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LIPSCHITZ ESTIMATES FOR MULTILINEAR COMMUTATOR OF MARCINKIEWICZ OPERATOR

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Abstract

In this paper, we will study the continuity of multilinear commutator generated by Marcinkiewicz operator and b on Triebel-Lizorkin space, Hardy space, and Herz-Hardy space, where the function b belongs to Lipschitz space.

1. Introduction

The Marcinkiewicz operator μ_Ω (introduced below) was first defined by Stein [14]. Stein proved that μ_Ω is bounded on $L^p(R^n)$ for $1 < p \le 2$, when Ω is continuous and satisfies a $Lip_\alpha(S^{n-1})(0 < \alpha \le 1)$ condition. Many authors have improved the result. On the other hand, Torchinsky and Wang (see [15]) considered the boundedness for the commutator

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of μ_Ω. They proved the commutator $[\mu_\Omega, b]$ is bounded on $L^p(w)$ for $1 < p < \infty$, $b \in BMO$, $w \in A_p$, when Ω is continuous and satisfies a $Lip_\alpha(S^{n-1})(0 < \alpha \le 1)$ condition. Following them, the main purpose of this paper is to discuss the boundedness of multilinear commutator generated by Marcinkiewicz operator and b on Triebel-Lizorkin space, Hardy space, and Herz-Hardy space, where $b \in Lip_\beta$.

2. Preliminaries and Definitions

Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of f, and write $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$, Q will denote a cube of R^n with sides parallel to the axes. Let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x) = \sup_{Q \ni x}|Q|^{-1}\int_Q |f(y) - f_Q|dy$. Denote the Hardy spaces by $H^p(R^n)$. It is well known that $H^p(R^n)(0 < p \le 1)$ has the atomic decomposition characterization (see [2], [8], [11]). For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}$ be the homogeneous Triebel-Lizorkin space. The Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$\|f\|_{Lip_\beta} = \sup_{\substack{x, y \in R^n \\ x \ne y}} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty.$$

Lemma 1 (See [10]). *For* $0 < \beta < 1, 1 < p < \infty$, *we have*

$$\|f\|_{\dot{F}_p^{\beta,\infty}} \approx \left\| \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| dx \right\|_{L^p}$$

$$\approx \left\| \sup_{\cdot \in Q} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c| dx \right\|_{L^p}.$$

Lemma 2 (See [10])**.** *For* $0 < \beta < 1$, $1 \le p \le \infty$, *we have*

$$\|f\|_{Lip_\beta} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| dx$$

$$\approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p dx \right)^{1/p}.$$

Lemma 3 (See [1])**.** *For* $1 \le r < \infty$ *and* $\beta > 0$, *let*

$$M_{\beta, r}(f)(\dot{x}) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-\beta r/n}} \int_Q |f(y)|^r dx \right)^{1/r},$$

suppose that $r < p < \beta / n$ *and* $1 / q = 1 / p - \beta / n$, *then*

$$\|M_{\beta, r}(f)\|_{L^q} \le C\|f\|_{L^p}.$$

Lemma 4 (See [14])**.** *If* $y \in B(x_0, r)$ *and* $x \in (2B(x_0, r))^C$. *Then*

$$\left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x_0 - y)}{|x_0 - y|^{n-1}} \right| \le C\left(\frac{|x - x_0|}{|x_0 - y|^n} + \frac{|x - x_0|^\gamma}{|x_0 - y|^{n-1+\gamma}} \right).$$

Definition 1. Let $0 < p \le 1$. A function $a(x)$ on R^n is called an H^p-atom, if

(1) Supp$a \subset B(x_0, r)$ for some x_0 and for some $r > 0$ (or for some $r \ge 1$);

(2) $\|a\|_{L^\infty} \le |B(x_0, r)|^{-1/p}$;

(3) $\int_{R^n} a(x) dx = 0$.

Lemma 5 (See [7], [11]). *Let $0 < p \le 1$. A distribution f on R^n is in $H^p(R^n)$ if and only if f can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ in the distributional sense, where each a_j is H^p-atom and each λ_j is a constant, $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$. Moreover,*

$$\|f\|_{H^p} \approx \inf \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p},$$

with the infimum take over all decompositions of f as above.

Definition 2. Let $0 < p, q < \infty$, $\alpha \in R$, $B_k = \{x \in R^n, |x| \le 2^k\}$, $E_k = B_k \setminus B_{k-1}$, and $\chi_k = \chi_{E_k}$ for $k \in Z$.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha, p}(R^n) = \{f \in L_{\mathrm{Loc}}^q(R^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha, p}} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha, p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q}^p \right]^{1/p}.$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha, p}(R^n) = \{f \in L_{\mathrm{Loc}}^q(R^n) : \|f\|_{K_q^{\alpha, p}} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha, p}(R^n)} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q}^p + \|f\chi_{B_0}\|_{L^q}^p \right]^{1/p}.$$

Definition 3. Let $\alpha \in R$, $0 < p, q < \infty$.

(1) The homogeneous Herz type Hardy space is defined by

$$H\dot{K}_q^{\alpha, p}(R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_q^{\alpha, p}(R^n)\},$$

and

$$\|f\|_{H\dot{K}_q^{\alpha,p}} = \|G(f)\|_{\dot{K}_q^{\alpha,p}}.$$

(2) The nonhomogeneous Herz type Hardy space is defined by

$$HK_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in K_q^{\alpha,p}(R^n)\},$$

and

$$\|f\|_{HK_q^{\alpha,p}} = \|G(f)\|_{K_q^{\alpha,p}},$$

where $G(f)$ (see [11]) is the grand maximal function of f.

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 4. Let $\alpha \in R$, $1 < q < \infty$. A function $a(x)$ on R^n is called a central (α, q)-atom (or a central (α, q)-atom of restrict type), if

(1) $\operatorname{Supp} a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$);

(2) $\|a\|_{L^q} \leq |B(0, r)|^{-\alpha/n}$;

(3) $\int_{R^n} a(x)x^{\eta}dx = 0$ for any $|\eta| \leq [\alpha - n(1 - 1/q)]$.

