraints in all directions. This is connected with employment of neighbourhoods of $\mathbb{Y}_1$ and $\mathbb{Y}_2$ defined in terms of norms.

Now, we consider another weakening of constraints. For this, we use (3.20). Recall that

$$((\mathbf{n}_1,\ r_1) - \text{step})[E_1;\ \mathcal{L}_1;\ S^{(1)}]$$

$$= \left\{M \in \mathcal{P}(\overline{1,\ \mathbf{n}_1}) \mid S^{(1)}_{i,j} \in B_0(E_1,\ \mathcal{L}_1) \quad \forall i \in M \quad \forall j \in \overline{1,\ r_1}\right\}, \quad (4.15)$$

and

$$((\mathbf{n}_2,\ r_2) - \text{step})[E_2;\ \mathcal{L}_2;\ S^{(2)}]$$

$$= \left\{M \in \mathcal{P}(\overline{1,\ \mathbf{n}_2}) \mid S^{(2)}_{i,j} \in B_0(E_2,\ \mathcal{L}_2) \quad \forall i \in M \quad \forall j \in \overline{1,\ r_2}\right\}. \quad (4.16)$$

We fix sets $M_1 \in ((\mathbf{n}_1, r_1) - \text{step})[E_1; \mathcal{L}_1; S^{(1)}]$ and $M_2 \in ((\mathbf{n}_2,\ r_2) - \text{step})$ $[E_2;\ \mathcal{L}_2;\ S^{(2)}]$. Of course, by (4.15) and (4.16),

$$(M_1 \subset \overline{1,\ \mathbf{n}_1}) \quad \& \quad (M_2 \subset \overline{1,\ \mathbf{n}_2}).$$

In addition, we have two index sets defining the «step» directions of our matriciants. Now, we introduce two variants of the family (4.19)

$$\mathfrak{B}_1 \triangleq ((\mathbf{n}_1,\ r_1) - \mathbf{ADM})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{Y}_1;\ M_1;\ S^{(1)}]$$

$$= \left\{((\mathbf{n}_1,\ r_1) - \text{ADM})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1;\ \widehat{\mathcal{O}}^{(\mathbf{n}_1)}_{\zeta}[\mathbb{Y}_1 \mid M_1];\ S^{(1)}] : \zeta \in]0,\ \infty[\right\}$$

$$\in \beta[(r_1 - \text{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1]], \quad (4.17)$$

$$\mathfrak{B}_2 \triangleq ((\mathbf{n}_2,\ r_2) - \mathbf{ADM})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{Y}_2;\ M_2;\ S^{(2)}]$$

$$= \left\{((\mathbf{n}_2,\ r_2) - \text{ADM})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2;\ \widehat{\mathcal{O}}^{(\mathbf{n}_2)}_{\zeta}[\mathbb{Y}_2 \mid M_2];\ S^{(2)}] : \zeta \in]0,\ \infty[\right\}$$

$$\in \beta[(r_2 - \text{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2]]. \quad (4.18)$$

Of course, $\mathfrak{A}_1$ and $\mathfrak{B}_1$ are realized two types of «asymptotic constraints» for player I. Analogously, $\mathfrak{A}_2$ and $\mathfrak{B}_2$ are realized two types of «asymptotic constraints» for player II. In the following, we will consider immediate variants of constraints too. But, now we are restricted consideration of the above-mentioned extreme variants $\mathfrak{A}_1$, $\mathfrak{B}_1$, $\mathfrak{A}_2$, and $\mathfrak{B}_2$. From Proposition 3.1, we obtain the following two chains of equalities:

$$\begin{aligned}
\mathfrak{M} &= (\mathbf{as})[(r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1];\ \Sigma_{r_1}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1]; \\
&\qquad \tau^*_\Sigma[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1 \mid r_1];\ \mathbf{I};\ \mathfrak{A}_1] \\
&= (\mathbf{as})[(r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1];\ \Sigma_{r_1}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1]; \\
&\qquad \tau^*_\Sigma[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1 \mid r_1];\ \mathbf{I};\ \mathfrak{B}_1] \\
&= (\mathbf{as})[(r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1];\ \Sigma_{r_1}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1]; \\
&\qquad \tau^0_\Sigma[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1 \mid r_1];\ \mathbf{I};\ \mathfrak{A}_1] \\
&= (\mathbf{as})[(r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1];\ \Sigma_{r_1}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1]; \\
&\qquad \tau^0_\Sigma[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1 \mid r_1];\ \mathbf{I};\ \mathfrak{B}_1],
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
\mathfrak{N} &= (\mathbf{as})[(r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2];\ \Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2]; \\
&\qquad \tau^*_\Sigma[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2 \mid r_2];\ \mathbf{J};\ \mathfrak{A}_2] \\
&= (\mathbf{as})[(r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2];\ \Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2]; \\
&\qquad \tau^*_\Sigma[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2 \mid r_2];\ \mathbf{J};\ \mathfrak{B}_2] \\
&= (\mathbf{as})[(r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2];\ \Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2]; \\
&\qquad \tau^0_\Sigma[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2 \mid r_2];\ \mathbf{J};\ \mathfrak{A}_2]
\end{aligned}$$

$$= (\mathbf{as})[(r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2];\ \Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2];$$

$$\tau_{\Sigma}^{0}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2 | r_2];\ \mathbf{J};\ \mathfrak{B}_2]. \tag{4.20}$$

In the following, we fix $\mathbf{k}_1 \in \mathbb{N}$ and $\mathbf{k}_2 \in \mathbb{N}$. Moreover, we fix the mappings

$$A^{(1)} : \overline{1, \mathbf{k}_1} \times \overline{1, r_1} \to B(E_1, \mathcal{L}_1); \quad A^{(2)} : \overline{1, \mathbf{k}_2} \times \overline{1, r_2} \to B(E_2, \mathcal{L}_2). \tag{4.21}$$

By $A^{(1)}$ and $A^{(2)}$, we define two matriciants. In addition, we follow to stipulations:

$$\left(A_{i,j}^{(1)} \triangleq A^{(1)}(i, j)\ \forall i \in \overline{1, \mathbf{k}_1}\ \forall j \in \overline{1, r_1}\right) \& \left(A_{i,j}^{(2)} \triangleq A^{(2)}(i, j)\ \forall i \in \overline{1, \mathbf{k}_2}\ \forall j \in \overline{1, r_2}\right). \tag{4.22}$$

Of course, in (4.22), the natural renamings are realized. With (4.21), the «traditional» vector-functionals are connected: The vector-functional

$$\widehat{\mathcal{A}_1} : (r_1 - \mathrm{adm})[\,\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1\,] \to \mathbb{R}^{\mathbf{k}_1}, \tag{4.23}$$

of player I is defined by the rule

$$\widehat{\mathcal{A}}_1((f_j)_{j \in \overline{1, r_1}}) \triangleq \left(\sum_{j=1}^{r_1} \int_{E_1} A_{i,j}^{(1)} f_j \, d\eta_1\right)_{i \in \overline{1, \mathbf{k}_1}} \forall (f_j)_{j \in \overline{1, r_1}} \in (r_1 - \mathrm{adm})[\,\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1\,]. \tag{4.24}$$

By analogy with (4.23) and (4.24), we introduce the vector-functional

$$\widehat{\mathcal{A}_2} : (r_2 - \mathrm{adm})[\,\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2\,] \to \mathbb{R}^{\mathbf{k}_2}, \tag{4.25}$$

of player II by the rule

$$\widehat{\mathcal{A}}_2((f_j)_{j \in \overline{1, r_2}}) \triangleq \left(\sum_{j=1}^{r_2} \int_{E_2} A_{i,j}^{(2)} f_j \, d\eta_2\right)_{i \in \overline{1, \mathbf{k}_2}} \forall (f_j)_{j \in \overline{1, r_2}} \in (r_2 - \mathrm{adm})$$

$$\times [\,\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2\,]. \tag{4.26}$$

