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APPLICATIONS OF THE RETRACTION METHOD ON ANALYSIS OF EXISTENCE AND APPROXIMATION OF SOLUTIONS OF QUASILINEAR DIFFERENTIAL EQUATIONS

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Abstract

The paper presents some result on the existence and behaviour of parameter classes of solutions for system of quasilinear differential equations. Behaviour of integral curves in neighbourhoods of solution of corresponding linear system is considered. The obtained results contain the answer to the question on approximation of solutions, whose existence is established. The errors of the approximation are defined by the functions that can be sufficiently small.

The theory of qualitative analysis of differential equations and topological retraction method are used.

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1. Introduction

In [5] are presented some results on existence and approximation of solutions for system of quasilinear differential equations in neighbourhoods of an arbitrary curve. Now, we consider behaviour of solutions of the system of quasilinear differential equations in neighbourhoods of integral curve of corresponding linear system.

Let us consider the systems of differential equations:

$$\dot{x} = A(x, t)x + F(x, t), \tag{1.1}$$

$$\dot{x} = C(t)x + G(t), \tag{1.2}$$

where

$$\left.\begin{aligned} A(x, t) &= C(t) + D(x, t) \\ F(x, t) &= G(t) + H(x, t) \end{aligned}\right\}, \tag{1.3}$$

$x(t) = (x_1(t), \dots, x_n(t))^T$, $n \geq 2$; $t \in I = \langle a, \infty \rangle$, $a \in \mathbb{R}$; $D \subset \mathbb{R}^n$ is open set, $\Omega = D \times I$, $A(x, t) = (a_{ij}(x, t))_{n \times n}$ is the matrix-function with elements $a_{ij} \in C(\Omega, \mathbb{R})$ $(i, j = 1, \dots, n)$, $F(x, t) = (f_1(x, t), \dots, f_n(x, t))^T$ is the vector-function with elements $f_i \in C(\Omega, \mathbb{R})$. The matrix-functions $C(t) = (c_{ij}(t))_{n \times n}$, $G(t) = (g_{ij}(t))_{n \times 1}$, $D(x, t) = (d_{ij}(x, t))_{n \times n}$, and $H(x, t) = (h_{ij}(x, t))_{n \times 1}$ are given by (1.3). Moreover, $A(x, t)$ and $F(x, t)$ satisfy sufficient conditions for existence and uniqueness of solution of any system (1.1) in Ω.

Let

$$\Gamma = \{(x, t) \in \Omega : x = \varphi(t), \quad t \in I\},$$

where $\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))$, $\varphi_i(t) \in C^1(I, \mathbb{R})$, is an integral curve of (1.2)

In this paper, the behaviour of the solutions of system (1.1) in neighbourhood of curve Γ is considered. The qualitative analysis theory of differential equations and the topological retraction method of Wažewski [10], are used.

2. Notation and Preliminaries

We shall consider the behaviour of integral curves $(x(t), t), t \in I,$ of system (1.1) with respect to the set

$$\omega = \{(x, t) \in \Omega : |x_i - \varphi_i(t)| < r_i(t), \quad i = 1, \dots, n\},$$

where $r_i \in C^1(I, \mathbb{R}^+), i = 1, \dots, n.$ The boundary surfaces of the set ω with respect to the set Ω

$$W_i^k = \left\{(x, t) \in Cl\omega \cap \Omega : B_i^k(x, t) := (-1)^k (x_i - \varphi_i(t)) - r_i(t) = 0\right\},$$

where $i = 1, \dots, n; \ k = 1, 2.$ In direct x_i-axis, we have $W_i^k, \ k = 1, 2.$

Let us denote the tangent vector field to an integral curve $(x(t), t), t \in I,$ of (1.1) by T. The vectors ∇B_i^k are the external normal on surfaces W_i^k. We have

$$T = \left(\sum_{j=1}^{n} a_{1j} x_j + f_1, \dots, \sum_{j=1}^{n} a_{ij} x_j + f_i, \dots, \sum_{j=1}^{n} a_{nj} x_j + f_n, 1\right),$$

$$\nabla B_i^k = \left(\delta_{1i}, \dots, \delta_{ni}, (-1)^{k+1} \varphi_i' - r_i'\right),$$

where δ_{ji} is the Kronecker delta symbol.

