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SOME TOEPLITZ TYPE TRANSFORMS RELATED TO GENERAL SINGULAR INTEGRAL OPERATOR

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Abstract

In this paper, we establish the sharp maximal function estimates for some Toeplitz type transforms related to the singular integral operator with general kernel. As an application, we obtain the boundedness of the transforms on weighted Lebesgue and Triebel-Lizorkin spaces.

1. Introduction and Preliminaries

As the development of singular integral operators (see [7, 17, 18]), their commutators have been well studied. In [5, 15, 16], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [3]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [4, 9, 14], the boundedness for the

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commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [1, 8], the boundedness for the commutators generated by the singular integral operators and the weighted BMO and Lipschitz functions on $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [2], some singular integral operators with general kernel are introduced, and the boundedness for the operators and their commutators generated by BMO and Lipschitz functions are obtained (see [2, 12]). In [10, 11], some Toeplitz type operators related to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by BMO and Lipschitz functions are obtained. In this paper, we will study the Toeplitz type transforms generated by the singular integral operators with general kernel and the weighted Lipschitz functions. First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well-known that (see [7, 17])

$$M^{\#}(f)(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For $\eta > 0$, let $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < 1$ and $1 \le r < \infty$, set

$$M_{\eta, r}(f)(x) = \sup_{Q \ni x}\left(\frac{1}{|Q|^{1-r\eta/n}} \int_Q |f(y)|^r dy\right)^{1/r}.$$

The A_p weight is defined by (see [7])

$$A_p = \left\{w \in L^1_{\text{loc}}(R^n) : \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty\right\},$$

$$1 < p < \infty,$$

and

$$A_1 = \{w \in L^p_{\text{loc}}(R^n) : M(w)(x) \le Cw(x), \text{ a.e.}\}.$$

The $A(p, r)$ weight is defined by (see [13]), for $1 < p, r < \infty$,

$$A(p, r) =$$

$$\left\{w > 0 : \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)^r dx\right)^{1/r}\left(\frac{1}{|Q|}\int_Q w(x)^{-p/(p-1)}dx\right)^{(p-1)/p} < \infty\right\}.$$

Given a non-negative weight function w. For $1 \le p < \infty$, the weighted Lebesgue space $L^p(w)$ is the space of functions f such that

$$\|f\|_{L^p(w)} = \left(\int_{R^n} |f(x)|^p w(x)dx\right)^{1/p} < \infty.$$

For $\beta > 0$, $p > 1$ and the non-negative weight function w, let $\dot{F}_p^{\beta, \infty}(w)$ be the weighted homogeneous Triebel-Lizorkin space (see [14]).

For $0 < \beta < 1$ and the non-negative weight function w, the weighted Lipschitz space $Lip_\beta(w)$ is the space of functions b such that

$$\|b\|_{Lip_\beta(w)} = \sup_Q \frac{1}{w(Q)^{1+\beta/n}} \int_Q |b(y) - b_Q| dy < \infty.$$

Remark. (1) It has been known that, for $b \in Lip_\beta(w)$, $w \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^j Q}| \leq Cj\|b\|_{Lip_\beta(w)} w(x) w(2^j Q)^{\beta/n}.$$

(2) Let $b \in Lip_\beta(w)$ and $w \in A_1$. By [6], we know that spaces $Lip_\beta(w)$ coincide and the norms $\|b\|_{Lip_\beta(w)}$ are equivalent with respect to different values $1 \leq p \leq \infty$.

In this paper, we will study some singular integral operators as following (see [2]):

Definition. Let $T : S \to S'$ be a linear operator such that T is bounded on $L^2(R^n)$ and there exists a locally integrable function $K(x, y)$ on $R^n \times R^n \setminus \{(x, y) \in R^n \times R^n : x = y\}$ such that

$$T(f)(x) = \int_{R^n} K(x, y) f(y) dy,$$

for every bounded and compactly supported function f, where K satisfies: There is a sequence of positive constant numbers $\{C_j\}$ such that for any $j \geq 1$,

$$\int_{2|y-z|<|x-y|} (|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)|) dx \leq C,$$

and

$$\left(\int_{2^j|z-y|\leq|x-y|<2^{j+1}|z-y|} (|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)|)^q dy\right)^{1/q}$$
$$\leq C_j (2^j |z - y|)^{-n/q'},$$

where $1 < q' < 2$ and $1/q + 1/q' = 1$.