We can consider (4.23), (4.24) and (4.25), (4.26) as goal operators in [9]-[11]. Moreover, we use $A^{(1)}$ for constructing of generalized vector-functional of player I. Namely, we introduce the generalized vector-functional

$$\mathbb{P} : \Sigma_{r_1}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1] \to \mathbb{R}^{\mathbf{k}_1}, \tag{4.27}$$

by the following natural rule:

$$\mathbb{P}((\mu_j)_{j\in\overline{1,r_1}}) \triangleq \left(\sum_{j=1}^{r_1}\int_{E_1} A_{i,j}^{(1)} d\mu_j\right)_{i\in\overline{1,\mathbf{k}_1}} \quad \forall (\mu_j)_{j\in\overline{1,r_1}} \in \Sigma_{r_1}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1]. \tag{4.28}$$

Analogously, we define the generalized vector-functional of player II. Namely, we suppose that

$$\mathbb{Q} : \Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2] \to \mathbb{R}^{\mathbf{k}_2}, \tag{4.29}$$

is defined by the rule

$$\mathbb{Q}((\nu_j)_{j\in\overline{1,r_2}}) \triangleq \left(\sum_{j=1}^{r_2}\int_{E_2} A_{i,j}^{(2)} d\nu_j\right)_{i\in\overline{1,\mathbf{k}_2}} \quad \forall (\nu_j)_{j\in\overline{1,r_2}} \in \Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2]. \tag{4.30}$$

Under employment of the sets $\mathfrak{M}$ and $\mathfrak{N}$ in the capacity of sets of generalized admissible controls of players I and II, we consider $\mathbb{P}^1(\mathfrak{M})$ and $\mathbb{Q}^1(\mathfrak{N})$ as distinctive attainability domains.

Of course, the mapping $\mathbb{P}$ is generalized analogue of $\widehat{\mathcal{A}}_1$, since

$$\mathbb{P} \circ \mathbf{I} = \widehat{\mathcal{A}}_1. \tag{4.31}$$

The testing of (4.31) is realized with employment of simplest properties of indefinite integral; see [8, (3.4.11)]. Moreover, we note that similarly to (3.28)

$$\mathbb{P} \in C\Big(\Sigma_{r_1}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1],\ \tau_{\Sigma}^{*}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1 |\ r_1],\ \mathbb{R}^{\mathbf{k}_1},\ \tau_{\mathbb{R}}^{(\mathbf{k}_1)}\Big). \quad (4.32)$$

So, we obtain the collection $(\Sigma_{r_1}[E_1; \mathcal{L}_1; \eta_1; \mathbb{F}_1], \tau_{\Sigma}^{*}[E_1; \mathcal{L}_1; \eta_1; \mathbb{F}_1 | r_1]; \mathbf{I}, \mathbb{P})$ with the properties (4.31) and (4.32). Moreover, we have the property similar to the compactness property of TS (3.12). Namely, $(\Sigma_{r_1}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1],\ \tau_{\Sigma}^{*}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1 |\ r_1])$ is a nonempty compactum. Then by [12, Corollary 3.1], the equality

$$\begin{aligned}(\mathbf{as})[(r_1 - \mathrm{adm})[\mathbb{F}_1 \,|\, E_1;\ \mathcal{L}_1;\ \eta_1];\ \mathbb{R}^{\mathbf{k}_1},\ \tau_{\mathbb{R}}^{(\mathbf{k}_1)};\ \widehat{\mathcal{A}}_1;\ \mathcal{E}_1] \\ = \mathbb{P}^1((\mathbf{as})[(r_1 - \mathrm{adm})[\mathbb{F}_1 \,|\, E_1;\ \mathcal{L}_1;\ \eta_1];\ \Sigma_{r_1}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1]; \\ \tau_{\Sigma}^{*}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1 |\ r_1];\ \mathbf{I};\ \mathcal{E}_1]) \\ \forall \mathcal{E}_1 \in \mathcal{P}'(\mathcal{P}((r_1 - \mathrm{adm})[\mathbb{F}_1 \,|\, E_1;\ \mathcal{L}_1;\ \eta_1])). \quad (4.33)\end{aligned}$$

Of course, we obtain the concrete variant of (3.29). We can use in (4.33) the families $\mathfrak{A}_1$ and $\mathfrak{B}_1$ instead of arbitrary family of subsets of $(r_1 - \mathrm{adm})[\mathbb{F}_1 \,|\, E_1;\ \mathcal{L}_1;\ \eta_1]$. In these cases, (4.19) and (4.33) are used. Then,

$$\begin{aligned}(\mathbf{as})[(r_1 - \mathrm{adm})[\mathbb{F}_1 \,|\, E_1;\ \mathcal{L}_1;\ \eta_1];\ \mathbb{R}^{\mathbf{k}_1},\ \tau_{\mathbb{R}}^{(\mathbf{k}_1)};\ \widehat{\mathcal{A}_1};\ \mathfrak{A}_1] \\ = (\mathbf{as})[(r_1 - \mathrm{adm})[\mathbb{F}_1 \,|\, E_1;\ \mathcal{L}_1;\ \eta_1];\ \mathbb{R}^{\mathbf{k}_1},\ \tau_{\mathbb{R}}^{(\mathbf{k}_1)};\ \widehat{\mathcal{A}}_1;\ \mathfrak{B}_1] = \mathbb{P}^1(\mathfrak{M}). \quad (4.34)\end{aligned}$$

In reality, the more general property takes place. We keep in mind the following concrete variant of Theorem 3.1: $\forall \mathcal{Z} \in \mathcal{P}'(\mathcal{P}((r_1 - \mathrm{adm})[\mathbb{F}_1 \,|E_1;$ $\mathcal{L}_1;\ \eta_1]))$

$$\begin{aligned}((\mathfrak{A}_1 \dashv \mathcal{Z})\,\&\,(\mathcal{Z} \dashv \mathfrak{B}_1)) \\ \Rightarrow \Big((\mathbf{as})[(r_1 - \mathrm{adm})[\mathbb{F}_1 \,|\, E_1;\ \mathcal{L}_1;\ \eta_1];\ \mathbb{R}^{\mathbf{k}_1},\ \tau_{\mathbb{R}}^{(\mathbf{k}_1)};\ \widehat{\mathcal{A}}_1;\ \mathcal{Z}] = \mathbb{P}^1(\mathfrak{M})\Big). \quad (4.35)\end{aligned}$$

So, $\mathbb{P}^1(\mathfrak{M})$ is very universal attraction set in $\mathbb{R}^{\mathbf{k}_1}$: We can use different families $\mathcal{Z}$ of subsets of $(r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1]$ with properties $\mathfrak{A}_1 \dashv \mathcal{Z}$ and $\mathcal{Z} \dashv \mathfrak{B}_1$.

