

SOME STABILITY THEOREMS ON COMMON FIXED POINTS IN HAUSDORFF UNIFORM SPACES

ALFRED OLUFEMI BOSEDE

Department of Mathematics
Lagos State University
Ojo, Lagos State
Nigeria
e-mail: aolubosedede@yahoo.co.uk

Abstract

In this paper, we establish some stability results for some common fixed points for a pair of self-mappings in Hausdorff uniform spaces. These results are proved by using the concepts of A -distance and E -distance as well as the notion of comparison functions. Our results generalize and improve some of the known stability results in literature.

1. Introduction

Within the last two decades, several authors such as Berinde [2], Jachymski [12], Kada et al. [13], Rhoades [18, 19], Rus [21], Wang et al. [23], and Zeidler [24] studied the theory of fixed point or common fixed point for contractive self-mappings in complete metric spaces or Banach spaces in general.

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Later, Kang [14], Rodriguez-Montes and Charris [20] established some results on fixed and coincidence points of maps by means of appropriate W -contractive or W -expansive assumptions in uniform spaces.

The following definition of a uniform space shall be required in the sequel:

Definition 1.1. Let X be a nonempty set and Φ be a nonempty family of subsets of $X \times X$. The pair (X, Φ) is called a *uniform space*, if it satisfies the following properties:

- (i) if G is in Φ , then G contains the diagonal $\{(x, x) \mid x \in X\}$;
- (ii) if G is in Φ and H is a subset of $X \times X$, which contains G , then H is in Φ ;
- (iii) if G and H are in Φ , then $G \cap H$ is in Φ ;
- (iv) if G is in Φ , then there exists H in Φ , such that, whenever (x, y) and (y, z) are in H , then (x, z) is in H ;
- (v) if G is in Φ , then $\{(y, x) \mid (x, y) \in G\}$ is also in Φ .

Φ is called the *uniform structure* of X and its elements are called *entourages* or *neighbourhoods* or *surroundings*.

If property (v) is omitted, then (X, Φ) is called a *quasiuniform space* (for example, see Bourbaki [9] and Zeidler [24]).

Let (X, Φ) be a uniform space and $(X, \tau(\Phi))$ be a topological space whenever topological concepts are mentioned in the context of a uniform space (X, Φ) . Definitions 1.2-1.6 are contained in Aamri and El Moutawakil [1].

Definition 1.2. If $H \in \Phi$ and $(x, y) \in H, (y, x) \in H, x$ and y are said to be *H -close*. A sequence $\{x_n\}_{n=0}^{\infty} \subset X$ is said to be a *Cauchy sequence* for Φ if for any $H \in \Phi$, there exists $N \geq 1$ such that x_n and x_m are *H -close* for $n, m \geq N$.

Definition 1.3. A function $p : X \times X \rightarrow R^+$ is said to be an *A-distance*, if for any $H \in \Phi$, there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in H$.

Definition 1.4. A function $p : X \times X \rightarrow R^+$ is said to be an *E-distance*, if

(p₁) p is an *A-distance*;

(p₂) $p(x, y) \leq p(x, z) + p(z, y), \quad \forall x, y \in X$.

Definition 1.5. A uniform space (X, Φ) is said to be *Hausdorff*, if and only if the intersection of all $H \in \Phi$ reduces to the diagonal $\{(x, x) \mid x \in X\}$, i.e., if $(x, y) \in H$ for all $H \in \Phi$ implies $x = y$. This guarantees the uniqueness of limits of sequences. $H \in \Phi$ is said to be *symmetrical*, if $H = H^{-1} = \{(y, x) \mid (x, y) \in H\}$.

Definition 1.6. Let (X, Φ) be a uniform space and p be an *A-distance* on X .

(i) Sequence $\{x_n\}_{n=0}^{\infty}$ is *p-Cauchy* if given $\epsilon > 0$, there exists N such that if $m, n > N$, then $p(x_m, x_n) < \epsilon$.

(ii) X is said to be *S-complete* if for every *p-Cauchy* sequence $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ with $\lim_{n \rightarrow \infty} p(x_n, x) = 0$.

(iii) X is said to be *p-Cauchy complete* if for every *p-Cauchy* sequence $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ with $\lim_{n \rightarrow \infty} x_n = x$ with respect to $\tau(\Phi)$.