Let b be a locally integrable function on R^n. The Toeplitz type transform related to T is defined by

$$T_b = \sum_{k=1}^{m} T^{k,1} M_b T^{k,2},$$

where $T^{k,1}$ are T or $\pm I$ (the identity operator), $T^{k,2}$ are the bounded linear operators on $L^p(R^n)$ for $1 < p < \infty$ and $k = 1, \ldots, m$, $M_b(f) = bf$.

Note that the classical Calderón-Zygmund singular integral operator satisfies Definition 1 with $C_j = 2^{-j\delta}$ (see [7, 17]). And note that the commutator $[b, T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz type operators T_b. The Toeplitz type operators T_b are the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [15, 16]). The main purpose of this paper is to prove the sharp maximal inequalities for the Toeplitz type transforms T_b. As the application, we obtain the weighted L^p-norm inequality and Triebel-Lizorkin spaces boundedness for the Toeplitz type transforms T_b.

2. Theorems

We shall prove the following theorems:

Theorem 1. *Let T be the singular integral operator as Definition, the sequence $\{jC_j\} \in l^1$, $w \in A_1$, $0 < \beta < 1$, $q' \le s < \infty$, and $b \in Lip_\beta(w)$. If $g \in L^p(R^n)(1 < p < \infty)$ and $T_1(g) = 0$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$M^{\#}(T_b(f))(\tilde{x}) \le C\|b\|_{Lip_\beta(w)} w(\tilde{x})^{1+\beta/n} \sum_{k=1}^{m} M_{\beta,s}(T^{k,2}(f))(\tilde{x}).$$

Theorem 2. *Let T be the singular integral operator as Definition, the sequence* $\{j2^{j\beta}C_j\} \in l^1$, $w \in A_1$, $0 < \beta < 1$, $q' \le s < \infty$, *and* $b \in Lip_\beta(w)$. *If* $g \in L^p(R^n)(1 < p < \infty)$ *and* $T_1(g) = 0$, *then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$\sup_{Q \ni \tilde{x}} \inf_{c \in R^n} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T_b(f)(x) - c| dx$$

$$\le C\|b\|_{Lip_\beta(w)} w(\tilde{x})^{1+\beta/n} \sum_{k=1}^{m} M_{\beta,s}(T^{k,2}(f))(\tilde{x}).$$

Theorem 3. *Let T be the singular integral operator as Definition, the sequence* $\{jC_j\} \in l^1$, $w \in A_1$, $0 < \beta < \min(1, n/q')$, $q' < p < n/\beta$, $1/r = 1/p - \beta/n$, *and* $b \in Lip_\beta(w)$. *If* $g \in L^p(R^n)(1 < p < \infty)$ *and* $T_1(g) = 0$, *then* T_b *is bounded from* $L^p(w)$ *to* $L^r(w^{r/p - r(1+\beta/n)})$.

Theorem 4. *Let T be the singular integral operator as Definition, the sequence* $\{j2^{j\beta}C_j\} \in l^1$, $w \in A_1$, $0 < \beta < \min(1, n/q')$, $q' < p < n/\beta$, $1/r = 1/p - \beta/n$, *and* $b \in Lip_\beta(w)$. *If* $g \in L^p(R^n)(1 < p < \infty)$ *and* $T_1(g) = 0$, *then* T_b *is bounded from* $L^p(w)$ *to* $\dot{F}_r^{\beta,\infty}(w^{r/p - r(1+\beta/n)})$.

3. Proofs of Theorems

To prove the theorems, we need the following lemmas:

Lemma 1 (See [2]). *Let T be the singular integral operator as Definition, the sequence* $\{C_j\} \in l^1$. *Then T is bounded on* $L^p(w)$ *for* $w \in A_\infty$ *with* $1 < p < \infty$.