Now, we consider $\mathbb{Q}$ as generalized analogue of $\widehat{\mathcal{A}}_2$. This interpretation is natural since

$$\widehat{\mathcal{A}}_2 = \mathbb{Q} \circ \mathbf{J}. \tag{4.36}$$

In addition, similar to (4.32), we have the continuity property

$$\mathbb{Q} \in C\Big(\Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2],\ \tau^*_{\Sigma}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2 |\ r_2],\ \mathbb{R}^{\mathbf{k}_2},\ \tau_{\mathbb{R}}^{(\mathbf{k}_2)}\Big). \tag{4.37}$$

We obtain the collection $(\Sigma_{r_2}[E_2; \mathcal{L}_2; \eta_2; \mathbb{F}_2], \tau^*_{\Sigma}[E_2; \mathcal{L}_2; \eta_2; \mathbb{F}_2 | r_2], \mathbf{J}, \mathbb{Q})$ with the properties (4.36) and (4.37). In addition, TS

$$(\Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2],\ \tau^*_{\Sigma}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2 |\ r_2]), \tag{4.38}$$

is a nonempty compactum. By [12, Corollary 3.1],

$$\begin{aligned}
&(\mathbf{as})[(r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2];\ \mathbb{R}^{\mathbf{k}_2},\ \tau_{\mathbb{R}}^{(\mathbf{k}_2)};\ \widehat{\mathcal{A}}_2;\ \mathcal{E}_2] \\
&\quad = \mathbb{Q}^1((\mathbf{as})[(r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2];\ \Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2]; \\
&\qquad\qquad \tau^*_{\Sigma}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2 |\ r_2];\ \mathbf{J};\ \mathcal{E}_2]) \\
&\qquad\qquad\qquad \forall\, \mathcal{E}_2 \in \mathcal{P}'(\mathcal{P}((r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2])).
\end{aligned} \tag{4.39}$$

Of course, we can consider the variants of (4.39) corresponding to employment of the families $\mathfrak{A}_2$ and $\mathfrak{B}_2$. Then, by (4.20) and (4.39),

$$\begin{aligned}
&(\mathbf{as})[(r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2];\ \mathbb{R}^{\mathbf{k}_2},\ \tau_{\mathbb{R}}^{(\mathbf{k}_2)},\ \widehat{\mathcal{A}}_2;\ \mathfrak{A}_2] \\
&\quad = (\mathbf{as})[(r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2];\ \mathbb{R}^{\mathbf{k}_2},\ \tau_{\mathbb{R}}^{(\mathbf{k}_2)};\ \widehat{\mathcal{A}}_2;\ \mathfrak{B}_2] = \mathbb{Q}^1(\mathfrak{N}).
\end{aligned} \tag{4.40}$$

Of course, we can supplement the relation (4.40). For this, we use the corresponding concrete variant of Theorem 3.1: $\forall \mathcal{Z} \in \mathcal{P}'(\mathcal{P}((r_2 - \mathrm{adm})$ $[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2]))$

$((\mathfrak{A}_2 \dashv \mathcal{Z})\,\&\,(\mathcal{Z} \dashv \mathfrak{B}_2))$

$$\Rightarrow \Big((\mathbf{as})[(r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2];\ \mathbb{R}^{\mathbf{k}_2},\ \tau_{\mathbb{R}}^{(\mathbf{k}_2)};\ \widehat{\mathcal{A}}_2;\ \mathcal{Z}] = \mathbb{Q}^1(\mathfrak{N})\Big). \quad (4.41)$$

We obtain that $\mathbb{Q}^1(\mathfrak{N})$ is very universal attraction set in $\mathbb{R}^{\mathbf{k}_2}$: Using $\mathbb{Q}^1(\mathfrak{N})$, we realize the attraction set for any family $\mathcal{Z}$ such that $\mathfrak{A}_2 \dashv \mathcal{Z}$ and $\mathcal{Z} \dashv \mathfrak{B}_2$.

On the other hand, $\mathbb{P}^1(\mathfrak{M})$ and $\mathbb{Q}^1(\mathfrak{N})$ are generalized «attainability domains». We can consider (in the following) some generalized game for which player I is realized the choice of vector measure

$$(\mu_j)_{j \in \overline{1, r_1}} \in \Sigma_{r_1}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1], \quad (4.42)$$

and player II is realized the choice of vector measure

$$(\nu_j)_{j \in \overline{1, r_2}} \in \Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2]. \quad (4.43)$$

Of course, vector measures in (4.42) and (4.43) play the role of strategies. In the following, we will introduce the corresponding cost function. But, this generalized game problem (we consider only maximin problem) will be used for realization of asymptotics of usual maximin values under employment of sets $Z_1 \in \mathcal{Z}_1$ and $Z_2 \in \mathcal{Z}_2$ in the capacity of constraints on the choice of vector-functions

$$\left(f_j^{(1)}\right)_{j \in \overline{1, r_1}} \in (r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1],$$

and

$$\left(f_j^{(2)}\right)_{j \in \overline{1, r_2}} \in (r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2],$$

respectively; here $\mathcal{Z}_1 \in \mathcal{P}'(\mathcal{P}((r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1]))$ and $\mathcal{Z}_2 \in \mathcal{P}'(\mathcal{P}((r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2]))$ satisfy to conditions

$$(\mathfrak{A}_1 \dashv \mathcal{Z}_1)\ \&\ (\mathcal{Z}_1 \dashv \mathfrak{B}_1) \text{ and } (\mathfrak{A}_2 \dashv \mathcal{Z}_2)\ \&\ (\mathcal{Z}_2 \dashv \mathfrak{B}_2),$$

respectively. The more detailed constructing will be reduced in the next section.

We recall that by (1.2), (1.3), and (4.3),

$$((\mathbf{n}_1,\ r_1)-\mathrm{ADM})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1;\ \widehat{\mathcal{O}}_\zeta^{(\mathbf{n}_1)}[\mathbb{Y}_1 \mid M_1];\ S^{(1)}]$$

$$\subset ((\mathbf{n}_1,\ r_1)-\mathrm{ADM})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathcal{O}_\zeta^{(\mathbf{n}_1)}[\mathbb{Y}_1];\ S^{(1)}] \quad \forall \zeta \in]0,\ \infty[.$$

As a corollary, from (4.13) and (4.17), we obtain that

$$\forall\, A \in \mathfrak{A}_1 \quad \exists B \in \mathfrak{B}_1 : B \subset A. \tag{4.44}$$

Now, from (3.31) and (4.44), we obtain the property

$$\mathfrak{A}_1 \dashv \mathfrak{B}_1. \tag{4.45}$$

Analogously, by (1.2), (1.3), and (4.3), we obtain that

$$((\mathbf{n}_2,\ r_2)-\mathrm{ADM})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2;\ \widehat{\mathcal{O}}_\zeta^{(\mathbf{n}_2)}[\mathbb{Y}_2 \mid M_2];\ S^{(2)}]$$

$$\subset ((\mathbf{n}_2,\ r_2)-\mathrm{ADM})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathcal{O}_\zeta^{(\mathbf{n}_2)}[\mathbb{Y}_2];\ S^{(2)}] \quad \forall \zeta \in]0,\ \infty[. \tag{4.46}$$

From (4.14), (4.18), and (4.46), we obtain that

$$\forall\, A \in \mathfrak{A}_2 \quad \exists\, B \in \mathfrak{B}_2 : B \subset A. \tag{4.47}$$

From (3.31) and (4.47), we obtain the following property:

$$\mathfrak{A}_2 \dashv \mathfrak{B}_2. \tag{4.48}$$

Using (4.45) and (4.48), we obtain that

$$(\mathfrak{A}_1 \dashv \mathfrak{B}_1)\ \&\ (\mathfrak{A}_2 \dashv \mathfrak{B}_2). \tag{4.49}$$

