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BOUNDEDNESS OF TOEPLITZ TYPE OPERATOR ASSOCIATED TO SINGULAR INTEGRAL OPERATOR WITH VARIABLE CALDERÓN-ZYGMUND KERNELS ON L^p SPACES WITH VARIABLE EXPONENT

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Abstract

In this paper, the boundedness for some Toeplitz type operator related to some singular integral operator with variable Calderón-Zygmund kernels on L^p spaces with variable exponent is obtained by using a sharp estimate of the operator.

1. Introduction

As the development of the singular integral operators (see [6, 19]), their commutators have been well studied (see [2, 17, 18]). In [1], some singular integral operators with variable Calderón-Zygmund kernels are introduced, and the boundedness for the operators and their

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commutators are obtained (see [11, 12, 13, 15, 20]). In [8, 10, 14], some Toeplitz type operators related to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators are obtained. In the last years, a theory of L^p spaces with variable exponent has been developed because of its connections with some questions in fluid dynamics, calculus of variations, differential equations, and elasticity (see [3, 4, 5, 16] and their references). Karlovich and Lerner study the boundedness of the commutators of singular integral operators on L^p spaces with variable exponent (see [7]). Motivated by these papers, the main purpose of this paper is to introduce some Toeplitz type operator related to some singular integral operator with variable Calderón-Zygmund kernels and prove the boundedness for the operator on L^p spaces with variable exponent by using a sharp estimate of the operator.

2. Preliminaries and Results

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f and $\delta > 0$, the sharp function of f is defined by

$$f_\delta^{\#}(x) = \sup_{Q \ni x}\left(\frac{1}{|Q|}\int_Q |f(y) - f_Q|^\delta dy\right)^{1/\delta},$$

where, and in what follows, $f_Q = |Q|^{-1}\int_Q f(x)dx$. It is well-known that (see [6, 19])

$$f_\delta^{\#}(x) \approx \sup_{Q \ni x}\inf_{c \in C}\left(\frac{1}{|Q|}\int_Q |f(y) - c|^\delta dy\right)^{1/\delta}.$$

We write $f^{\#} = f_\delta^{\#}$ if $\delta = 1$. We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^\infty(R^n)$ and define $\|f\|_{BMO} = \|f^{\#}\|_{L^\infty}$. Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)| dy.$$

For $k \in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f)(x) = M(f)(x)$ and

$$M^k(f)(x) = M(M^{k-1}(f))(x) \text{ when } k \geq 2.$$

Let Φ be a Young function and $\widetilde{\Phi}$ be the complementary associated to Φ, we denote that the Φ-average by, for a function f,

$$\|f\|_{\Phi, Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right) dy \leq 1\right\},$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x) = \sup_{Q \ni x} \|f\|_{\Phi, Q}.$$

The Young functions to be using in this paper are $\Phi(t) = t(1 + \log t)^r$ and $\widetilde{\Phi}(t) = \exp(t^{1/r})$, the corresponding average and maximal functions denoted by $\|\cdot\|_{L(\log L)^r, Q}$, $M_{L(\log L)^r}$, and $\|\cdot\|_{\exp L^{1/r}, Q}$, $M_{\exp L^{1/r}}$. Following [17, 18], we know the generalized Hölder's inequality:

$$\frac{1}{|Q|}\int_Q |f(y)g(y)| dy \leq \|f\|_{\Phi, Q}\|g\|_{\widetilde{\Phi}, Q},$$

and the following inequality, for $r, r_j \geq 1$, $j = 1, \cdots, l$ with $1/r = 1/r_1 + \cdots + 1/r_l$, and any $x \in R^n$, $b \in BMO(R^n)$,

$$\|f\|_{L(\log L)^{1/r}, Q} \leq M_{L(\log L)^{1/r}}(f) \leq CM_{L(\log L)^l}(f) \leq CM^{l+1}(f),$$

$$\|f - f_Q\|_{\exp L^r, Q} \leq C\|f\|_{BMO},$$

$$|f_{2^{k+1}Q} - f_{2Q}| \leq Ck\|f\|_{BMO}.$$

The non-increasing rearrangement of a measurable function *f* on R^n is defined by

$$f^*(t) = \inf\{\lambda > 0 : |\{x \in R^n : |f(x)| > \lambda\}| \leq t\} \ (0 < t < \infty).$$