Lemma 2 (See [6]). *For any cube* Q, $b \in Lip_\beta(w)$, $0 < \beta < 1$ *and* $w \in A_1$, *we have*

$$\sup_{x \in Q} |b(x) - b_Q| \leq C \|b\|_{Lip_\beta(w)} w(Q)^{1+\beta/n} |Q|^{-1}.$$

Lemma 3 (See [14]). *For* $0 < \beta < 1$, $1 < p < \infty$ *and* $w \in A_\infty$, *we have*

$$\|f\|_{\dot{F}_p^{\beta,\infty}(w)} \approx \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| dx \right\|_{L^p(w)}$$

$$\approx \left\| \sup_{Q \ni \cdot} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c| dx \right\|_{L^p(w)}.$$

Lemma 4 (See [7]). *Let* $0 < p < \infty$ *and* $w \in \bigcup_{1 \leq r < \infty} A_r$. *Then, for any smooth function f for which the left-hand side is finite,*

$$\int_{R^n} M(f)(x)^p w(x) dx \leq C \int_{R^n} M^{\#}(f)(x)^p w(x) dx.$$

Lemma 5 (See [13]). *Suppose that* $0 \leq \eta < n$, $1 < s < p < n/\eta$, $1/r = 1/p - \eta/n$ *and* $w \in A(p, r)$. *Then*

$$\|M_{\eta,s}(f)\|_{L^r(w^r)} \leq C \|f\|_{L^p(w^p)}.$$

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|} \int_Q |T_b(f)(x) - C_0| dx \leq C \|b\|_{Lip_\beta(w)} w(\tilde{x})^{1+\beta/n} \sum_{k=1}^m M_{\beta,s}(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T^{k,1}$ are $T (k = 1, \ldots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Write

$$T_b(f)(x) = T_{b-b_Q}(f)(x) = T_{(b-b_Q)\chi_{2Q}}(f)(x) + T_{(b-b_Q)\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x).$$

Then

$$\frac{1}{|Q|}\int_Q |T_b(f)(x) - f_2(x_0)|\,dx$$

$$\leq \frac{1}{|Q|}\int_Q |f_1(x)|\,dx + \frac{1}{|Q|}\int_Q |f_2(x) - f_2(x_0)|\,dx = I_1 + I_2.$$

For I_1, by Hölder's inequality and Lemma 2, we obtain

$$\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|\,dx$$

$$\leq \left(\frac{1}{|Q|}\int_{R^n} |T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|^s\,dx\right)^{1/s}$$

$$\leq C|Q|^{-1/s}\left(\int_{2Q} |M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|^s\,dx\right)^{1/s}$$

$$\leq C|Q|^{-1/s}\left(\int_{2Q} (|b(x) - b_Q|\,|T^{k,2}(f)(x)|)^s\,dx\right)^{1/s}$$

$$\leq C|Q|^{-1/s}\sup_{x\in 2Q}|b(x) - b_{2Q}|\left(\int_Q |T^{k,2}(f)(x)|^s\,dx\right)^{1/s}$$

$$\leq C|Q|^{-1/s}\|b\|_{Lip_\beta(w)}\frac{w(2Q)^{1+\beta/n}}{|2Q|}|Q|^{1/s-\beta/n}\left(\frac{1}{|Q|^{1-s\beta/n}}\int_Q |T^{k,2}(f)(x)|^s\,dx\right)^{1/s}$$

$$\leq C\|b\|_{Lip_\beta(w)}\left(\frac{w(Q)}{|Q|}\right)^{1+\beta/n}M_{\beta,s}(T^{k,2}(f))(\tilde{x})$$

$$\leq C\|b\|_{Lip_\beta(w)}w(\tilde{x})^{1+\beta/n}M_{\beta,s}(T^{k,2}(f))(\tilde{x}),$$

thus,

$$I_1 \leq \sum_{k=1}^{m}\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|\,dx$$

$$\leq C\|b\|_{Lip_\beta(w)} w(\widetilde{x})^{1+\beta/n} \sum_{k=1}^{m} M_{\beta,s}(T^{k,2}(f))(\widetilde{x}).$$