Lemma 6 (See [9]). *Let* $0 < p < \infty$, $1 < q < \infty$, *and* $\alpha \geq n(1 - 1/q)$. *A temperate distribution f belongs to* $H\dot{K}_q^{\alpha,p}(R^n)$ (*or* $HK_q^{\alpha,p}(R^n)$), *if and only if there exist central* (α, q)-*atoms* (*or central* (α, q)-*atoms of restrict type*) a_j *supported on* $B_j = B(0, 2^j)$ *and constants* λ_j, $\sum_j |\lambda_j|^p < \infty$ *such that* $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (*or* $f = \sum_{j=0}^{\infty} \lambda_j a_j$) *in the* $S'(R^n)$ *sense, and*

$$\|f\|_{H\dot{K}_q^{\alpha,p}} \left(or\ \|f\|_{HK_q^{\alpha,p}}\right) \sim \left(\sum_j |\lambda_j|^p\right)^{1/p}.$$

Definition 5 (See [13]). Let $\alpha \in R$, $1 < p, q < \infty$.

(1) A measure function f is said to belong to homogeneous weak Herz space $W\dot{K}_q^{\alpha, p(R^n)}$, if

$$\|f\|_{W\dot{K}_q^{\alpha, p}} = \sup_{\lambda > 0} \lambda (\sum_{-\infty}^{+\infty} 2^{k\alpha p} |\{x \in E_k : |f(x)| > \lambda\}|^{p/q})^{1/q} < \infty.$$

(2) A measure function f is said to belong to inhomogeneous weak Herz space $WK_q^{\alpha, p(R^n)}$, if

$$\|f\|_{WK_q^{\alpha, p}} = \sup_{\lambda > 0} \lambda (\sum_{k=1}^{+\infty} 2^{k\alpha p} |\{x \in E_k : |f(x)| > \lambda\}|^{p/q}$$

$$+ |\{x \in B_0 : |f(x) > \lambda|^{p/q}\}|)^{1/p} < \infty.$$

Definition 6. Let $0 < \gamma \leq 1$ and Ω be homogeneous of degree zero on R^n such that $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leq M|x - y|^\gamma$. The Marcinkiewicz multilinear commutator is defined by

$$\mu_\Omega^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \left[\prod_{j=1}^m (b_j(x) - b_j(y)) \right] f(y) dy.$$

Set

$$F_t(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy,$$

we also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

which is the Marcinkiewicz operator (see [12]).

Let H be the space $H = \left\{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t^3)^{1/2} < \infty\right\}$. Then, it is clear that

$$\mu_\Omega(f)(x) = \|F_t(f)(x)\| \text{ and } \mu_\Omega^{\vec{b}}(f)(x) = \|F_t^{\vec{b}}(f)(x)\|.$$

Note that when $b_1 = \cdots = b_m$, $\mu_\Omega^{\vec{b}}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [5], [6], [9], [12], [13]). Our main purpose is to establish the boundedness of the multilinear commutator on Triebel-Lizorkin space, Hardy space, and Herz-Hardy space.

Given a positive integer m and $1 \le j \le m$, we set $\|\vec{b}\|_{Lip_\beta} = \prod_{j=1}^m \|b_j\|_{Lip_\beta}$ and denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements, $|\sigma| = j$ is the element number of σ. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$, and $\|\vec{b}_\sigma\|_{Lip_\beta} = \|b_{\sigma(1)}\|_{Lip_\beta} \cdots \|b_{\sigma(j)}\|_{Lip_\beta}$.

3. Theorems and Proofs

Theorem 1. *Let* $0 < \beta < \min(1/2m, \gamma/m)$, $1 < p < \infty$, $\vec{b} = (b_1, \cdots, b_m)$ *with* $b_j \in Lip_\beta(R^n)$ *for* $1 \le j \le m$ *and* $\mu_\Omega^{\vec{b}}$ *be the multilinear commutator of Marcinkiewicz operator as in Definition* 6. *Then*

(a) $\mu_\Omega^{\vec{b}}$ *is bounded from* $L^p(R^n)$ *to* $\dot{F}_p^{m\beta,\infty}(R^n)$.

(b) $\mu_\Omega^{\vec{b}}$ *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$ *for* $1/p - 1/q = m\beta/n$ *and* $1/p > m\beta/n$.

Proof. (a) Fixed a cube $Q = Q(x_0, l)$ and $\tilde{x} \in Q$. Set $\vec{b}_Q = ((b_1)_Q, \cdots, (b_m)_Q)$, where $(b_j)_Q = |Q|^{-1}\int_Q b_j(y)dy$, $1 \le j \le m$. Write $f = f_1 + f_2$, where $f_1 = f\chi_{2Q}$, $f_2 = f\chi_{R^n \setminus 2Q}$, we have

$$F_t^{\vec{b}}(f)(x) = \int_{R^n}(b_1(x) - b_1(y))\cdots(b_m(x) - b_m(y))\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)dy$$

$$= (b_1(x) - (b_1)_Q)\cdots(b_m(x) - (b_m)_Q)F_t(f)(x)$$

$$+ (-1)^m F_t((b_1 - (b_1)_Q)\cdots(b_m - (b_m)_Q)f)(x)$$

$$+ \sum_{j=1}^{m-1}\sum_{\sigma \in C_j^m}(-1)^{m-j}(b(x) - \vec{b}_Q)_\sigma$$

$$\times \int_{|x-y|\le t}(b(y) - \vec{b}_Q)_{\sigma^c}\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)dy$$

$$= (b_1(x) - (b_1)_Q)\cdots(b_m(x) - (b_m)_Q)F_t(f)(x)$$

$$+ (-1)^m F_t((b_1 - (b_1)_Q)\cdots(b_m - (b_m)_Q)f_1)(x)$$

$$+ (-1)^m F_t((b_1 - (b_1)_Q)\cdots(b_m - (b_m)_Q)f_2)(x)$$

$$+ \sum_{j=1}^{m-1}\sum_{\sigma \in C_j^m}(-1)^{m-j}(b(x) - \vec{b}_Q)_\sigma F_t((b - \vec{b}_Q)_{\sigma^c}f)(x),$$

then

$$
\begin{aligned}
&|\mu_\Omega^{\vec{b}}(f)(x) - \mu_\Omega(((b_1)_Q - b_1)\cdots((b_m)_Q - b_m)f_2)(x_0)| \\
&\quad \le \|F_t^{\vec{b}}(f)(x) - (-1)^m F_t(((b_1)_Q - b_1)\cdots((b_m)_Q - b_m)f_2)(x_0)\| \\
&\quad \le \|(b_1(x) - (b_1)_Q)\cdots(b_m(x) - (b_m)_Q)F_t(f)(x) \\
&\qquad + \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|(b(x) - \vec{b}_Q)_\sigma F_t((b - \vec{b}_Q)_{\sigma^c} f)(x)\| \\
&\qquad + \|F_t((b_1 - (b_1)_Q)\cdots(b_m - (b_m)_Q)f_1)(x)\| \\
&\qquad + \|F_t((b_1 - (b_1)_Q)\cdots(b_m - (b_m)_Q)f_2)(x) \\
&\qquad - F_t((b_1 - (b_1)_Q)\cdots(b_m - (b_m)_Q)f_2)(x_0)\| \\
&\quad = I_1(x) + I_2(x) + I_3(x) + I_4(x),
\end{aligned}
$$