By means of the scalar products

$$P_i^k(x, t) = (\nabla B_i^k, T) \quad \text{on} \quad W_i^k, \quad k = 1, 2, \quad i = 1, \dots, n,$$

we shall establish the behaviour of integral curve of (1.1) with respect to the set ω.

Let us denote by $S^p(I)$, $p \in \{0, 1, \dots, n\}$, a class of solutions of the system (1.1), defined on I, which depends on p parameters. We shall simply say that the class of solutions $S^p(I)$ belongs to the set ω, if graphs of function in $S^p(I)$ are contained in ω. In that case, we shall write $S^p(I) \subset \omega$. For $p = 0$, we have the notation $S^0(I)$, which means that exist at least one solution $(x(t), t)$ on I of the system (1.1), whose graph belongs to the set ω.

The results of this paper are based on the following lemma (see [4], [7]). In the following, $(n_1, \cdots, n_n)$ denote a permutation of the indices $(1, \cdots, n)$.

Lemma 2.1. *If*, *for the system* (1.1), *the scalar product*

$$P_i^k(x, t) = (\nabla B_i^k, T) < 0 \quad on \quad W_i^k, \quad k = 1, 2, \quad i = n_1, \dots, n_p; \tag{2.4}$$

and

$$P_i^k(x, t) = (\nabla B_i^k, T) > 0 \quad on \quad W_i^k, \quad k = 1, 2, \quad i = n_{p+1}, \dots, n_n, \tag{2.5}$$

where $p \in \{0, 1, \dots, n\}$, *then the system* (1.1) *has a class of solutions* $S^p(I)$ *belonging to the set* σ *for all* $t \in I$, *i.e.,* $S^p(I) \subset \sigma, p \in \{0, 1, \dots, n\}$.

Notice that, according to this lemma, the case $p = 0$ means that the system (1.1) has at least one solution belonging to the set ω for all $t \in I$.

The conditions (2.4) and (2.5) imply that the set $U = \bigcup_{i=n_1}^{n_p} (W_i^1 \cup W_i^2)$ has no point of exit and $V = \bigcup_{i=n_{p+1}}^{n_n} (W_i^1 \cup W_i^2)$ is the set of points of strict exit from set ω with respect to the set Ω, for integral curves of system (1.1), which according to the retraction method [10], makes the statement of lemma valid (see [6], [7], [9]). In the case $p = n$, this lemma gives the statement of Lemma 1, and for $p = 0$, the statement of Lemma 2 in [8].

3. The Main Results

Theorem 3.1. *Let* Γ *be an integral curve of the system* (1.2) *and* $r_i \in C^1(I, \mathbb{R}^+)$. *If*

$$\left| \sum_{j=1(j\neq i)}^{n} a_{ij}(x, t)x_j + d_{ii}(x, t)\varphi_i(t) + h_i(x, t) \right| < -a_{ii}(x, t)r_i(t) + r_i'(t) \quad (3.6)$$

on W_i^k, $k = 1, 2$, $i = n_1, \ldots, n_p$; *and*

$$\left| \sum_{j=1(j\neq i)}^{n} a_{ij}(x, t)x_j + d_{ii}(x, t)\varphi_i(t) + h_i(x, t) \right| < a_{ii}(x, t)r_i(t) - r_i'(t) \quad (3.7)$$

on W_i^k, $k = 1, 2$, $i = n_{p+1}, \ldots, n_n$, *then the system* (1.1) *has a p-parameter class of solutions* $S^p(I)$, *which belongs the set* ω *for all* $t \in I$, *i.e.,* $S^p(I) \subset \omega$.