For I_2, by the boundedness of T and recalling that $s > q'$, we get, for $x \in Q$,

$$|T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x) - T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x_0)|$$

$$\leq \int_{(2Q)^c} |b(y) - b_{2Q}||K(x, y) - K(x_0, y)||T^{k,2}(f)(y)|dy$$

$$\leq \sum_{j=1}^{\infty} \int_{2^j d \leq |y-x_0| < 2^{j+1} d} |K(x, y) - K(x_0, y)||b(y) - b_{2^{j+1}Q}||T^{k,2}(f)(y)|dy$$

$$+ \sum_{j=1}^{\infty} \int_{2^j d \leq |y-x_0| < 2^{j+1} d} |K(x, y) - K(x_0, y)||b_{2^{j+1}Q} - b_{2Q}||T^{k,2}(f)(y)|dy$$

$$\leq C \sum_{j=1}^{\infty} \left(\int_{2^j d \leq |y-x_0| < 2^{j+1} d} |K(x, y) - K(x_0, y)|^q dy \right)^{1/q}$$

$$\times \sup_{y \in 2^{j+1}Q} |b(y) - b_{2^{j+1}Q}| \left(\int_{2^{j+1}Q} |T^{k,2}(f)(y)|^{q'} dy \right)^{1/q'}$$

$$+ C \sum_{j=1}^{\infty} |b_{2^{j+1}Q} - b_{2Q}| \left(\int_{2^j d \leq |y-x_0| < 2^{j+1} d} |K(x, y) - K(x_0, y)|^q dy \right)^{1/q}$$

$$\times \left(\int_{2^{j+1}Q} |T^{k,2}(f)(y)|^{q'} dy \right)^{1/q'}$$

$$\leq C \sum_{j=1}^{\infty} C_j (2^j d)^{-n/q'} \frac{w(2^{j+1}Q)^{1+\beta/n}}{|2^{j+1}Q|} \|b\|_{Lip_\beta(w)} |2^{j+1}Q|^{1/q'-1/s} |2^{j+1}Q|^{1/s-\beta/n}$$

$$\times \left(\frac{1}{|2^{j+1}Q|^{1-s\beta/n}} \int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s dy \right)^{1/s}$$

$$+ C\sum_{j=1}^{\infty} j\|b\|_{Lip_\beta(w)} w(\tilde{x}) w(2^j Q)^{\beta/n} C_j (2^j d)^{-n/q'} |2^{j+1}Q|^{1/q'-1/s} |2^{j+1}Q|^{1/s-\beta/n}$$

$$\times \left(\frac{1}{|2^{j+1}Q|^{1-s\beta/n}} \int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s dy \right)^{1/s}$$

$$\leq C\|b\|_{Lip_\beta(w)} \sum_{j=1}^{\infty} C_j \left(\frac{w(2^{j+1}Q)}{|2^{j+1}Q|} \right)^{1+\beta/n} M_{\beta,s}(T^{k,2}(f))(\tilde{x})$$

$$+ C\|b\|_{Lip_\beta(w)}(w)(\tilde{x}) \sum_{j=1}^{\infty} jC_j \left(\frac{w(2^{j+1}Q)}{|2^{j+1}Q|} \right)^{\beta/n} M_{\beta,s}(T^{k,2}(f))(\tilde{x})$$

$$\leq C\|b\|_{Lip_\beta(w)} w(\tilde{x})^{1+\beta/n} M_{\beta,s}(T^{k,2}(f))(\tilde{x}) \sum_{j=1}^{\infty} (j+1) C_j$$

$$\leq C\|b\|_{Lip_\beta(w)} w(\tilde{x})^{1+\beta/n} M_{\beta,s}(T^{k,2}(f))(\tilde{x}),$$

thus,

$$I_2 \leq \frac{1}{|Q|} \int_Q \sum_{k=1}^{m} |T^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x) - T^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x_0)| dx$$

$$\leq C\|b\|_{Lip_\beta(w)} w(\tilde{x})^{1+\beta/n} \sum_{k=1}^{m} M_{\beta,s}(T^{k,2}(f))(\tilde{x}).$$