thus,

$$
\begin{aligned}
&\frac{1}{|Q|^{1+m\beta/n}}\int_Q|\mu_\Omega^{\vec{b}}(f)(x) - \mu_\Omega((b_1)_Q - b_1)\cdots((b_m)_Q - b_m)f_2)(x_0)|dx \\
&\qquad \le \frac{1}{|Q|^{1+m\beta/n}}\int_Q I_1(x)dx + \frac{1}{|Q|^{1+m\beta/n}}\int_Q I_2(x)dx \\
&\qquad\quad + \frac{1}{|Q|^{1+m\beta/n}}\int_Q I_3(x)dx + \frac{1}{|Q|^{1+m\beta/n}}\int_Q I_4(x)dx \\
&\qquad = I + II + III + IV.
\end{aligned}
$$

For I, by using Lemma 2, we have

$$
\begin{aligned}
I &\le \frac{1}{|Q|^{1+m\beta/n}}\sup_{x\in Q}|b_1(x) - (b_1)_Q|\cdots|b_m(x) - (b_m)_Q|\int_Q|\mu_\Omega(f)(x)|dx \\
&\le C\|\vec{b}\|_{Lip_\beta}\frac{1}{|Q|^{1+m\beta/n}}|Q|^{m\beta/n}\int_Q|\mu_\Omega(f)(x)|dx
\end{aligned}
$$

$$\leq C\|\vec{b}\|_{Lip_\beta} M(\mu_\Omega(f))(\tilde{x}).$$

Fix $1 < r < p$. For II, by using the Hölder's inequality and the boundedness of μ_Ω on L^r and Lemma 2, we get

$$II \leq \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|^{1+m\beta/n}} \int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma| |\mu_\Omega((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x)| dx$$

$$\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|^{1+m\beta/n}} \left(\int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma|^{r'} dx \right)^{1/r'}$$

$$\times \left(\int_Q |\mu_\Omega((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x)|^r dx \right)^{1/r}$$

$$\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|^{1+m\beta/n}} \left(\int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma|^{r'} dx \right)^{1/r'}$$

$$\times \left(\int_Q |(\vec{b}(x) - \vec{b}_Q)_{\sigma^c} f(x)|^r dx \right)^{1/r}$$

$$\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|^{1+m\beta/n}} |Q|^{1/r'} \|\vec{b}_\sigma\|_{Lip_\beta} |Q|^{|\sigma|\beta/n} \|\vec{b}_{\sigma^c}\|_{Lip_\beta} |Q|^{|\sigma^c|\beta/n} |Q|^{1/r}$$

$$\times \left(\frac{1}{|Q|} \int_Q |f(x)|^r dx \right)^{1/r}$$

$$\leq C\|\vec{b}\|_{Lip_\beta} M_r(f)(\tilde{x}).$$

For III, by Hölder's inequality, we have

$$III = \frac{1}{|Q|^{1+m\beta/n}} \int_Q |\mu_\Omega((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q) f_1)(x)| dx$$

$$\leq C\frac{1}{|Q|^{1+m\beta/n}}\left(\int_{R^n}|\mu_\Omega(\prod_{j=1}^m(b_j-(b_j)_Q)f_1)(x)|^r dx\right)^{1/r}|Q|^{1-1/r}$$

$$\leq C\frac{1}{|Q|^{1+m\beta/n}}|Q|^{1-1/r}\left(\int_{2Q}|\prod_{j=1}^m(b_j(x)-(b_j)_Q)f(x)|^r dx\right)^{1/r}$$

$$\leq C\frac{1}{|Q|^{1+m\beta/n}}|Q|^{1-1/r}\|\vec{b}\|_{Lip_\beta}|Q|^{m\beta/n}|Q|^{1/r}\left(\frac{1}{|2Q|}\int_{2Q}|f(x)|^r dx\right)^{1/r}$$

$$\leq C\|\vec{b}\|_{Lip_\beta}M_r(f)(\tilde{x}).$$

For *IV*, since $|x_0-y|\approx|x-y|$ for $y\in(2Q)^c$, we have

$$I_4=\|F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x)$$

$$-F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x_0)\|$$

$$=(\int_0^\infty|\int_{|x-y|\leq t}\frac{\Omega(x-y)f_2(y)}{|x-y|^{n-1}}\left[\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right]dy$$

$$-\int_{|x_0-y|\leq t}\frac{\Omega(x_0-y)f_2(y)}{|x_0-y|^{n-1}}\left[\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right]dy|^2\frac{dt}{t^3})^{1/2}$$

$$\leq\left(\int_0^\infty\left[\int_{|x_0-y|\leq t,|x_0-y|>t}\frac{|\Omega(x-y)||f_2(y)|}{|x-y|^{n-1}}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|dy\right]^2\frac{dt}{t^3}\right)^{1/2}$$

$$+\left(\int_0^\infty\left[\int_{|x-y|>t,|x_0-y|\leq t}\frac{|\Omega(x_0-y)||f_2(y)|}{|x_0-y|^{n-1}}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|dy\right]^2\frac{dt}{t^3}\right)^{1/2}$$

$$+\left(\int_0^\infty\left[\int_{|x-y|\leq t,|x_0-y|\leq t}\left|\frac{|\Omega(x-y)|}{|x-y|^{n-1}}-\frac{|\Omega(x_0-y)|}{|x_0-y|^{n-1}}\right|\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right||f_2(y)|dy\right]^2\frac{dt}{t^3}\right)^{1/2}$$

$$\equiv J_1+J_2+J_3.$$

For J_1, since $m\beta < 1/2$,

$$J_1 \le C\int_{(2Q)^c}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)|}{|x-y|^{n-1}}\left(\int_{|x-y|\le t<|x_0-y|}\frac{dt}{t^3}\right)^{1/2}dy$$

$$\le C\int_{(2Q)^c}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)|}{|x-y|^{n-1}}\left|\frac{1}{|x-y|^2}-\frac{1}{|x_0-y|^2}\right|^{1/2}dy$$