5. Maximin Problem with Constraints Weakening (Informative Setting)

We recall that $\mathbb{F}_1$ and $\mathbb{F}_2$ are nonempty finite-dimensional compactums. Therefore, for some $\mathbf{a}_1 \in]0, \infty[$ and $\mathbf{a}_2 \in]0, \infty[$, the properties

$$(\| \tilde{x} \|^{(\mathbf{k}_1)} \leqslant \mathbf{a}_1 \ \forall \tilde{x} \in \mathbb{F}_1) \quad \& \quad (\| \hat{x} \|^{(\mathbf{k}_2)} \leqslant \mathbf{a}_2 \ \forall \hat{x} \in \mathbb{F}_2), \tag{5.1}$$

take place. Using (4.4) and the definition of $\| \cdot \|^{(\mathbf{k}_1)}$, we obtain that

$$|\mu_l(E_1)| \leqslant \mathbf{a}_1 \ \forall (\mu_j)_{j \in \overline{1, r_1}} \in \Sigma_{r_1}[E_1; \mathcal{L}_1; \eta_1; \mathbb{F}_1] \quad \forall l \in \overline{1, r_1}. \tag{5.2}$$

By $\| \cdot \|_1$, we denote the natural sup-norm of the space $\mathbb{B}(E_1)$ of all bounded real-valued functions on E_1: We use the concrete variant of sup-norm $\| \cdot \|$ of the space $\mathbb{B}(E)$ in Section 2. Then, by (4.28),

$$|\mathbb{P}((\mu_j)_{j \in \overline{1, r_1}})(i)| \leqslant \mathbf{a}_1 \sum_{j=1}^{r_1} \left\| A_{i,j}^{(1)} \right\|_1 \forall (\mu_j)_{j \in \overline{1, r_1}} \in \Sigma_{r_1}[E_1; \mathcal{L}_1; \eta_1; \mathbb{F}_1] \quad \forall i \in \overline{1, \mathbf{k}_1}.$$

As a corollary, we have the following estimates. Namely, for $\alpha_1 \triangleq \mathbf{a}_1 \max\limits_{i \in \overline{1, \mathbf{k}_1}} \sum\limits_{j=1}^{r_1} \left\| A_{i,j}^{(1)} \right\|_1 \in [0, \infty[$, the inequality system takes place

$$\|\mathbb{P}((\mu_j)_{j \in \overline{1, r_1}})\|^{(\mathbf{k}_1)} \leqslant \alpha_1 \ \forall (\mu_j)_{j \in \overline{1, r_1}} \in \Sigma_{r_1}[E_1; \mathcal{L}_1; \eta_1; \mathbb{F}_1]. \tag{5.3}$$

In connection with (5.3), we introduce the following ball:

$$U \triangleq \{x \in \mathbb{R}^{\mathbf{k}_1} \mid \|x\|^{(\mathbf{k}_1)} \leqslant \alpha_1\}. \tag{5.4}$$

Then by (5.3) and (5.4), we obtain the obvious inclusion

$$\mathbb{P}^1(\Sigma_{r_1}[E_1; \mathcal{L}_1; \eta_1; \mathbb{F}_1]) \subset U. \tag{5.5}$$

We equip the nonempty set U (5.4) with the metric ρ_1 defined in the form

$$(x',\ x'') \mapsto \|x' - x''\|^{(\mathbf{k}_1)} : U \times U \to [0,\ \infty[.$$

In addition, topology $\tau_{\mathbb{R}}^{(\mathbf{k}_1)}|_U$ of the set U induced from $\left(\mathbb{R}^{\mathbf{k}_1},\ \tau_{\mathbb{R}}^{(\mathbf{k}_1)}\right)$ is generated by metric ρ_1. Then, $(U,\ \rho_1)$ is a compact metric space. In addition, by (4.11) and (5.5),

$$\mathbb{P}^1(\mathfrak{M}) \subset \mathbb{P}^1(\ \Sigma_{r_1}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1]) \subset U. \tag{5.6}$$

Now, we consider analogous estimates for player II (we keep in mind the inclusion chain similar to (5.6)). For this, we use the second statement of (5.1). By analogy with (5.2),

$$|\nu_l(E_2)| \leqslant \mathbf{a}_2\ \ \forall\,(\nu_j)_{j\in\overline{1,\,r_2}} \in \Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2] \quad \forall l \in \overline{1,\ r_2}. \tag{5.7}$$

We introduce sup-norm $\|\cdot\|_2$ of the space $\mathbb{B}(E_2)$ of all bounded real-valued function on E_2. By (4.30) and (5.7), we obtain that

$$|\mathbb{Q}((\nu_j)_{j\in\overline{1,\,r_2}})(i)| \leqslant \mathbf{a_2}\sum_{j=1}^{r_2}\left\|A_{i,\,j}^{(2)}\right\|_2 \qquad \forall(\nu_j)_{j\in\overline{1,\,r_2}} \in \Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2]$$

$$\forall i \in \overline{1,\ \mathbf{k}_2}. \tag{5.8}$$

We suppose that $\alpha_2 \in [0,\ \infty[$ is the number

$$\alpha_2 \triangleq \mathbf{a}_2 \max_{i\in\overline{1,\ \mathbf{k}_2}} \sum_{j=1}^{r_2}\left\|A_{i,\,j}^{(2)}\right\|_2.$$

By (5.8), we obtain the following estimates:

$$\|\mathbb{Q}((\nu_j)_{j\in\overline{1,\,r_2}})\|^{(\mathbf{k_2})} \leqslant \alpha_2\ \ \forall(\nu_j)_{j\in\overline{1,\,r_2}} \in \Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2]. \tag{5.9}$$

In connection with (5.9), we introduce the following ball:

$$V \triangleq \{x \in \mathbb{R}^{\mathbf{k}_2}\,|\,\|x\|^{(\mathbf{k}_2)} \leqslant \alpha_2\}. \tag{5.10}$$

By (5.9) and (5.10), we obtain the obvious inclusion

$$\mathbb{Q}^1(\Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2]) \subset V. \tag{5.11}$$

Using (5.10), we define the metric ρ_2 in the form $(x',\ x'') \mapsto \|x' - x''\|^{(\mathbf{k}_2)} : V \times V \to [0,\ \infty[$. Then topology $\tau_{\mathbb{R}}^{(\mathbf{k}_2)}|_V$ of the set V is induced by metric ρ_2. Of course, $(V,\ \rho_2)$ is a compact metric space. In addition, by (4.12) and (5.11),

$$\mathbb{Q}^1(\mathfrak{N}) \subset \mathbb{Q}^1(\Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2]) \subset V. \tag{5.12}$$

In the following, we use the natural combination of (5.6) and (5.12). Moreover, we use operators **I** and **J** of Section 4. In this connection, we note that by (4.7), (4.31), and (5.6),

$$\widehat{\mathcal{A}}_1((f_j')_{j\in\overline{1,r_1}}) = \mathbb{P}((f_j' * \eta_1)_{j\in\overline{1,r_1}}) \in U$$

$$\forall (f_j')_{j\in\overline{1,r_1}} \in (r_1 - \text{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1].$$

Therefore, we obtain the following property:

$$\widehat{\mathcal{A}}_1 : (r_1 - \text{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1] \to U. \tag{5.13}$$

On the other hand, by (4.8), (4.36), and (5.12),

$$\widehat{\mathcal{A}}_2((f_j'')_{j\in\overline{1,r_2}}) = \mathbb{Q}((f_j'' * \eta_2)_{j\in\overline{1,r_2}}) \in V$$

$$\forall (f_j'')_{j\in\overline{1,r_2}} \in (r_2 - \text{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2].$$

As a corollary, we have the obvious property

$$\widehat{\mathcal{A}}_2 : (r_2 - \text{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2] \to V. \tag{5.14}$$