These complete the proof of Theorem 1. □

Proof of Theorem 2. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |T_b(f)(x) - C_0| dx \leq C\|b\|_{Lip_\beta(w)} w(\tilde{x})^{1+\beta/n} \sum_{k=1}^{m} M_s(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1, \ldots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similar to the proof of main Lemma 1, we have

$$T_b(f)(x) = T_{b-b_Q}(f)(x) = T_{(b-b_Q)\chi_{2Q}}(f)(x) + T_{(b-b_Q)\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x),$$

and

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |T_b(f)(x) - f_2(x_0)| dx$$

$$\leq \frac{1}{|Q|^{1+\beta/n}} \int_Q |f_1(x)| dx + \frac{1}{|Q|^{1+\beta/n}} \int_Q |f_2(x) - f_2(x_0)| dx = I_3 + I_4.$$

By using the same argument as in the proof of Theorem 1, we get

$$I_3 \leq \sum_{k=1}^{m} \frac{C}{|Q|^{\beta/n}} \sup_{x \in 2Q} |b(x) - b_{2Q}| \, |Q|^{-1/s} \left(\int_{2Q} |T^{k,2}(f)(x)|^s dx \right)^{1/s}$$

$$\leq \sum_{k=1}^{m} \|b\|_{Lip_\beta(w)} \left(\frac{w(Q)}{|Q|} \right)^{1+\beta/n} \left(\frac{1}{|2Q|} \int_{2Q} |T^{k,2}(f)(x)|^s dx \right)^{1/s}$$

$$\leq C \|b\|_{Lip_\beta(w)} w(\tilde{x})^{1+\beta/n} \sum_{k=1}^{m} M_s(T^{k,2}(f))(\tilde{x});$$

$$I_4 \leq \sum_{k=1}^{m} \frac{1}{|Q|^{1+\beta/n}} \int_Q \sum_{j=1}^{\infty} \int_{2^j d \leq |y-x_0| < 2^{j+1} d} |K(x, y) - K(x_0, y)|$$

$$\times |b(y) - b_{2^{j+1}Q}| \, |T^{k,2}(f)(y)| dy dx$$

$$+ \sum_{k=1}^{m} \frac{1}{|Q|^{1+\beta/n}} \int_Q \sum_{j=1}^{\infty} \int_{2^j d \leq |y-x_0| < 2^{j+1} d} |K(x, y) - K(x_0, y)|$$

$$\times |b_{2^{j+1}Q} - b_{2Q}| \, |T^{k,2}(f)(y)| dy dx$$

$$\leq \sum_{k=1}^{m} \frac{C}{|Q|^{1+\beta/n}} \int_Q \sum_{j=1}^{\infty} \left(\int_{2^j d \leq |y-x_0| < 2^{j+1} d} |K(x,\ y) - K(x_0,\ y)|^q \, dy \right)^{1/q}$$

$$\times \sup_{y \in 2^{j+1}Q} |b(y) - b_{2^{j+1}Q}| \left(\int_{2^{j+1}Q} |T^{k,2}(f)(y)|^{q'} \, dy \right)^{1/q'} dx$$

$$+ \sum_{k=1}^{m} \frac{C}{|Q|^{1+\beta/n}} \int_Q \sum_{j=1}^{\infty} |b_{2^{j+1}Q} - b_{2Q}| \left(\int_{2^j d \leq |y-x_0| < 2^{j+1} d} |K(x,\ y) - K(x_0,\ y)|^q \, dy \right)^{1/q}$$

$$\times \left(\int_{2^{j+1}Q} |T^{k,2}(f)(y)|^{q'} \, dy \right)^{1/q'} dx$$

$$\leq C \sum_{k=1}^{m} |Q|^{-\beta/n} \sum_{j=1}^{\infty} C_j (2^j d)^{-n/q'} \frac{w(2^{j+1}Q)^{1+\beta/n}}{|2^{j+1}Q|} \|b\|_{Lip_\beta(w)} |2^{j+1}Q|^{1/q'}$$

$$\times \left(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s \, dy \right)^{1/s}$$