$$\le C\int_{(2Q)^c}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)|}{|x-y|^{n-1}}\frac{|x_0-x|^{1/2}}{|x-y|^{3/2}}dy$$

$$\le C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\frac{|Q|^{1/2n}|f(y)|}{|x_0-y|^{n+1/2}}dy$$

$$\le C\|\vec{b}\|_{Lip_\beta}\sum_{k=1}^{\infty}|2^{k+1}Q|^{m\beta/n}|Q|^{1/2n}|2^kQ|^{(-n-1/2)/n}\int_{2^{k+1}Q}|f(y)|dy$$

$$\le C\|\vec{b}\|_{Lip_\beta}|Q|^{m\beta/n}\sum_{k=1}^{\infty}2^{-k(1/2-m\beta)+m\beta+n}M(f)(\tilde{x})$$

$$\le C\|\vec{b}\|_{Lip_\beta}M(f)(\tilde{x})|Q|^{m\beta/n}.$$

For J_2, similar to J_1, we have

$$J_2 \le C\|\vec{b}\|_{Lip_\beta}M(f)(\tilde{x})|Q|^{m\beta/n}.$$

For J_3, by using Lemmas 2, 4 and $m\beta < \gamma$,

$$J_3 \le C\int_{(2Q)^c}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)||x-x_0|}{|x_0-y|^n}\left(\int_{|x-y|\le t,|x_0-y|\le t}\frac{dt}{t^3}\right)^{1/2}dy$$

$$+C\int_{(2Q)^c}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)||x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\left(\int_{|x-y|\le t,|x_0-y|\le t}\frac{dt}{t^3}\right)^{1/2}dy$$

$$\leq C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\left(\frac{|Q|^{1/(n)}}{|x_0-y|^{n+1}}+\frac{|Q|^{\gamma/(n)}}{|x_0-y|^{n+\gamma}}\right)|f(y)|dy$$

$$\leq C\|\vec{b}\|_{Lip_\beta}\sum_{k=1}^{\infty}|2^{k+1}Q|^{m\beta/n}\left(\frac{|Q|^{1/n}}{|2^kQ|^{(n+1)/n}}+\frac{|Q|^{\gamma/n}}{|2^kQ|^{(n+\gamma)/n}}\right)\int_{2^{k+1}Q}|f(y)|dy$$

$$\leq C\|\vec{b}\|_{Lip_\beta}|Q|^{m\beta/n}\sum_{k=1}^{\infty}(2^{-k(1-m\beta)+m\beta+n}+2^{-k(\gamma-m\beta)+m\beta+n})M(f)(\tilde{x})$$

$$\leq C\|\vec{b}\|_{Lip_\beta}M(f)(\tilde{x})|Q|^{m\beta/n}.$$

Thus,

$$IV\leq C\|\vec{b}\|_{Lip_\beta}M(f)(\tilde{x}).$$

We put these estimates together, by using Lemma 1 and taking the supremum over all Q such that $x\in Q$, we obtain

$$\|\mu_\Omega^{\vec{b}}(f)(x)\|_{\dot{F}_p^{m\beta,\infty}}\leq C\|\vec{b}\|_{Lip_\beta}\|f\|_{L^p}.$$

This complete the proof of (a).

(b) By some argument as in the proof of (a), we have

$$\frac{1}{|Q|}\int_Q|\mu_\Omega^{\vec{b}}(f)(x)-\mu\Omega(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)|dx$$

$$\leq\frac{1}{|Q|}\int_Q I_1(x)dx+\frac{1}{|Q|}\int_Q I_2(x)dx+\frac{1}{|Q|}\int_Q I_3(x)dx+\frac{1}{|Q|}\int_Q I_4(x)dx$$

$$\leq C\|\vec{b}\|_{Lip_\beta}(M_{m\beta,1}(\mu_\Omega(f))+M_{m\beta,r}(f)+M_{m\beta,1}(f)),$$

thus,

$$(\mu_\Omega^{\vec{b}}(f))^{\#}\leq C\|\vec{b}\|_{Lip_\beta}(M_{m\beta,1}(\mu_\Omega(f))+M_{m\beta,r}(f)+M_{m\beta,1}(f)).$$

By using Lemma 3 and the boundedness of μ_Ω, we have

$$\|\mu_\Omega^{\vec{b}}(f)\|_{L^q} \le C\|(\mu_\Omega^{\vec{b}}(f))^{\#}\|_{L^q}$$

$$\le C\|\vec{b}\|_{Lip_\beta}(\|M_{m\beta,1}(\mu_\Omega(f))\|_{L^q} + \|M_{m\beta,r}(f)\|_{L^q} + \|M_{m\beta,1}(f)\|_{L^q})$$

$$\le C\|f\|_{L^p}.$$

This complete the proof of (b).

Theorem 2. *Let* $0 < \beta < \min(\gamma/m, 1/2m)$, $n/(n+\beta) < p \le 1$, $1/q = 1/p - m\beta/n$, $\vec{b} = (b_1, \cdots, b_m)$ *with* $b_j \in Lip_\beta(R^n)$ *for* $1 \le j \le m$. *Then* $\mu_\Omega^{\vec{b}}$ *is bounded from* $H^p(R^n)$ *to* $L^q(R^n)$.

Proof. By Lemma 5, it suffices to show that there exists a constant $C > 0$, such that for every H^p-atom a,

$$\|\mu_\Omega^{\vec{b}}(a)\|_{L^q} \le C.$$

Write

$$\|\mu_\Omega^{\vec{b}}(a)(x)\|_{L^q} \le \left(\int_{|x-x_0|\le 2r} |\mu_\Omega^{\vec{b}}(a)(x)|^q dx\right)^{1/q} + \left(\int_{|x-x_0|>2r} |\mu_\Omega^{\vec{b}}(a)(x)|^q dx\right)^{1/q}$$

$$= I + II.$$

For I, choose $1 < p_1 < n/m\beta$ and q_1 such that $1/q_1 = 1/p_1 - m\beta/n$. By the boundedness of $\mu_\Omega^{\vec{b}}$ from $L^{p_1}(R^n)$ to $L^{q_1}(R^n)$ (see Theorem 1), the size condition of a and Hölder's inequality, we get

$$I \le C\|\mu_\Omega^{\vec{b}}(a)\|_{L^{q_1}} r^{n(1/q-1/q_1)} \le C\|a\|_{L^{q_1}} \|\vec{b}\|_{Lip_\beta} r^{n(1/q-1/q_1)} \le C\|\vec{b}\|_{Lip_\beta}.$$