So, if player I chooses the vector-function $(f_j')_{j\in\overline{1,r_1}} \in (r_1 - \text{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1]$ and player II chooses $(f_j'')_{j\in\overline{1,r_2}} \in (r_2 - \text{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2]$, then the pair

$$\left(\widehat{\mathcal{A}}_1((f_l')_{l\in\overline{1,r_1}}),\ \widehat{\mathcal{A}}_2((f_s'')_{s\in\overline{1,r_2}})\right) \in U \times V,$$

is realized. Analogous situation is realized under the choice of generalized controls. We consider the set $U \times V$ as a metric space. Namely, we introduce $\rho_3 : (U \times V) \times (U \times V) \to [0, \infty[$ by the following rule: If $u_1 \in U$, $v_1 \in V$, $u_2 \in U$, and $v_2 \in V$, then $\rho_3((u_1, v_1), (u_2, v_2)) = \sup(\{\rho_1(u_1, u_2); \rho_2(v_1, v_2)\})$. Then, ρ_3 is the concrete metric on $U \times V$ generating the natural product of the topologies of U and V generated by ρ_1 and ρ_2, respectively. Of course, $(U \times V, \rho_3)$ is a compact metric space.

In the following, we fix a function

$$\mathbf{f} : U \times V \to \mathbb{R}. \tag{5.15}$$

We consider $\mathbf{f}$ as the cost function in the corresponding game problem. Namely, we consider games

$$\downarrow \mathbf{f}\left(\widehat{\mathcal{A}}_1\left(\left(f_l'\right)_{l\in\overline{1, r_1}}\right), \quad \widehat{\mathcal{A}}_2\left(\left(f_s''\right)_{s\in\overline{1, r_2}}\right)\right)\uparrow, \tag{5.16}$$

with some constraints on the choice $(f_l')_{l\in\overline{1, r_1}}$ and $(f_s'')_{s\in\overline{1, r_2}}$. We suppose that these constraints are realized by weakening of the initial precise conditions. As a result, we obtain constraints of asymptotic character for which the game problems (5.16) are considered. We investigate maximin problems. Moreover, under $(\mu_l)_{l\in\overline{1, r_1}} \in \Sigma_{r_1}[E_1; \mathcal{L}_1; \eta_1; \mathbb{F}_1]$ and $(\nu_s)_{s\in\overline{1, r_2}} \in \Sigma_{r_2}[E_2; \mathcal{L}_2; \eta_2; \mathbb{F}_2]$, we obtain (see (5.5) and (5.11)) the pair

$$(\mathbb{P}((\mu_l)_{l\in\overline{1, r_1}}), \mathbb{Q}((\nu_s)_{s\in\overline{1, r_2}})) \in U \times V.$$

Therefore, we can consider the generalized game

$$\downarrow \mathbf{f}\,(\mathbb{P}((\mu_l)_{l\in\overline{1, r_1}}), \quad \mathbb{Q}((\nu_s)_{s\in\overline{1, r_2}}))\uparrow, \tag{5.17}$$

under some fixed constraints on the choice of $(\mu_l)_{l\in\overline{1, r_1}}$ and $(\nu_s)_{s\in\overline{1, r_2}}$:

$$((\mu_l)_{l\in\overline{1, r_1}} \in \mathfrak{M}) \quad \& \quad ((\nu_s)_{s\in\overline{1, r_2}} \in \mathfrak{N}). \tag{5.18}$$

Here, we investigate the maximin problem too. In the following, it is established that this maximin problem defines important variants of maximin asymptotics for game problems of type (5.16). So, the generalized problem (5.17), (5.18) defines the «true» result for game problems of type (5.16) under weakening of the initial precise conditions.

6. The Generalized Maximin Problem

In this section, we consider the game problem (5.17) and (5.18). Of course, this problem is correct under $\mathfrak{M} \neq \varnothing$ and $\mathfrak{N} \neq \varnothing$. In the following, we consider only this case; so, we investigate the case of compatible constraints of the generalized maximin problem: In the following, we suppose that

$$(\mathfrak{M} \neq \varnothing) \quad \& \quad (\mathfrak{N} \neq \varnothing). \tag{6.1}$$

Proposition 6.1. *Each of families* $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{B}_1,$ *and* $\mathfrak{B}_2$ *consists of nonempty sets.*

We use (4.19) and (4.20); moreover, we take into account [8, (2.5.1)].

□

Corollary 6.1. *If* $\mathcal{Z} \in \mathcal{P}'(\mathcal{P}((r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1; \mathcal{L}_1; \eta_1]))$ *and* $\mathcal{Z} \dashv \mathfrak{B}_1$, *then* $\varnothing \notin \mathcal{Z}$. *Moreover, if* $\tilde{\mathcal{Z}} \in \mathcal{P}'(\mathcal{P}((r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2; \mathcal{L}_2; \eta_2]))$ *and* $\tilde{\mathcal{Z}} \dashv \mathfrak{B}_2$, *then* $\varnothing \notin \tilde{\mathcal{Z}}$.

Proof follows from (3.31). We omit this obvious reasoning. From (4.13), (4.14), and Proposition 6.1, we obtain that

$$\left.\begin{array}{l} (\mathfrak{A}_1 \in \beta_0[(r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1]]), \\ \text{and} \\ (\mathfrak{A}_2 \in \beta_0[(r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2]]). \end{array}\right\} \tag{6.2}$$

Analogously, from (4.17), (4.18), and Proposition 6.1, the following properties imply:

$$\left.\begin{array}{l}(\mathfrak{B}_1 \in \beta_0[(r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1]]), \\ \text{and} \\ (\mathfrak{B}_2 \in \beta_0[(r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2]]).\end{array}\right\} \quad (6.3)$$

Finally, from Corollary 6.1, we obtain that

$$\left.\begin{array}{l}(\forall \mathcal{Z} \in \beta[(r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1]](\mathcal{Z} \dashv \mathfrak{B}_1) \\ \qquad \Rightarrow (\mathcal{Z} \in \beta_0[(r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1]])), \\ \text{and} \\ (\forall \widetilde{\mathcal{Z}} \in \beta[(r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2]](\widetilde{\mathcal{Z}} \dashv \mathfrak{B}_2) \\ \qquad \Rightarrow (\widetilde{\mathcal{Z}} \in \beta_0[(r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2]])).\end{array}\right\} (6.4)$$

So, in (6.2)-(6.4), we have the required variants of asymptotic compatibility. Now, we return to (6.1) and will consider the generalized game problem (5.17) and (5.18).

In the following, suppose that the cost function $\mathbf{f}$ (5.15) is continuous with respect to topology generated by metric ρ_3. By definition of ρ_3, we obtain that $\mathbf{f}$ is a function of two variables continuous with respect to totality of the above-mentioned variables. Then, for any vector $y \in V$, the function $\mathbf{f}(\cdot,\ y)$ defined in the form

$$x \mapsto \mathbf{f}(x,\ y) : U \to \mathbb{R}, \quad (6.5)$$

is continuous. In addition, the mapping (4.9) is continuous in the sense of topologies $\tau_{\Sigma}^{*}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1 | r_1]$ and $\tau_{\mathbb{R}}^{(\mathbf{n}_1)}$. Therefore, by (4.11), we obtain that $\mathfrak{M}$ is a closed set in topology $\tau_{\Sigma}^{*}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1 | r_1]$ (recall that $\mathbb{Y}_1$ is a closed set) and, as a corollary, a compact set in this topology.