$$+ C \sum_{k=1}^{m} |Q|^{-\beta/n} \sum_{j=1}^{\infty} j \|b\|_{Lip_\beta(w)} w(\tilde{x}) w(2^j Q)^{\beta/n} C_j (2^j d)^{-n/q'} |2^{j+1}Q|^{1/q'}$$

$$\times \left(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s \, dy \right)^{1/s}$$

$$\leq C \sum_{k=1}^{m} \|b\|_{Lip_\beta(w)} \sum_{j=1}^{\infty} 2^{j\beta} C_j \left(\frac{w(2^{j+1}Q)}{|2^{j+1}Q|} \right)^{1+\beta/n} M_s(T^{k,2}(f))(\tilde{x})$$

$$+ C \sum_{k=1}^{m} \|b\|_{Lip_\beta(w)} w(\tilde{x}) \sum_{j=1}^{\infty} j 2^{j\beta} C_j \left(\frac{w(2^{j+1}Q)}{|2^{j+1}Q|} \right)^{\beta/n} M_s(T^{k,2}(f))(\tilde{x})$$

$$\leq C \|b\|_{Lip_\beta(w)} w(\tilde{x})^{1+\beta/n} \sum_{k=1}^{m} M_s(T^{k,2}(f))(\tilde{x}) \sum_{j=1}^{\infty} (j+1) 2^{j\beta} C_j$$

$$\leq C\|b\|_{Lip_\beta(w)} w(\tilde{x})^{1+\beta/n} \sum_{k=1}^{m} M_s(T^{k,2}(f))(\tilde{x}).$$

This completes the proof of Theorem 2. □

Proof of Theorem 3. Choose $q' < s < p$ in Theorem 1, notice $w^{r/p-r(1+\beta/n)} \in A_\infty$ and $w^{1/p} \in A(p, r)$ we have, by Lemmas 1, 4, and 5,

$$\|T_b(f)\|_{L^q(w^{r/p-r(1+\beta/n)})}$$

$$\leq \|M(T_b(f))\|_{L^r(w^{r/p-r(1+\beta/n)})}$$

$$\leq C\|M^{\#}(T_b(f))\|_{L^r(w^{r/p-r(1+\beta/n)})}$$

$$\leq C\|b\|_{Lip_\beta(w)} \sum_{k=1}^{m} \|M_{\beta,s}(T^{k,2}(f)) w^{1+\beta/n}\|_{L^r(w^{r/p-r(1+\beta/n)})}$$

$$= C\|b\|_{Lip_\beta(w)} \sum_{k=1}^{m} \|M_{\beta,s}(T^{k,2}(f)))\|_{L^r(w^{r/p})}$$

$$\leq C\|b\|_{Lip_\beta(w)} \sum_{k=1}^{m} \|T^{k,2}(f)\|_{L^p(w)}$$

$$\leq C\|b\|_{Lip_\beta(w)} \|f\|_{L^p(w)}.$$

This completes the proof of Theorem 3. □

Proof of Theorem 4. Choose $q' < s < p$ in Theorem 2, notice that $w^{r/p-r(1+\beta/n)} \in A_\infty$ and $w^{1/p} \in A(p, r)$. By using Lemma 3, we obtain

$$\|T_b(f)\|_{\dot{F}_r^{\beta,\infty}(w^{r/p-r(1+\beta/n)})}$$

$$\leq C\left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T_b(f)(x) - C_0|\, dx \right\|_{L^r(w^{r/p-r(1+\beta/n)})}$$

$$\leq C\|b\|_{Lip_\beta(w)}\sum_{k=1}^{m}\|M_s(T^{k,2}(f))w^{1+\beta/n}\|_{L^r(w^{r/p-r(1+\beta/n)})}$$

$$= C\|b\|_{Lip_\beta(w)}\sum_{k=1}^{m}\|M_s(T^{k,2}(f))\|_{L^r(w^{r/p})}$$

$$\leq C\|b\|_{Lip_\beta(w)}\sum_{k=1}^{m}\|T^{k,2}(f)\|_{L^p(w)}$$

$$\leq C\|b\|_{Lip_\beta(w)}\|f\|_{L^p(w)}.$$

This completes the proof of Theorem 4. □

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