For *II*, since $|x - x_0| > 2r$, we have

$$|\mu_\Omega^{\vec{b}}(a)(x)| \le \left(\int_0^{|x-x_0|+2r} \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \prod_{j=1}^m (b_j(x) - b_j(y)) a(y) dy\right|^2 \frac{dt}{t^3}\right)^{1/2}$$

$$+ \left(\int_{|x-x_0|+2r}^{\infty} \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \prod_{j=1}^m (b_j(x) - b_j(y)) a(y) dy\right|^2 \frac{dt}{t^3}\right)^{1/2}$$

$$= J_1 + J_2.$$

Observe that from $|x - x_0| > 2r$ and $y \in B = B(x_0, r)$, it follows that $|x - y| \sim |x - x_0| \sim |x - x_0| + 2r$. By the Minkowski inequality and $Lip_\gamma(S^{n-1}) \subset L^\infty(S^{n-1})$, we obtain

$$J_1 \le C\int_{R^n} \left(\int_{|x-y|}^{|x-x_0|+2r} \frac{dt}{t^3}\right)^{\frac{1}{2}} \frac{|a(y)|}{|x-y|^{n-1}} \prod_{j=1}^m |b_j(x) - b_j(y)| dy$$

$$\le C\|\vec{b}\|_{Lip_\beta} \int_B |x-y|^{m\beta} \frac{|a(y)|}{|x-y|^{n-1}} \frac{r^{1/2}}{|x-x_0|^{3/2}} dy$$

$$\le C\|\vec{b}\|_{Lip_\beta} \int_B |x-x_0|^{m\beta} \frac{|a(y)|}{|x-x_0|^{n-1}} \frac{r^{1/2}}{|x-x_0|^{3/2}} dy$$

$$\le C\|\vec{b}\|_{Lip_\beta} |x-x_0|^{m\beta-n-1/2} r^{1/2} \int_B |a(y)| dy$$

$$\le C\|\vec{b}\|_{Lip_\beta} |x-x_0|^{m\beta-n-1/2} r^{1/2+n(1-1/p)}.$$

Notice that from $t \ge |x - x_0| + 2r$, and $y \in B$, it follows $t \ge |x - x_0| + |y - x_0| \ge |x - y|$, we obtain

$$J_2 \le C\left(\int_{|x-x_0|+2r}^{\infty} \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \prod_{j=1}^m (b_j(x) - b_j(y)) a(y) dy\right|^2 \frac{dt}{t^3}\right)^{1/2}$$

$$\leq C\Big(\int_{|x-x_0|+2r}^{\infty}\Big|\sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}(x_0))_\sigma|\int_{R^n}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}$$

$$\times |(\vec{b}(y)-\vec{b}(x_0))_{\sigma_c}|\,|a(y)|dy\Big|^2\frac{dt}{t^3}\Big)^{1/2}$$

$$+\ C\Big(\int_{|x-x_0|+2r}^{\infty}\Big|\prod_{j=1}^{m}(b_j(x)-b_j(x_0))\int_{R^n}\Big(\frac{\Omega(x-y)}{|x-y|^{n-1}}$$

$$-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big)a(y)dy\Big|^2\frac{dt}{t^3}\Big)^{1/2}$$

$$\leq C\sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}(x_0))_\sigma|\int_B\frac{1}{|x-y|^{n-1}}$$

$$\times |(\vec{b}(y)-\vec{b}(x_0))_{\sigma_c}|\,|a(y)|dy\Big(\int_{|x-x_0|+2r}^{\infty}\frac{dt}{t^3}\Big)^{1/2}$$

$$+\ C\prod_{j=1}^{m}|b_j(x)-b_j(x_0)|\int_B\Big(\frac{r}{|x-x_0|^n}$$

$$+\frac{r^\gamma}{|x-x_0|^{n-1+\gamma}}\Big)|a(y)|dy\Big(\int_{|x-x_0|+2r}^{\infty}\frac{dt}{t^3}\Big)^{1/2}$$

$$\leq C\|\vec{b}\|_{Lip_\beta}\sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}|x-x_0|^{\beta|\sigma|-n}\int_B|y-x_0|^{\beta|\sigma_c|}|a(y)|dy$$

$$+\ C\|\vec{b}\|_{Lip_\beta}\big(|x-x_0|^{\beta m-n-1}r^{1+n(1-1/p)}+|x-x_0|^{\beta m-n-\gamma}r^{\gamma+n(1-1/p)}\big)$$

$$\leq C\sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}\|_{Lip_\beta}|x-x_0|^{\beta|\sigma|-n}r^{\beta|\sigma_c|+n(1-1/p)}$$

$$+ C\|\vec{b}\|_{Lip_\beta}(|x-x_0|^{\beta m-n-1} r^{1+n(1-1/p)} + |x-x_0|^{\beta m-n-\gamma} r^{\gamma+n(1-1/p)}).$$

Therefore, since $p > n/(n+\beta) > n/(n+1/2) > n(n+1)$ and $p > n/(n+\beta) > n/(n+\gamma)$.

$$II \le C\|\vec{b}\|_{Lip_\beta} \cdot r^{1/2+n(1-1/p)}\left(\int_{|x-x_0|>2r} |x-x_0|^{(m\beta-n-1/2)_q}\, dx\right)^{1/q}$$

$$+ C\sum_{j=0}^{m}\sum_{\sigma\in C_j^m} \|\vec{b}\|_{Lip_\beta} \cdot r^{\beta|\sigma_c|+n(1-1/p)}\left(\int_{|x-x_0|>2r} |x-x_0|^{(\beta|\sigma|-n)_q}\, dx\right)^{1/q}$$

$$+ C\|\vec{b}\|_{Lip_\beta} \cdot r^{1+n(1-1/p)}\left(\int_{|x-x_0|>2r} |x-x_0|^{(m\beta-n-1)_q}\, dx\right)^{1/q}$$

$$+ C\|\vec{b}\|_{Lip_\beta} \cdot r^{\gamma+n(1-1/p)}\left(\int_{|x-x_0|>2r} |x-x_0|^{(m\beta-n-\gamma)_q}\, dx\right)^{1/q}$$

$$\le C\|\vec{b}\|_{Lip_\beta}.$$

Combining the estimates for I and II, then leads to the desired result.

It is well-known that the dual space of $H^1(R^n)$ is $BMO(R^n)$. From this and Theorem 2, by a dual argument, we easily deduce the following conclusion:

Corollary 1. *Let* $0 < \beta < \min(\gamma/m, 1/2m)$, $b_j \in Lip_\beta(R^n)$, $1 \le j \le m$. *Then* $\mu_\Omega^{\vec{b}}$ *maps* $L^{n/m\beta}(R^n)$ *continuously into* $BMO(R^n)$.