For $\lambda \in (0, 1)$ and a measurable function f on R^n, the local sharp maximal function of *f* is defined by

$$M_\lambda^{\#}(f)(x) = \sup_{Q \ni x} \inf_{c \in C} ((f - c)\chi_Q)^*(\lambda |Q|).$$

Let $p : R^n \to [1, \infty)$ be a measurable function. Denote by $L^{p(\cdot)}(R^n)$ the sets of all Lebesgue measurable functions f on R^n such that $m(\lambda f, p) < \infty$ for some $\lambda = \lambda(f) > 0$, where

$$m(f, p) = \int_{R^n} |f(x)|^{p(x)} dx.$$

The sets become a Banach spaces with respect to the following norm:

$$\|f\|_{L^{p(\cdot)}} = \inf\{\lambda > 0 : m(f / \lambda, p) \leq 1\}.$$

Denote by $M(R^n)$ the sets of all measurable functions $p : R^n \to [1, \infty)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(R^n)$ and the following holds:

$$1 < p_- = ess \inf_{x \in R^n} p(x), \quad ess \sup_{x \in R^n} p(x) = p_+ < \infty. \tag{1}$$

In recent years, the boundedness of classical operators on spaces $L^{p(\cdot)}(R^n)$ have attracted a great attention (see [3, 4, 5, 16] and their references).

In this paper, we will study some singular integral operator as following (see [1]):

Definition 1. Let $K(x) = \Omega(x)/|x|^n : R^n \setminus \{0\} \to R$. K is said to be a Calderón-Zygmund kernels, if

(a) $\Omega \in C^\infty(R^n \setminus \{0\})$;

(b) Ω is homogeneous of degree zero;

(c) $\int_\Sigma \Omega(x)x^\alpha d\sigma(x) = 0$ for all multi-indices $\alpha \in (N \cup \{0\})^n$ with $|\alpha| = N$, where $\Sigma = \{x \in R^n : |x| = 1\}$ is the unit sphere of R^n.

Definition 2. Let $K(x, y) = \Omega(x, y)/|y|^n : R^n \times (R^n \setminus \{0\}) \to R$. K is said to be a variable Calderón-Zygmund kernels, if

(d) $K(x, \cdot)$ is a Calderón-Zygmund kernels for a.e. $x \in R^n$;

(e) $\max_{|\gamma| \le 2n} \left\| \frac{\partial^{|\gamma|}}{\partial^\gamma y} \Omega(x, y) \right\|_{L^\infty(R^n \times \Sigma)} = L < \infty$.

Moreover, let b be a locally integrable function on R^n and T be the singular integral operator with variable Calderón-Zygmund kernels as

$$T(f)(x) = \int_{R^n} K(x, x - y) f(y) dy,$$

where $K(x, x - y) = \frac{\Omega(x, x - y)}{|x - y|^n}$ and that $\Omega(x, y)/|y|^n$ is a variable Calderón-Zygmund kernels.

Let b be a locally integrable function on R^n and T be the singular integral operator with variable Calderón-Zygmund kernels. The Toeplitz type operator associated to T are defined by

$$T_b = \sum_{k=1}^{m} T^{k,1} M_b T^{k,2},$$

where $T^{k,1}$ are the singular integral operator T with variable Calderón-Zygmund kernels or $\pm I$ (the identity operator), $T^{k,2}$ are the linear operators for $k = 1, \ldots, m$ and $M_b(f) = bf$.

Note that the commutator $[b, T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz type operator T_b. The Toeplitz type operators are the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [18]). In [1, 15], the boundedness of the singular integral operator with variable Calderón-Zygmund kernels and their commutator are obtained. Our works are motivated by these papers. The main purpose of this paper has twofold, first, we establish a sharp estimate for the operator T_b, and second, we prove the boundedness for the operator on L^p spaces with variable exponent by using the sharp estimate.