With employment of (6.1), we obtain that $\mathfrak{M}$ is a nonempty set compact in topology $\tau_{\Sigma}^{*}[E_1;\ \mathcal{L}_1;\ \eta_1;\ \mathbb{F}_1|\ r_1]$. From (4.32), the compactness property of $\mathbb{P}^1(\mathfrak{M})$ is realized; of course, we keep in mind compactness in $\left(\mathbb{R}^{\mathbf{k}_1},\ \tau_{\mathbb{R}}^{(\mathbf{k}_1)}\right)$. Therefore, from (5.6), we obtain the compactness of $\mathbb{P}^1(\mathfrak{M})$ in topology $\tau_{\mathbb{R}}^{(\mathbf{k}_1)}|_U$, which is generated by metric ρ_1. Then, $\mathbb{P}^1(\mathfrak{M})$ is a nonempty compact set in metric space $(U,\ \rho_1)$. Therefore, by Weierstrass theorem and continuity of the functions (6.5), we obtain that, under $y \in V$, $\min\limits_{x \in \mathbb{P}^1(\mathfrak{M})} \mathbf{f}(x,\ y) \in \mathbb{R}$ is defined correctly. As a result, the function

$$y \mapsto \min_{x \in \mathbb{P}^1(\mathfrak{M})} \mathbf{f}(x,\ y) : V \to \mathbb{R}, \tag{6.6}$$

is defined correctly. In addition, under $y \in V$, the function

$$\mu \mapsto \mathbf{f}(\mathbb{P}(\mu),\ y) : \mathfrak{M} \to \mathbb{R},$$

is defined correctly and attains the minimum; as what is more,

$$\min_{\mu \in \mathfrak{M}} \mathbf{f}(\mathbb{P}(\mu),\ y) = \min_{x \in \mathbb{P}^1(\mathfrak{M})} \mathbf{f}(x,\ y). \tag{6.7}$$

From (6.7), we obtain that (6.6) coincides with the function

$$y \mapsto \min_{\mu \in \mathfrak{M}} \mathbf{f}(\mathbb{P}(\mu),\ y) : V \to \mathbb{R}. \tag{6.8}$$

So, we can use (6.8) instead of the above-mentioned function (6.6).

Proposition 6.2. *The function* (6.6) *and* (6.8) *is continuous*: $\left(\min\limits_{\mu \in \mathfrak{M}} \mathbf{f}(\mathbb{P}(\mu),\ y)\right)_{y \in V} \in \mathbb{C}(V,\ \tau_{\mathbb{R}}^{(\mathbf{k}_2)}|_V)$.

The proof is obvious: We use the coincidence of the functions (6.6) and (6.8) and uniform continuity of $\mathbf{f}$, since $(U \times V,\ \rho_3)$ is a compact metric space (in this connection, see [18, (3.4)]).

Recall that

$$\mathcal{S}_2 \in C\Big(\Sigma_{r_2}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2],\ \tau_{\Sigma}^{*}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2|\ r_2],\ \mathbb{R}^{\mathbf{n}_2},\ \tau_{\mathbb{R}}^{(\mathbf{n}_2)}\Big).$$

Since $\mathbb{Y}_2$ is a closed set, we obtain (see (4.12)) that $\mathfrak{N}$ is a closed set in the sense of topology $\tau_{\Sigma}^{*}[E_2;\ \mathcal{L}_2;\ \eta_2;\ \mathbb{F}_2|\ r_2]$. Using the compactness property of this topology, in the form of $\mathfrak{N}$, we have a nonempty compact set in TS (4.38). As a corollary (see (4.37)), the set $\mathbb{Q}^1(\mathfrak{N})$ is a nonempty compact set in $\Big(\mathbb{R}^{\mathbf{k}_2},\ \tau_{\mathbb{R}}^{(\mathbf{k}_2)}\Big)$; of course, we use the known property of image of a continuous mapping; for example, [7, p. 199]. With employment of (5.12) and the transitivity property of operation of the passage to a subspace of TS, we obtain that $\mathbb{Q}^1(\mathfrak{N})$ is the nonempty compact set in $(V,\ \tau_{\mathbb{R}}^{(\mathbf{k}_2)}|_V)$. In the other words, $\mathbb{Q}^1(\mathfrak{N})$ is the nonempty compact set in metric space $(V,\ \rho_2)$. Therefore, from Proposition 6.1, the function (6.8) attains the maximum on the set $\mathbb{Q}^1(\mathfrak{N})$; in addition,

$$\max_{y \in \mathbb{Q}^1(\mathfrak{N})} \min_{\mu \in \mathfrak{M}} \mathbf{f}(\mathbb{P}(\mu),\ y) = \max_{\nu \in \mathfrak{N}} \min_{\mu \in \mathfrak{M}} \mathbf{f}(\mathbb{P}(\mu),\ \mathbb{Q}(\nu)). \tag{6.9}$$

In (6.9), we use the following property: The image of $\mathbb{Q}^1(\mathfrak{N})$ under operation of function (6.8) coincides with the image of $\mathfrak{N}$ under operation of function

$$\nu \mapsto \min_{\mu \in \mathfrak{M}} \mathbf{f}(\mathbb{P}(\mu),\ \mathbb{Q}(\nu)) : \mathfrak{N} \to \mathbb{R};$$

the above-mentioned coincidence follows from definition of an image. We consider the number

$$\mathbf{V} = \max_{\nu \in \mathfrak{N}} \min_{\mu \in \mathfrak{M}} \mathbf{f}(\mathbb{P}(\mu),\ \mathbb{Q}(\nu)) \in \mathbb{R}, \tag{6.10}$$

as generalized maximin or the maximin in generalized problem. In the following, it will established that (6.10) defines asymptotics of realizable values of maximin for variants of «constraints» considered in premises of implications (4.35) and (4.41). In connection with (6.10), we recall (4.19) and (4.20). Moreover, from (6.7) and (6.9), we obtain that

$$\mathbf{V} = \max_{y \in \mathbb{Q}^1(\mathfrak{N})} \min_{x \in \mathbb{P}^1(\mathfrak{M})} \mathbf{f}(x,\ y). \tag{6.11}$$

In connection with (6.11), we use (4.34), (4.35), (4.40), and (4.41).

7. Asymptotics of Maximin

In this section, we fix families

$$(\mathcal{Z}_1 \in \mathcal{P}'(\mathcal{P}((r_1 - \text{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1]))) \,\&$$

$$(\mathcal{Z}_2 \in \mathcal{P}'(\mathcal{P}((r_2 - \text{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2]))), \tag{7.1}$$

with the following properties:

$$(\mathfrak{A}_1 \dashv \mathcal{Z}_1) \,\&\, (\mathcal{Z}_1 \dashv \mathfrak{B}_1) \,\&\, (\mathfrak{A}_2 \dashv \mathcal{Z}_2) \,\&\, (\mathcal{Z}_2 \dashv \mathfrak{B}_2). \tag{7.2}$$

Then by (4.35), (4.41), and (7.2), we obtain the following two equalities:

$$(\mathbf{as})[(r_1 - \text{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1];\ \mathbb{R}^{\mathbf{k}_1};\ \tau_{\mathbb{R}}^{(\mathbf{k}_1)};\ \widehat{\mathcal{A}}_1;\ \mathcal{Z}_1] = \mathbb{P}^1(\mathfrak{M}), \tag{7.3}$$

$$(\mathbf{as})[(r_2 - \text{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2];\ \mathbb{R}^{\mathbf{k}_2};\ \tau_{\mathbb{R}}^{(\mathbf{k}_2)};\ \widehat{\mathcal{A}}_2;\ \mathcal{Z}_2] = \mathbb{Q}^1(\mathfrak{N}). \tag{7.4}$$