Theorem 3. *Let* $0 < \beta \le 1$, $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = m\beta/n$, $n(1-1/q_1) \le \alpha < n(1-1/q_1) + \beta$, $\vec{b} = (b_1, \cdots, b_m)$ *with* $b_j \in Lip_\beta(R^n)$ *for* $1 \le j \le m$. *Then* $\mu_\Omega^{\vec{b}}$ *is bounded from* $H\dot{K}_{q_1}^{\alpha,p}(R^n)$ *to* $\dot{K}_{q_2}^{\alpha,p}$.

Proof. By Lemma 6, let $f \in H\dot{K}_{q_1}^{\alpha, p}(R^n)$ and $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, $\operatorname{supp} a_k \subset B_k = B(0, 2^k)$, a_k be a central (α, q_1)-atom, and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$. Then, we have

$$\begin{aligned}
\|\mu_\Omega^{\vec{b}}(f)\|_{\dot{K}_{q_2}^{\alpha, p}}^p &\le C \sum_{l=-\infty}^{\infty} 2^{l\alpha p} \left(\sum_{k=l-1}^{\infty} |\lambda_k| \, \|\mu_\Omega^{\vec{b}}(a_k)\chi_l\|_{L_{q_2}} \right)^p \\
&\quad + C \sum_{l=-\infty}^{\infty} 2^{l\alpha p} \left(\sum_{k=-\infty}^{l-2} |\lambda_k| \, \|\mu_\Omega^{\vec{b}}(a_k)\chi_l\|_{L_{q_2}} \right)^p \\
&= I_1 + I_2.
\end{aligned}$$

For I_1, by the boundedness of $\mu_\Omega^{\vec{b}}$ on (L^{q_1}, L^{q_2}) (see Theorem 1), it is easy to verify that

$$\begin{aligned}
I_1 &\le C\|\vec{b}\|_{Lip_\beta}^p \sum_{l=-\infty}^{\infty} 2^{l\alpha p} \Big(\sum_{k=l-1}^{\infty} |\lambda_k| \, \|a_k\|_{L^{q_1}} \Big)^p \\
&\le C\|\vec{b}\|_{Lip_\beta}^p \sum_{l=-\infty}^{\infty} \Big(\sum_{k=l-1}^{\infty} |\lambda_k| \cdot 2^{(l-k)\alpha} \Big)^p \\
&\le C\|\vec{b}\|_{Lip_\beta}^p \\
&\quad \begin{cases} \sum_{k=-\infty}^{\infty} |\lambda_k|^p \sum_{l=-\infty}^{k+1} \cdot 2^{(l-k)\alpha p}, & 0 < p \le 1 \\ \sum_{l=-\infty}^{\infty} \Big(\sum_{k=l-1}^{\infty} |\lambda_k|^p \cdot 2^{(l-k)\alpha p/2}\Big) \Big(\sum_{k=l-1}^{\infty} 2^{(l-k)\alpha p'/2}\Big)^{p/p'}, & 1 < p < \infty \end{cases} \\
&\le C\|\vec{b}\|_{Lip_\beta}^p \sum_{k=-\infty}^{\infty} |\lambda_k|^p.
\end{aligned}$$

For I_2, note that

$$|\mu_\Omega^{\vec{b}}(a_k)(x)| \le \left(\int_0^{|x|+2^{k+1}} \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \prod_{j=1}^m (b_j(x)-b_j(y))a_k(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}$$

$$+ \left(\int_{|x|+2^{k+1}}^{\infty} \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \prod_{j=1}^m (b_j(x)-b_j(y))a_k(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}$$

$$= J_1 + J_2.$$

When $x \in E_l$ and $|x-y| < t$ with $t < |x| + 2^{k+1}$, it follows from $l \ge k+2$ that $|x-y| \sim |x| \sim |x| + 2^{k+1}$. Then, by the Minkowski inequality,

$$J_1 \le C\int_{R^n} \left(\int_{|x-y|}^{|x|+2^{k+1}} \frac{dt}{t^3}\right)^{1/2} \frac{\prod_{j=1}^m |b_j(x)-b_j(y)|}{|x-y|^{n-1}} |a_k(y)|dy$$

$$\le C\|\vec{b}\|_{Lip_\beta} \int_{R^n} \frac{|x-y|^{m\beta}|a_k(y)|}{|x-y|^{n-1}} \frac{(2^{k+1})^{1/2}}{|x|^{3/2}} dy$$

$$\le C\|\vec{b}\|_{Lip_\beta} 2^{(k+1)/2} \int_{B_k} |x|^{m\beta-n-1/2} |a_k(y)|dy$$

$$\le C\|\vec{b}\|_{Lip_\beta} 2^{(k+1)/2} |x|^{m\beta-n-1/2} 2^{kn[(1-1/q_1)-\alpha/n]}.$$

By the method to the estimate for J_2 in the proof of the Theorem 2, we get

$$J_2 \le C\sum_{j=0}^{m-1} \sum_{\sigma\in C_j^m} \|\vec{b}\|_{Lip_\beta} |x|^{\beta|\sigma|-n} (2^k)^{\beta|\sigma_c|+n(1-1/q_1)-\alpha}$$

$$+ C\|\vec{b}\|_{Lip_\beta} \left(|x|^{\beta m-n-1}(2^k)^{1+n(1-1/q_1)-\alpha} + |x|^{\beta m-n-\gamma}(2^k)^{\gamma+n(1-1/q_1)-\alpha}\right),$$

since $\alpha < n(1-1/q_1) + \beta$, then

$$\|\mu_\Omega^{\vec{b}}(a_k)\chi_l\|_{L_{q_2}} \le \|J_1\|_{L_{q_2}} + \|J_2\|_{L_{q_2}}$$

$$\le C\|\vec{b}\|_{Lip_\beta}[\int_{E_l}(2^{(k+1)/2}|x|^{m\beta-n-1/2}2^{kn(1-1/q_1)-k\alpha})^{q_2}dx]^{1/q_2}$$

$$+ C\sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}\|_{Lip_\beta}[\int_{E_l}(|x|^{\beta|\sigma|-n}(2^k)^{\beta|\sigma_c|+n(1-1/q_1)-\alpha})^{q_2}dx]^{1/q_2}$$