We shall prove the following theorems:

Theorem 1. *Let T be the singular integral operators with variable Calderón-Zygmund kernel as Definition* 2, $0 < \delta < 1$ *and* $b \in BMO(R^n)$. *If* $T_1(g) = 0$ *for any* $g \in L^u(R^n)(1 < u < \infty)$, *then there exists a constant* $C > 0$ *such that for any* $f \in L_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$(T_b(f))_\delta^{\#}(\tilde{x}) \leq C\|b\|_{BMO} \sum_{k=1}^{m} M^2(T^{k,2}(f))(\tilde{x}).$$

Theorem 2. *Let T be the singular integral operators with variable Calderón-Zygmund kernels as Definition* 2, $p(\cdot) \in M(R^n)$ *and* $b \in BMO(R^n)$. *If* $T_1(g) = 0$ *for any* $g \in L^u(R^n)(1 < u < \infty)$ *and* $T^{k,2}$ *are the bounded operators on* $L^{p(\cdot)}(R^n)$ *for* $k = 1, \ldots, m$, *then* T_b *is bounded on* $L^{p(\cdot)}(R^n)$, *that is,*

$$\|T_b(f)\|_{L^{p(\cdot)}} \leq C\|b\|_{BMO}\|f\|_{L^{p(\cdot)}}.$$

Corollary. *Let* $[b, T](f) = bT(f) - T(bf)$ *be the commutator generated by the singular integral operator* T *with variable Calderón-Zygmund kernels and* b. *Then Theorems* 1 *and* 2 *hold for* $[b, T]$.

3. Proof of Theorems

To prove the theorems, we need the following lemmas:

Lemma 1 ([6, p.485]). *Let* $0 < p < q < \infty$. *We define that, for any function* $f \geq 0$ *and* $1/r = 1/p - 1/q$,

$$\|f\|_{WL^q} = \sup_{\lambda>0} \lambda|\{x \in R^n : f(x) > \lambda\}|^{1/q},\ N_{p,q}(f) = \sup_E \|f\chi_E\|_{L^p} / \|\chi_E\|_{L^r},$$

where the sup is taken for all measurable sets E *with* $0 < |E| < \infty$. *Then*

$$\|f\|_{WL^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p}\|f\|_{WL^q}.$$

Lemma 2 ([18]). *Let* $r_j \geq 1$ *for* $j = 1, \cdots, l$, *we denote that* $1/r = 1/r_1 + \cdots + 1/r_l$. *Then*

$$\frac{1}{|Q|}\int_Q |f_1(x)\cdots f_l(x)g(x)|dx \leq \|f\|_{\exp L^{r_1}, Q} \cdots \|f\|_{\exp L^{r_l}, Q}\|g\|_{L(\log L)^{1/r}, Q}.$$

Lemma 3 ([1]). *Let* T *be the singular integral operators with variable Calderón-Zygmund kernels as Definition* 2. *Then* T *is bounded from* $L^1(R^n)$ *to* $WL^1(R^n)$.

Lemma 4 ([16]). *Let* $p : R^n \to [1, \infty)$ *be a measurable function satisfying* (1). *Then* $L_0^\infty(R^n)$ *is dense in* $L^{p(\cdot)}(R^n)$.

Lemma 5 ([7, 9]). *Let* $\delta > 0$, $0 < \lambda < 1$, *and* $f \in L_{\text{loc}}^\delta(R^n)$. *Then*

$$M_\lambda^\#(f)(x) \leq (1/\lambda)^{1/\delta} f_\delta^\#(x).$$

Lemma 6 ([16]). *Let* $f \in L^1_{loc}(R^n)$ *and* g *be a measurable function satisfying*

$$|\{x \in R^n : |g(x)| > \alpha\}| < \infty \text{ for all } \alpha > 0.$$

Then

$$\int_{R^n} |f(x)g(x)|dx \leq C_n \int_{R^n} M^{\#}_{\lambda_n}(f)(x)M(g)(x)dx.$$