In the following, we suppose that the families $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are directed:

$$\left.\begin{aligned} &(\mathcal{Z}_1 \in \beta[(r_1 - \text{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1]]), \\ \text{and}\quad & \\ &(\mathcal{Z}_2 \in \beta[(r_2 - \text{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2]]). \end{aligned}\right\} \tag{7.5}$$

Remark 7.1. Of course, we can use the cases

$$((\mathcal{Z}_1 = \mathfrak{A}_1) \,\&\, (\mathcal{Z}_2 = \mathfrak{A}_2)) \vee ((\mathcal{Z}_1 = \mathfrak{A}_1) \,\&\, (\mathcal{Z}_2 = \mathfrak{B}_2))$$

$$\vee ((\mathcal{Z}_1 = \mathfrak{B}_1) \,\&\, (\mathcal{Z}_2 = \mathfrak{A}_2)) \vee ((\mathcal{Z}_1 = \mathfrak{B}_1) \,\&\, (\mathcal{Z}_2 = \mathfrak{B}_2)).$$

These possibilities follow from (4.13), (4.14), (4.17), (4.18), and (4.49). □

From (1.8), (7.3), and (7.5), we obtain that

$$\mathbb{P}^1(\mathfrak{M}) = \bigcap_{H \in \mathcal{Z}_1} \mathrm{cl}\left(\widehat{\mathcal{A}}_1^1(H),\ \tau_{\mathbb{R}}^{(\mathbf{k}_1)}\right). \tag{7.6}$$

Analogously, from (1.8), (7.4), and (7.5), the equality

$$\mathbb{Q}^1(\mathfrak{N}) = \bigcap_{H \in \mathcal{Z}_2} \mathrm{cl}\left(\widehat{\mathcal{A}}_2^1(H),\ \tau_{\mathbb{R}}^{(\mathbf{k}_2)}\right), \tag{7.7}$$

is realized. Of course, by (6.1),

$$(\mathbb{P}^1(\mathfrak{M}) \neq \varnothing)\ \&\ (\mathbb{Q}^1(\mathfrak{N}) \neq \varnothing). \tag{7.8}$$

We note obvious corollaries of (7.6)-(7.8). Really, by (7.6) and (7.8), under $H \in \mathcal{Z}_1$, the property $H \neq \varnothing$ is realized. From (7.5), we have the property

$$\mathcal{Z}_1 \in \beta_0[(r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1;\ \mathcal{L}_1;\ \eta_1]]. \tag{7.9}$$

Analogously, from (7.7) and (7.8), we obtain that $\widetilde{H} \neq \varnothing\ \forall\ \widetilde{H} \in \mathcal{Z}_2$. From (7.5), the property

$$\mathcal{Z}_2 \in \beta_0[(r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2;\ \mathcal{L}_2;\ \eta_2]], \tag{7.10}$$

is realized. Now, we will use [18, Proposition 3]. In this connection, we recall (5.13). Then, under $S \in \mathcal{Z}_1$, we have the inclusion

$$\widehat{\mathcal{A}}_1^1(S) \subset U;$$

as a corollary, the following equality chain is realized:

$$\mathrm{cl}\left(\widehat{\mathcal{A}}_1^1(S),\ \tau_{\mathbb{R}}^{(\mathbf{k}_1)}|_U\right) = \mathrm{cl}\left(\widehat{\mathcal{A}}_1^1(S),\ \tau_{\mathbb{R}}^{(\mathbf{k}_1)}\right) \cap U = \mathrm{cl}\left(\widehat{\mathcal{A}}_1^1(S),\ \tau_{\mathbb{R}}^{(\mathbf{k}_1)}\right), \tag{7.11}$$

(since $\mathrm{cl}\left(\widehat{\mathcal{A}}_1^1(S),\ \tau_{\mathbb{R}}^{(\mathbf{k}_1)}\right) \subset U$; really, U is closed in the sense of $\tau_{\mathbb{R}}^{(\mathbf{k}_1)}$). From (7.6) and (7.11), we obtain that

$$\mathbb{P}^1(\mathfrak{M}) = \bigcap_{H \in \mathcal{Z}_1} \mathrm{cl}\left(\widehat{\mathcal{A}}_1^1(H), \tau_{\mathbb{R}}^{(\mathbf{k}_1)}|_U\right). \tag{7.12}$$

In addition, by (1.8), (5.13), (7.5), and (7.12), the equality

$$\mathbb{P}^1(\mathfrak{M}) = (\mathbf{as})[(r_1 - \mathrm{adm})[\mathbb{F}_1 \mid E_1; \mathcal{L}_1; \eta_1]; U; \tau_{\mathbb{R}}^{(\mathbf{k}_1)}|_U; \widehat{\mathcal{A}}_1; \mathcal{Z}_1], \tag{7.13}$$

holds (in (7.13), we have analogue of the first equality of [18, (2.6)]).

Now, we recall (5.14): Under $H \in \mathcal{Z}_2$, the inclusion $\widehat{\mathcal{A}}_2^1(H) \subset V$; as a corollary,

$$\mathrm{cl}\left(\widehat{\mathcal{A}}_2^1(H), \tau_{\mathbb{R}}^{(\mathbf{k}_2)}|_V\right) = \mathrm{cl}\left(\widehat{\mathcal{A}}_2^1(H), \tau_{\mathbb{R}}^{(\mathbf{k}_2)}\right) \cap V = \mathrm{cl}\left(\widehat{\mathcal{A}}_2^1(H), \tau_{\mathbb{R}}^{(\mathbf{k}_2)}\right). \tag{7.14}$$

This property is analogous to (7.11). By (7.7) and (7.14),

$$\mathbb{Q}^1(\mathfrak{N}) = \bigcap_{H \in \mathcal{Z}_2} \mathrm{cl}\left(\widehat{\mathcal{A}}_2^1(H), \tau_{\mathbb{R}}^{(\mathbf{k}_2)}|_V\right) = (\mathbf{as})[(r_2 - \mathrm{adm})[\mathbb{F}_2 \mid E_2; \mathcal{L}_2; \eta_2];$$

$$V; \tau_{\mathbb{R}}^{(\mathbf{k}_2)}|_V; \widehat{\mathcal{A}}_2; \mathcal{Z}_2], \tag{7.15}$$

(we use (1.8), (7.5), and (7.14)).

In the following, we use [18, Proposition 4]. For this, we introduce maximins corresponding to the case, then the constraints of our players are defined by the pair (H_1, H_2) of sets for which $H_1 \in \mathcal{Z}_1$ and $H_2 \in \mathcal{Z}_2$. We note that, for $H \in \mathcal{Z}_1$, the inclusion $\widehat{\mathcal{A}}_1^1(H) \in \mathcal{P}'(U)$ takes place. Recall that, in our case, $\mathbf{f}$ is a bounded real-valued function (indeed, $\mathbf{f}$ is a continuous real-valued function on compact metric space $(U \times V, \rho_3)$). Then, under $H \in \mathcal{Z}_1$, the values

$$\inf_{y \in \widehat{\mathcal{A}}_1^1(H)} \mathbf{f}(y, z) = \inf\left(\{\mathbf{f}(y, z) : y \in \widehat{\mathcal{A}}_1^1(H)\}\right) = \inf_{h \in H} \mathbf{f}\left(\widehat{\mathcal{A}}_1(h), z\right) \in \mathbb{R} \;\; \forall z \in V. \tag{7.16}$$