$$+ C\|\vec{b}\|_{Lip_\beta}[\int_{E_l}(|x|^{m\beta-n-1}(2^k)^{1+n(1-1/q_1)-\alpha})^{q_2}dx]^{1/q_2}$$

$$+ C\|\vec{b}\|_{Lip_\beta}[\int_{E_l}(|x|^{m\beta-n-\gamma}(2^k)^{\gamma+n(1-1/q_1)-\alpha})^{q_2}dx]^{1/q_2}$$

$$\le C\|\vec{b}\|_{Lip_\beta}2^{(k+1)/2}|2^l|^{((m\beta-n-1/2)q_2+n)/q_2}2^{kn(1-1/q_1)-k\alpha}$$

$$+ C\sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}\|_{Lip_\beta}(2^l)^{((\beta|\sigma|-n)q_2+n)/q_2}(2^k)^{\beta|\sigma_c|+n(1-1/q_1)-\alpha}$$

$$+ C\|\vec{b}\|_{Lip_\beta}(2^l)^{((m\beta-n-1)q_2+n)/q_2}(2^k)^{1+n(1-1/q_1)-\alpha}$$

$$+ C\|\vec{b}\|_{Lip_\beta}(2^l)^{((m\beta-n-\gamma)q_2+n)/q_2}(2^k)^{\gamma+n(1-1/q_1)-\alpha}$$

$$\le C\|\vec{b}\|_{Lip_\beta}2^{-l\alpha}2^{(l-k)(\alpha-n(1-1/q_1)-1/2)} + C\|\vec{b}\|_{Lip_\beta}2^{-l\alpha}2^{(l-k)(\alpha-n(1-1/q_1)-\beta)}$$

$$+ C\|\vec{b}\|_{Lip_\beta}2^{-l\alpha}2^{(l-k)(\alpha-n(1-1/q_1)-1)} + C\|\vec{b}\|_{Lip_\beta}2^{-l\alpha}2^{(l-k)(\alpha-n(1-1/q_1)-\gamma)}.$$

Thus,

$$I_2 \le C\|\vec{b}\|_{Lip_\beta}^p[\sum_{l=-\infty}^{\infty}(\sum_{k=-\infty}^{l-2}|\lambda_k|2^{(l-k)(\alpha-n(1-1/q_1)-1/2)})^p$$

$$+ \sum_{l=-\infty}^{\infty}(\sum_{k=-\infty}^{l-2}|\lambda_k|2^{(l-k)(\alpha-n(1-1/q_1)-\beta)})^p$$

$$+\sum_{l=-\infty}^{\infty}(\sum_{k=-\infty}^{l-2}|\lambda_k|2^{(l-k)(\alpha-n(1-1/q_1)-1)})^p$$

$$+\sum_{l=-\infty}^{\infty}(\sum_{k=-\infty}^{l-2}|\lambda_k|2^{(l-k)(\alpha-n(1-1/q_1)-\gamma)})^p].$$

When $p \le 1$,

$$I_2 \le C\|\vec{b}\|_{Lip_\beta}^p[\sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{l-2}|\lambda_k|^p 2^{(l-k)(\alpha-n(1-1/q_1)-1/2)p}$$

$$+\sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{l-2}|\lambda_k|^p 2^{(l-k)(\alpha-n(1-1/q_1)-\beta)p}$$

$$+\sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{l-2}|\lambda_k|^p 2^{(l-k)(\alpha-n(1-1/q_1)-1)p}$$

$$+\sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{l-2}|\lambda_k|^p 2^{(l-k)(\alpha-n(1-1/q_1)-\gamma)p}]$$

$$\le C\|\vec{b}\|_{Lip_\beta}^p\sum_{k=-\infty}^{\infty}|\lambda_k|^p.$$

When $\dot{p} > 1$,

$$I_2 \le C\|\vec{b}\|_{Lip_\beta}^p[\sum_{l=-\infty}^{\infty}(\sum_{k=-\infty}^{l-2}|\lambda_k|^p 2^{(l-k)(\alpha-n(1-1/q_1)-1/2)p/2})$$

$$\times(\sum_{k=-\infty}^{l-2}2^{(l-k)(\alpha-n(1-1/q_1)-1/2)p'/2})^{p/p'}$$

$$+\sum_{l=-\infty}^{\infty}(\sum_{k=-\infty}^{l-2}|\lambda_k|^p 2^{(l-k)(\alpha-n(1-1/q_1)-\beta)p/2})(\sum_{k=-\infty}^{l-2}2^{(l-k)(\alpha-n(1-1/q_1)-\beta)p'/2})^{p/p'}$$

$$+\sum_{l=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{l-2}|\lambda_k|^p 2^{(l-k)(\alpha-n(1-1/q_1)-1)p/2}\Big)\Big(\sum_{k=-\infty}^{l-2}2^{(l-k)(\alpha-n(1-1/q_1)-1)p'/2}\Big)^{p/p'}$$

$$+\sum_{l=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{l-2}|\lambda_k|^p 2^{(l-k)(\alpha-n(1-1/q_1)-\gamma)p/2}\Big)\Big(\sum_{k=-\infty}^{l-2}2^{(l-k)(\alpha-n(1-1/q_1)-\gamma)p'/2}\Big)^{p/p'}$$

$$\leq C\|\vec{b}\|_{Lip_\beta}^p\sum_{k=-\infty}^{\infty}|\lambda_k|^p.$$

The estimates for I_1 and I_2 lead to

$$\|\mu_\Omega^{\vec{b}}(f)\|_{\dot{K}_{q_2}^{\alpha,p}}\leq C\|\vec{b}\|_{Lip_\beta}\Big(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\Big)^{1/p},$$

and desired estimate follows from taking infimum over all decompositions of f.

When $\alpha = n(1-1/q_1)+\beta$, this kind of boundedness fails. In [15], Lu and Xu prove it when $m = 1$. Now, we give an estimate of weak type.

Theorem 4. *Let* $0<\beta<\min(\gamma/m, 1/2m)$, $0<p\leq 1$, $1<q_1$, $q_2<\infty$, $1/q_2 = 1/q_1 - m\beta/n$, $\vec{b}=(b_1,\cdots,b_m)$ *with* $b_j\in Lip_\beta(R^n)$ *for* $1\leq j\leq m$. *Then* $\mu_\Omega^{\vec{b}}$ *maps* $H\dot{K}_{q_1}^{n(1-1/q_1)+\beta,p}(R^n)$ *continuously into* $W\dot{K}_{q_2}^{n(1-1/q_1)+\beta,p}(R^n)$.