Lemma 7 ([9]). *Let* $p : R^n \to [1, \infty)$ *be a measurable function satisfying* (1). *If* $f \in L^{p(\cdot)}(R^n)$ *and* $g \in L^{p'(\cdot)}(R^n)$ *with* $p'(x) = p(x)/(p(x)-1)$. *Then fg is integrable on* R^n *and*

$$\int_{R^n} |f(x)g(x)|dx \leq C\|f\|_{L^{p(\cdot)}}\|g\|_{L^{p'(\cdot)}}.$$

Lemma 8 ([9]). *Let* $p : R^n \to [1, \infty)$ *be a measurable function satisfying* (1). *Set*

$$\|f\|'_{L^{p(\cdot)}} = \sup\left\{\int_{R^n} |f(x)g(x)|dx : f \in L^{p(\cdot)}(R^n),\ g \in L^{p'(\cdot)}(R^n)\right\}.$$

Then $\|f\|_{L^{p(\cdot)}} \leq \|f\|'_{L^{p(\cdot)}} \leq C\|f\|_{L^{p(\cdot)}}$.

Proof of Theorem 1. It suffices to prove for $f \in L^\infty_0(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q |T_b(f)(x) - C_0|^\delta dx\right)^{1/\delta} \leq C\|b\|_{BMO}\sum_{k=1}^m M^2(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1, \ldots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. We write, by $T_1(g) = 0$,

$$\begin{aligned} T_b(f)(x) &= T_{b-b_{2Q}}(f)(x) \\ &= T_{(b-b_{2Q})\chi_{2Q}}(f)(x) + T_{(b-b_{2Q})\chi_{(2Q)^c}}(f)(x) \\ &= f_1(x) + f_2(x). \end{aligned}$$

Then

$$\left(\frac{1}{|Q|}\int_Q |T_b(f)(x) - f_2(x_0)|^\delta dx\right)^{1/\delta}$$

$$\leq C\left(\frac{1}{|Q|}\int_Q |f_1(x)|^\delta dx\right)^{1/\delta} + C\left(\frac{1}{|Q|}\int_Q |f_2(x) - f_2(x_0)|^\delta dx\right)^{1/\delta} = I + II.$$

For I, by Lemmas 1, 2, and 3, we obtain

$$\left(\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|^\delta dx\right)^{1/\delta}$$

$$\leq |Q|^{-1}\frac{\|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\chi_Q\|_{L^\delta}}{|Q|^{1/\delta-1}}$$

$$\leq C|Q|^{-1}\|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\|_{WL^1}$$

$$\leq C|Q|^{-1}\|M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\|_{L^1}$$

$$\leq C|Q|^{-1}\int_{2Q}|b(x) - b_{2Q}||T^{k,2}(f)(x)|dx$$

$$\leq C\|b - b_{2Q}\|_{\exp L, 2Q}\|T^{k,2}(f)\|_{L(\log L), 2Q}$$

$$\leq C\|b\|_{BMO}M^2(T^{k,2}(f))(\tilde{x}),$$

thus,

$$I \leq \sum_{k=1}^{m}\left(\frac{C}{|Q|}\int_Q |T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|^\delta dx\right)^{1/\delta}$$

$$\leq C\|b\|_{BMO}\sum_{k=1}^{m}M^2(T^{k,2}(f))(\tilde{x}).$$

For II, by [1], we know that

$$T(f)(x)=\sum_{u=1}^{\infty}\sum_{v=1}^{g_u}a_{uv}(x)\int_{R^n}\frac{Y_{uv}(x-y)}{|x-y|^{n+m}}f(y)dy,$$

where $g_u\leq Cu^{n-2}$, $\|a_{uv}\|_{L^\infty}\leq Cu^{-2n}$, $|Y_{uv}(x-y)|\leq Cu^{n/2-1}$, and

$$\left|\frac{Y_{uv}(x-y)}{|x-y|^n}-\frac{Y_{uv}(x_0-y)}{|x_0-y|^n}\right|\leq Cu^{n/2}|x-x_0|/|x_0-y|^{n+1},$$