Proof. For the scalar product $P_i^k(x, t) = (\nabla B_i^k, T)$ on W_i^k, $k = 1, 2$, $i = 1, \ldots, n$, we have

$$P_i^k(x, t) = a_{ii}r_i + (-1)^k \left(\sum_{j=1(j\neq i)}^{n} a_{ij}x_j + (c_{ii} + d_{ii})\varphi_i + g_i + h_i - \varphi_i' \right) - r_i'$$

$$= a_{ii}r_i + (-1)^k \left(\sum_{j=1(j\neq i)}^{n} a_{ij}x_j + d_{ii}\varphi_i + h_i \right) - r_i'.$$

Now, according to (3.6) and (3.7), the following estimates valid, respectively,

$$P_i^k(x, t) \le a_{ii}r_i + \left| \sum_{j=1(j\neq i)}^{n} a_{ij}x_j + d_{ii}\varphi_i + h_i \right| - r_i' < 0$$

on W_i^k, $k = 1, 2$, $i = n_1, \ldots, n_p$; and

$$P_i^k(x, t) \geq a_{ii} r_i - \left| \sum_{j=1 (j \neq i)}^{n} a_{ij} x_j + d_{ii} \varphi_i + h_i \right| - r_i' > 0$$

on W_i^k, $k = 1, 2$, $i = n_{p+1}, \ldots, n_n$.

Hence, in direct p axis, we have $P_i^k(x, t) < 0$ on W_i^k, and in direct rest $n - p$ axis, we have $P_i^k(x, t) > 0$ on W_i^k, $k = 1, 2$. According to Lemma 2.1, the above estimates for $P_i^k(x, t)$ on W_i^k, confirms the statements of the theorem. □

Corollary 3.1. *Let $r_i \in C^1(I, \mathbb{R}^+)$ and C be a diagonal matrix (i.e., $c_{ij} = 0$ for $i \neq j$). If*

$$|d_{ii}(x, t)\varphi_i(t) + h_i(x, t)| < -a_{ii}(x, t) r_i(t) + r_i'(t)$$

on W_i^k, $k = 1, 2$, $i = n_1, \ldots, n_p$; and

$$|d_{ii}(x, t)\varphi_i(t) + h_i(x, t)| < a_{ii}(x, t) r_i(t) - r_i'(t)$$

on W_i^k, $k = 1, 2$, $i = n_{p+1}, \ldots, n_n$, then the system (1.1) *has a p-parameter class of solutions $S^p(I)$, which belongs the set ω for all $t \in I$, i.e., $S^p(I) \subset \omega$.*

Theorem 3.2. *Let Γ be an integral curve of the system* (1.2). *If*

$$\left| \sum_{j=1 (j \neq i)}^{n} c_{ij}(t)(x_j - \varphi_j(t)) + f_i(x, t) \right| < -c_{ii}(t) r_i(t) + r_i'(t) \tag{3.8}$$

on W_i^k, $k = 1, 2$, $i = n_1, \ldots, n_p$; and

$$\left| \sum_{j=1 (j \neq i)}^{n} c_{ij}(t)(x_j - \varphi_j(t)) + f_i(x, t) \right| < c_{ii}(t) r_i(t) - r_i'(t) \tag{3.9}$$

on W_i^k, $k = 1, 2$, $i = n_{p+1}, \ldots, n_n$, then the system (1.1) *has a p-parameter class of solutions $S^p(I) \subset \omega$.*

Proof. Here we have

$$P_i^k(x, t) = (-1)^k \left(\sum_{j=1}^{n} c_{ij} x_j + g_i + f_i - \varphi_i' \right) - r_i'$$

$$= (-1)^k \left(\sum_{j=1}^{n} c_{ij} (x_j - \varphi_j) + f_i + \sum_{j=1}^{n} c_{ij} \varphi_j + g_i - \varphi_i' \right) - r_i'$$

$$= (-1)^k \left(\sum_{j=1}^{n} c_{ij} (x_j - \varphi_j) + f_i \right) - r_i'$$

$$= (-1)^k c_{ii} (x_i - \varphi_i) + (-1)^k \left(\sum_{j=1 (j \neq i)}^{n} c_{ij} (x_j - \varphi_j) + f_i \right) - r_i'$$

$$= c_{ii} r_i + (-1)^k \left(\sum_{j=1 (j \neq i)}^{n} c_{ij} (x_j - \varphi_j) + f_i \right) - r_i'.$$

In view of (3.8) and (3.9), the following estimates valid, respectively,

$$P_i^k(x, t) \leq c_{ii} r_i + \left| \sum_{j=1 (j \neq i)}^{n} c_{ij} (x_j - \varphi_j) + f_i \right| - r_i' < 0$$

on W_i^k, $k = 1, 2$, $i = n_1, \ldots, n_p$; and

$$P_i^k(x, t) \geq c_{ii} r_i - \left| \sum_{j=1 (j \neq i)}^{n} c_{ij} (x_j - \varphi_j) + f_i \right| - r_i' > 0$$

on W_i^k, $k = 1, 2$, $i = n_{p+1}, \ldots, n_n$.