Proof. We write $f=\sum_{k=-\infty}^{\infty}\lambda_k a_k$, where each a_k is a central $(n(1-1/q_1)+\beta,\ q_1)$ atom supported on B_k and $\sum_{k=-\infty}^{\infty}|\lambda_k|^p<\infty$. Write

$$\|\mu_\Omega^{\vec{b}}\|_{W\dot{K}_{q_2}^{n(1-1/q_1)+\beta,p}}\leq\sup_{\lambda>0}\lambda\Big\{\sum_{l=-\infty}^{\infty}2^{l(n(1-1/q_1)+\beta)p}$$

$$\times\,|\{x\in E_l : |\mu_\Omega^{\vec{b}}\Big(\sum_{k=l-3}^{\infty}\lambda_k a_k\Big)(x)|>\lambda/2\}|^{p/q_2}\Big\}^{1/p}$$

$$+\sup_{\lambda>0}\lambda\{\sum_{l=-\infty}^{\infty}2^{l(n(1-1/q_1)+\beta)p}$$

$$\times|\{x\in E_l:|\mu_\Omega^{\vec{b}}(\sum_{k=-\infty}^{l-4}\lambda_k a_k)(x)|>\lambda/2\}|^{p/q_2}\}^{1/p}$$

$$=G_1+G_2.$$

By the (L^{q_1}, L^{q_2}) boundedness of $\mu_\Omega^{\vec{b}}$ and an estimate similar to that for I_1 in Theorem 3, we get

$$G_1^p\le C\sum_{l=-\infty}^{\infty}2^{lp(n(1-1/q_1)+\beta)}\|\mu_\Omega^{\vec{b}}(\sum_{l-3}^{\infty}\lambda_k a_k)(x)\chi_l\|_{q_2}^p\le C\|\vec{b}\|_{Lip_\beta}^p\sum_{k=-\infty}^{\infty}|\lambda_k|^p.$$

To estimate G_2, let us now use the estimate

$$|\mu_\Omega^{\vec{b}}(a_k)|\le C\|\vec{b}\|_{Lip_\beta}|x|^{m\beta-n-1/2}(2^k)^{1/2+n(1-1/q_1)-\alpha}$$

$$+C\sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}\|_{Lip_\beta}|x|^{\beta|\sigma|-n}(2^k)^{\beta|\sigma_c|+n(1-1/q_1)-\alpha}$$

$$+C\|\vec{b}\|_{Lip_\beta}|x|^{\beta m-n-1}(2^k)^{1+n(1-1/q_1)-\alpha}$$

$$+C\|\vec{b}\|_{Lip_\beta}|x|^{\beta m-n-\gamma}(2^k)^{\gamma+n(1-1/q_1)-\alpha},$$

which we get in the proof of Theorem 3. Note that when $x\in E_l$, $\alpha=n(1-1/q_1)+\beta$,

$$\lambda<\sum_{k=-\infty}^{l-4}|\lambda_k||\mu_\Omega^{\vec{b}}(a_k)|\le C\|\vec{b}\|_{Lip_\beta}\sum_{k=-\infty}^{l-4}|\lambda_k|[|x|^{m\beta-n-1/2}(2^k)^{1/2+n(1-1/q_1)-\alpha}$$

$$+ C\sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}|x|^{\beta|\sigma|-n}(2^k)^{\beta|\sigma_c|+n(1-1/q_1)-\alpha}$$

$$+ |x|^{\beta m-n-1}(2^k)^{1+n(1-1/q_1)-\alpha}$$

$$+ |x|^{\beta m-n-\gamma}(2^k)^{\gamma+n(1-1/q_1)-\alpha}\,]$$

$$\le C\|\vec{b}\|_{Lip_\beta}\sum_{k=-\infty}^{l-4}|\lambda_k|\,[(2^l)^{m\beta-n-1/2}\sum_{k=-\infty}^{l-4}(2^k)^{1/2-\beta}$$

$$+ \sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}(2^l)^{\beta|\sigma|-n}\sum_{k=-\infty}^{l-4}(2^k)^{\beta|\sigma_c|+n(1-1/q_1)-\alpha}$$

$$+ (2^l)^{m\beta-n-1}\sum_{k=-\infty}^{l-4}(2^k)^{1+n(1-1/q_1)-\alpha}$$

$$+ (2^l)^{m\beta-n-\gamma}\sum_{k=-\infty}^{l-4}(2^k)^{\gamma+n(1-1/q_1)-\alpha}\,]$$

$$\le C\|\vec{b}\|_{Lip_\beta}\sum_{k=-\infty}^{l-4}|\lambda_k|\,(2^l)^{((m-1)\beta-n)}$$

$$\le C\|\vec{b}\|_{Lip_\beta}2^{l((m-1)\beta-n)}(\sum_{k=-\infty}^{\infty}|\lambda_k|^p)^{1/p},$$

for $\lambda > 0$, let l_λ be the maximal positive integer satisfying

$$2^{l_\lambda(n-(m-1)\beta)} \le C\|\vec{b}\|_{Lip_\beta}\lambda^{-1}(\sum_{k=-\infty}^{\infty}|\lambda_k|^p)^{1/p},$$

then if $l > l_\lambda$, we have

$$|\{x\in E_l : |\mu_\Omega^{\vec{b}}(\sum_{k=-\infty}^{l-4}\lambda_k a_k)| > \lambda/2\}| = 0.$$

So, we obtain

$$G_2 \leq \sup_{\lambda>0} \lambda\{\sum_{l=-\infty}^{l_\lambda} 2^{l(n(1-1/q_1)+\beta)p}(2^l)^{np/q_2}\}^{1/p}$$

$$\leq \sup_{\lambda>0} \lambda\{\sum_{l=-\infty}^{l_\lambda}(2^l)^{(n-(m-1)\beta)}\}$$

$$\leq \sup_{\lambda>0} \lambda 2^{l_\lambda(n-(m-1)\beta)}$$

$$\leq C\|\vec{b}\|_{Lip_\beta}(\sum_{k=-\infty}^{\infty}|\lambda_k|^p)^{1/p}.$$

Now, combining the above estimates for G_1 and G_2, we obtain

$$\|\mu_\Omega^{\vec{b}}(f)\|_{W\dot{K}_{q_2}^{n(1-1/q_1)+\beta, p}} \leq C\|\vec{b}\|_{Lip_\beta}(\sum_{k=-\infty}^{\infty}|\lambda_k|^p)^{1/p}.$$

Theorem 4 follows by taking the infimum over all central atomic decompositions.

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