for $|x-y|>2|x_0-x|>0$. Then, we get, for $x\in Q$,

$$|T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}(f)(x)-T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}T^{k,2}(f)(x_0)|$$

$$\leq\int_{(2Q)^c}|b(y)-b_{2Q}||K(x,x-y)-K(x_0,x_0-y)||T^{k,2}(f)(y)|dy$$

$$=\sum_{j=1}^{\infty}\int_{2^jd\leq|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}||K(x,x-y)-K(x_0,x_0-y)||T^{k,2}(f)(y)|dy$$

$$\leq C\sum_{j=1}^{\infty}\int_{2^jd\leq|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\sum_{u=1}^{\infty}\sum_{v=1}^{g_u}|a_{uv}(x)|$$

$$\times\left|\frac{Y_{uv}(x-y)}{|x-y|^n}-\frac{Y_{uv}(x_0-y)}{|x_0-y|^n}\right||T^{k,2}(f)(y)|dy$$

$$\leq C\sum_{j=1}^{\infty}\int_{2^jd\leq|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\frac{|x-x_0|}{|x_0-y|^{n+1}}|T^{k,2}(f)(y)|dy$$

$$\leq C\sum_{j=1}^{\infty}\frac{d}{(2^{j+1}d)^{n+1}}\int_{2^{j+1}Q}|b(y)-b_{2Q}||T^{k,2}(f)(y)|dy$$

$$\leq C\sum_{j=1}^{\infty}2^{-j}\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|b(y)-b_{2Q}||T^{k,2}(f)(y)|dy$$

$$\leq C\sum_{j=1}^{\infty} 2^{-j}\|b - b_{2Q}\|_{\exp L, 2^{j+1}Q}\|T^{k,2}(f)\|_{L(\log L), 2^{j+1}Q}$$

$$\leq C\sum_{j=1}^{\infty} j2^{-j}\|b\|_{BMO} M^2(T^{k,2}(f))(\tilde{x})$$

$$\leq C\|b\|_{BMO} M^2(T^{k,2}(f))(\tilde{x}),$$

thus,

$$II \leq \frac{C}{|Q|}\int_Q \sum_{k=1}^{m} |T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}} T^{k,2}(f)(x) - T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}} T^{k,2}(f)(x_0)|dx$$

$$\leq C\|b\|_{BMO}\sum_{k=1}^{m} M^2(T^{k,2}(f))(\tilde{x}).$$

This completes the proof of Theorem 1.

Proof of Theorem 2. By Lemmas 4-7, we get, for $f \in L_0^\infty(R^n)$ and $g \in L^{p'(\cdot)}(R^n)$,

$$\int_{R^n} |T_b(f)(x)g(x)|dx \leq C\int_{R^n} M_{\lambda_n}^{\#}(T_b(f))(x)M(g)(x)|dx$$

$$\leq C\int_{R^n} (T_b(f))_\delta^{\#}(x)M(g)(x)dx$$

$$\leq C\|b\|_{BMO}\sum_{k=1}^{m}\int_{R^n} M^2(T^{k,2}(f))(x)M(g)(x)dx$$

$$\leq C\|b\|_{BMO}\sum_{k=1}^{m}\|M^2(T^{k,2}(f))\|_{L^{p(\cdot)}}\|M(g)\|_{L^{p'(\cdot)}}$$

$$\leq C\|b\|_{BMO}\sum_{k=1}^{m}\|T^{k,2}(f)\|_{L^{p(\cdot)}}\|M(g)\|_{L^{p'(\cdot)}}$$

$$\leq C\|b\|_{BMO}\|f\|_{L^{p(\cdot)}}\|g\|_{L^{p'(\cdot)}},$$

thus, by Lemma 8,

$$\|T_b(f)\|_{L^{p(\cdot)}} \leq \|b\|_{BMO}\|f\|_{L^{p(\cdot)}}.$$

This completes the proof of Theorem 2.

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