Hence, in direct p axis, we have $P_i^k(x, t) < 0$ on W_i^k, and in direct rest $n - p$ axis, we have $P_i^k(x, t) > 0$ on W_i^k, $k = 1, 2$. The estimates for $P_i^k(x, t)$ on W_i^k, according to Lemma 2.1, imply the statement of the theorem. □

Corollary 3.2. *If C be a diagonal matrix* $(i.e., c_{ij} = 0$ *for* $i \neq j)$ *and*

$$|f_i(x, t)| < -c_{ii}(t)r_i(t) + r_i'(t)$$

on W_i^k, $k = 1, 2$, $i = n_1, \dots, n_p$; *and*

$$|f_i(x, t)| < c_{ii}(t)r_i(t) - r_i'(t)$$

on W_i^k, $k = 1, 2$, $i = n_{p+1}, \dots, n_n$, *then the system* (1.1) *has a class of solutions* $S^p(I) \subset \omega$.

4. Applications

Let us consider the Lotka-Volterra model ([3])

$$\dot{x}_1 = x_1 - x_1x_2,$$

$$\dot{x}_2 = -x_2 + x_1x_2. \tag{4.10}$$

Theorem 4.3. *Let functions* $r_1, r_2 \in C^1(I, \mathbb{R}^+)$ *satisfy the conditions*

$$r_1(t) < 1 + \frac{r_2'(t)}{r_2(t)}, \quad r_2(t) < 1 - \frac{r_1'(t)}{r_1(t)}, \quad t \in I.$$

The system (4.10) *has a one-parameter class of solutions* $(x_1(t), x_2(t))$ *satisfying the condition*

$$|x_1(t)| < r_1(t), \quad |x_2(t)| < r_2(t), \quad t \in I.$$

Proof. Here, we consider the behaviour of integral curves $(x_1(t), x_2(t), t)$, $t \in I$, of system (4.10) with respect to the set

$$\omega = \{(x_1, x_2, t) \in \mathbb{R}^3 : |x_i| < r_i(t), \quad i = 1, 2\},$$

where $r_i \in C^1(I, \mathbb{R}^+)$, $i = 1, 2$. The boundary surfaces of the set ω

$$W_i^k = \left\{(x, t) \in Cl\omega \cap \mathbb{R}^3 : B_i^k(x, t) := (-1)^k x_i - r_i(t) = 0\right\},$$

where $i = 1, 2$; $k = 1, 2$. In direct x_i-axis, we have W_i^k, $k = 1, 2$. We have

$$P_2^k(x_1, x_2, t) = (x_1 - 1)r_2(t) - r_2'(t) \leq (r_1(t) - 1)r_2(t) - r_2'(t) < 0,$$

$$P_1^k(x_1, x_2, t) = (1 - x_2)r_1(t) - r_1'(t) \geq (1 - r_2(t))r_1(t) - r_1'(t) > 0.$$

According to Lemma 2.1, these estimates for $P_i^k(x_1, x_2, t)$ on W_i^k imply the statement of the theorem. □

For functions r_i, we can take, for example,

$$r_1(t) = r_2(t) = \alpha \exp(-\beta t), \quad \alpha, \beta \in \mathbb{R}^+, \alpha + \beta \leq 1, \quad t > 0.$$

Remark. We can note that the obtained results immediately possible to determine the approximate solutions with the assessment errors. The obtained results give the sufficient conditions for the existence of a class of the $S^p(I) \subset \omega$, where the set of ω precisely defined and whose "width" is given by the function of $2r(t)$, $t \in I$, which takes a sufficiently small value for every $t \in I$. In this case, each function $\xi(t)$ whose graph belongs to ω and for which $\xi(t_0) = x(t_0) = x_0$, $(x_0, t_0) \in \omega$, can serve as an approximation of the unknown solutions $x(t)$ on I with error of approximation exactly $2r(t)$.

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