From the boundedness of **f** and (7.16), for $H \in \mathcal{Z}_1$, we obtain that

$$z \mapsto \inf_{h \in H} \mathbf{f}\left(\widehat{\mathcal{A}}_1(h),\, z\right) : V \to \mathbb{R},$$

is a bounded real-valued function too. Therefore, under $H \in \mathcal{Z}_1$, for some $\mathbf{d} \in]0, \infty[$

$$\left\{\inf_{h \in H} \mathbf{f}\left(\widehat{\mathcal{A}}_1(h),\, z\right) : z \in \widehat{\mathcal{A}}_2^1(\widetilde{H})\right\} \subset]-\infty, \mathbf{d}] \quad \forall \widetilde{H} \in \mathcal{Z}_2,$$

(of course, we use (5.14)). As a corollary,

$$\sup_{z \in \widehat{\mathcal{A}}_2^1(\widetilde{H})} \inf_{h \in H} \mathbf{f}\left(\widehat{\mathcal{A}}_1(h),\, z\right) = \sup_{\tilde{h} \in \widetilde{H}} \inf_{h \in H} \mathbf{f}\left(\widehat{\mathcal{A}}_1(h),\, \widehat{\mathcal{A}}_2(\tilde{h})\right) \in \mathbb{R}$$

$$\forall H \in \mathcal{Z}_1 \ \forall \widetilde{H} \in \mathcal{Z}_2, \tag{7.17}$$

is defined correctly. And what is more, from [18, Theorem 1], the following statement takes place:

Proposition 7.1. *If* $\zeta \in]0, \infty[$, *then there exist* $\mathbf{H}_\zeta \in \mathcal{Z}_1$ *and* $\widetilde{\mathbf{H}}_\zeta \in \mathcal{Z}_2$ *such that*

$$\left|\sup_{\tilde{h} \in \widetilde{H}} \inf_{h \in H} \mathbf{f}\left(\widehat{\mathcal{A}}_1(h),\, \widehat{\mathcal{A}}_2(\tilde{h})\right) - \mathbf{V}\right| < \zeta \quad \forall H \in \mathcal{Z}_1 \cap \mathcal{P}(\mathbf{H}_\zeta) \quad \forall \widetilde{H} \in \mathcal{Z}_2 \cap \mathcal{P}(\widetilde{\mathbf{H}}_\zeta).$$

8. Particular Cases

In this section, we realize several corollaries of Proposition 7.1. Of course, (6.1) is supposed fulfilled. We recall that in the capacity of $\mathcal{Z}_1$ (see (7.1)), we can use $\mathfrak{A}_1$ and $\mathfrak{B}_1$. Analogously, for $\mathcal{Z}_2$ (in (7.1)), we can use $\mathfrak{A}_2$ and $\mathfrak{B}_2$. Respectively, in constructions of Section 7, the following variants of the pair $(\mathcal{Z}_1, \mathcal{Z}_2)$ can use:

$$(\mathfrak{A}_1, \mathfrak{A}_2), (\mathfrak{A}_1, \mathfrak{B}_2), (\mathfrak{B}_1, \mathfrak{A}_2), (\mathfrak{B}_1, \mathfrak{B}_2). \tag{8.1}$$

Then (of course, we use (4.49)), we have the corresponding variants of (7.16) and (7.17) realized for sets of the families $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{B}_1$, and $\mathfrak{B}_2$. From (4.13) and (4.14), the following statement takes place:

Proposition 8.1. *If* $\zeta \in]0, \infty[$, *then there exist* $æ \in]0, \infty[$ *such that, for any* $\varepsilon \in]0, æ[$ *and* $\delta \in]0, æ[$, *the inequality*

$$\left| \sup_{v \in \widetilde{H}} \inf_{u \in H} \mathbf{f}\left(\widehat{\mathcal{A}}_1(u), \widehat{\mathcal{A}}_2(v)\right) - \mathbf{V} \right| < \zeta, \tag{8.2}$$

holds, where $H = ((\mathbf{n}_1, r_1) - \text{ADM})[\mathbb{F}_1 \mid E_1; \mathcal{L}_1; \eta_1; \mathcal{O}_\varepsilon^{(\mathbf{n}_1)}[\mathbb{Y}_1]; S^{(1)}]$ *and* $\widetilde{H} = ((\mathbf{n}_2, r_2) - \text{ADM})[\mathbb{F}_2 \mid E_2; \mathcal{L}_2; \eta_2; \mathcal{O}_\delta^{(\mathbf{n}_2)}[\mathbb{Y}_2]; S^{(2)}]$.

So, we investigate first variant of (8.1). The second variant of (8.1) is extracted from (4.49) and Proposition 7.1.

Proposition 8.2. *If* $\zeta \in]0, \infty[$, *then there exist* $æ \in]0, \infty[$ *such that, for any* $\varepsilon \in]0, æ[$ *and* $\delta \in]0, æ[$, *the inequality* (8.2) *holds for* $H = ((\mathbf{n}_1, r_1) - \text{ADM})[\mathbb{F}_1 \mid E_1; \mathcal{L}_1; \eta_1; \mathcal{O}_\varepsilon^{(\mathbf{n}_1)}[\mathbb{Y}_1]; S^{(1)}]$ *and* $\widetilde{H} = ((\mathbf{n}_2, r_2) - \text{ADM})[\mathbb{F}_2 \mid E_2; \mathcal{L}_2; \eta_2; \widehat{\mathcal{O}}_\delta^{(\mathbf{n}_2)}[\mathbb{Y}_2 \mid M_2]; S^{(2)}]$.

Now, we consider third variant of (8.1) using (4.49) and Proposition 7.1 again.

Proposition 8.3. *If* $\zeta \in]0, \infty[$, *then there exist* $æ \in]0, \infty[$ *such that, for any* $\varepsilon \in]0, æ[$ *and* $\delta \in]0, æ[$, *the inequality* (8.2) *holds for* $H = ((\mathbf{n}_1, r_1) - \text{ADM})[\mathbb{F}_1 \mid E_1; \mathcal{L}_1; \eta_1; \widehat{\mathcal{O}}_\varepsilon^{(\mathbf{n}_1)}[\mathbb{Y}_1 \mid M_1]; S^{(1)}]$ *and* $\widetilde{H} = ((\mathbf{n}_2, r_2) - \text{ADM})[\mathbb{F}_2 \mid E_2; \mathcal{L}_2; \eta_2; \mathcal{O}_\delta^{(\mathbf{n}_2)}[\mathbb{Y}_2]; S^{(2)}]$.

Finally, for fourth variant of (8.1), we obtain the following statement (see (4.49)):

Proposition 8.4. *If* $\zeta \in]0, \infty[$, *then there exist* $æ \in]0, \infty[$ *such that, for any* $\varepsilon \in]0, æ[$ *and* $\delta \in]0, æ[$, *the inequality* (8.2) *holds for* $H = ((\mathbf{n}_1, r_1) - \text{ADM})[\mathbb{F}_1 | E_1; \mathcal{L}_1; \eta_1; \widehat{\mathcal{O}}_{\varepsilon}^{(\mathbf{n}_1)}[\mathbb{Y}_1 | M_1]; S^{(1)}]$ *and* $\widetilde{H} = ((\mathbf{n}_2, r_2) - \text{ADM})[\mathbb{F}_2 | E_2; \mathcal{L}_2; \eta_2; \widehat{\mathcal{O}}_{\delta}^{(\mathbf{n}_2)}[\mathbb{Y}_2 | M_2]; S^{(2)}]$.

Of course, Propositions 8.1-8.4 are simple corollaries of Proposition 7.1.

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