(iv) $f : X \rightarrow X$ is said to be *p-continuous* if $\lim_{n \rightarrow \infty} p(x_n, x) = 0$ implies that $\lim_{n \rightarrow \infty} p(f(x_n), f(x)) = 0$.

(v) $f : X \rightarrow X$ is $\tau(\Phi)$ -*continuous* if $\lim_{n \rightarrow \infty} x_n = x$ with respect to $\tau(\Phi)$ implies $\lim_{n \rightarrow \infty} f(x_n) = f(x)$ with respect to $\tau(\Phi)$.

(vi) X is said to be *p-bounded* if $\delta_p = \sup \{p(x, y) \mid x, y \in X\} < \infty$.

In 2004, Aamri and El Moutawakil [1] proved some common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the notions of an A -distance and an E -distance.

Aamri and El Moutawakil [1] introduced and employed the following contractive definition: Let $f, g : X \rightarrow X$ be self-mappings of X and p is an A -distance on X . Then, we have

$$p(f(x), f(y)) \leq \psi(p(g(x), g(y))), \quad \forall x, y \in X, \quad (1.1)$$

where $\psi : R^+ \rightarrow R^+$ is a nondecreasing function satisfying

- (i) for each $t \in (0, +\infty)$, $0 < \psi(t)$;
- (ii) $\lim_{n \rightarrow \infty} \psi^n(t) = 0$, $\forall t \in (0, +\infty)$.

ψ satisfies also the condition $\psi(t) < t$, for each $t > 0$.

In 2007, Olatinwo [15] established some common fixed point theorems by employing the following contractive definition: Let $f, g : X \rightarrow X$ be self-mappings of X . There exist $L \geq 0$ and a comparison function $\psi : R^+ \rightarrow R^+$ such that $\forall x, y \in X$, we have

$$p(f(x), f(y)) \leq Lp(x, g(x)) + \psi(p(g(x), g(y))), \quad \forall x, y \in X, \quad (1.2)$$

where p is an A -distance in X .

Recently, Bosede [6] proved some common fixed point theorems by employing the following contractive definition: Let $f, g : X \rightarrow X$ be self-mappings of X . There exist comparison functions $\psi_1 : R^+ \rightarrow R^+$ and $\psi_2 : R^+ \rightarrow R^+$ with $\psi_1(0) = 0$ such that $\forall x, y \in X$, we have

$$p(f(x), f(y)) \leq \psi_1(p(x, g(x))) + \psi_2(p(g(x), g(y))), \quad \forall x, y \in X, \quad (1.3)$$

where p is an A -distance in X .

In this paper, we shall establish some stability results for some common fixed points for a pair of self-mappings in Hausdorff uniform spaces by using a contractive condition more general than (1.1), (1.2), and (1.3).

We shall also employ the concepts of an A -distance, an E -distance as well as the notion of comparison functions in this paper.

2. Preliminaries

The following definition is contained in Aamri and El Moutawakil [1]:

Definition 2.1. Let (X, Φ) be a Hausdorff uniform space and p be an A -distance on X . Two self-mappings f and g on X are said to be p -compatible if, for each sequence $\{x_n\}_{n=0}^{\infty}$ of X such that $\lim_{n \rightarrow \infty} p(f(x_n), u) = \lim_{n \rightarrow \infty} p(g(x_n), u) = 0$ for some $u \in X$, then we have $\lim_{n \rightarrow \infty} p(f(g(x_n)), g(f(x_n))) = 0$.

The following definition, which is contained in Berinde [2], Rus [21], and Rus et al. [22] shall also be required in the sequel.

Definition 2.2. A function $\psi : R^+ \rightarrow R^+$ is called a *comparison function*, if

- (i) ψ is monotone increasing;
- (ii) $\lim_{n \rightarrow \infty} \psi^n(t) = 0, \quad \forall t \geq 0$.

Also, within the last two decades, many stability results have been obtained by various authors by using different contractive definitions. For example, Harder and Hicks [10] considered the following concept to obtain various stability results:

Let (X, d) be a complete metric space, $T : X \rightarrow X$ be a selfmap of X . Suppose that $F_T = \{u \in X : Tu = u\}$ is the set of fixed points of T in X .

Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iteration procedure involving the operator T , that is,

$$x_{n+1} = h(T, x_n), \quad n = 0, 1, 2, \dots, \quad (2.1)$$

where $x_0 \in X$ is the initial approximation and h is some function. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point u of T . Let $\{y_n\}_{n=0}^{\infty} \subset X$ and set

$$\epsilon_n = d(y_{n+1}, h(T, y_n)), \quad n = 0, 1, 2, \dots \quad (2.2)$$

Then, the iteration procedure (2.1) is said to be T -stable or stable with respect to T , if and only if $\lim_{n \rightarrow \infty} \epsilon_n = 0$ implies $\lim_{n \rightarrow \infty} y_n = u$.

Throughout this paper, h represents some function, while f and g shall denote two self-mappings of a uniform space (X, Φ) .

We shall employ the following definition of stability of iteration process, which is a natural extension of Harder and Hicks [10]:

Definition 2.3. Let (X, Φ) be a uniform space and $f, g : X \rightarrow X$ be two selfmaps of X . Suppose there exists $u \in F_f \cap F_g$, a common fixed point of f and g in X ; while F_f and F_g are the sets of fixed points of f and g in X , respectively.

Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iteration procedure involving the operators f and g , that is,

$$x_{n+1} = h(f, g, x_n), \quad n = 0, 1, 2, \dots, \quad (2.3)$$

where $x_0 \in X$ is the initial approximation and h is some function.

Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a common fixed point u of f and g in X .

Let $\{y_n\}_{n=0}^{\infty} \subset X$ and set

$$\epsilon_n = p(y_{n+1}, h(f, g, y_n)), \quad n = 0, 1, 2, \dots, \quad (2.4)$$

where p is an A -distance, which replaces the distance function d in (2.2).

Then, the iteration procedure (2.3) is said to be (f, g) -stable or stable with respect to f and g , if and only if $\lim_{n \rightarrow \infty} \epsilon_n = 0$ implies $\lim_{n \rightarrow \infty} y_n = u$.

Remark 2.1. If $f = g = T$ in (2.3), then we obtain the iteration procedure of Harder and Hicks [10].

Also, if $f = g = T$ and $p = d$ in (2.4), then we get (2.2); which was used by Harder and Hicks [10] and many other authors.

Our aim in this paper is to establish some stability results for a pair of self-mappings f and g in a Hausdorff uniform space X , by employing the following contractive condition:

Let $f, g : X \rightarrow X$ be self-mappings of X . There exist $M \geq 0$ and comparison functions $\psi_1 : R^+ \rightarrow R^+$ and $\psi_2 : R^+ \rightarrow R^+$ with $\psi_1(0) = 0$ such that $\forall x, y \in X$, we have

$$p(f(x), f(y)) \leq \frac{\psi_1(p(x, g(x))) + \psi_2(p(g(x), g(y)))}{1 + Mp(x, g(x))}, \quad \forall x, y \in X, \quad (2.5)$$

where p is an A -distance in X .

Remark 2.2. The contractive condition (2.5) is more general than (1.1), (1.2), and (1.3) in the sense that if $M = 0$ in (2.5), then we obtain

$$p(f(x), f(y)) \leq \psi_1(p(x, g(x))) + \psi_2(p(g(x), g(y))), \quad \forall x, y \in X,$$

which is the contractive condition employed by Bosede [6] in (1.3).

Moreover, if $M = 0$ and $\psi_1(u) = Lu$ in (2.5), for $L \geq 0, u \in R^+$, then we obtain

$$p(f(x), f(y)) \leq Lp(x, g(x)) + \psi_2(p(g(x), g(y))), \quad \forall x, y \in X,$$

which is the contractive condition employed by Olatinwo [15] in (1.2).

Furthermore, if $L = 0$ in the above inequality, then we obtain (1.1), which was employed by Aamri and El Moutawakil [1].

Therefore, the contractive condition (2.5) is a generalization of the contractive definitions (1.1), (1.2), and (1.3) of Aamri and El Moutawakil [1], Olatinwo [15], and Bosede [6], respectively.

3. Main Results

Theorem 3.1. *Let (X, Φ) be a Hausdorff uniform space and p be an A -distance on X such that X is p -bounded and S -complete. For arbitrary $x_0 \in X$, define a sequence $\{x_n\}_{n=0}^{\infty}$ iteratively by*

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \dots \quad (3.1)$$

Suppose that f and g are p -continuous or $\tau(\Phi)$ -continuous self-mappings of X , with a common fixed point u in X , satisfying

- (i) $f(X) \subseteq g(X)$;
- (ii) $p(f(x_i), f(x_i)) = 0, \quad \forall x_i \in X, \quad i = 0, 1, 2, \dots$ In particular, $p(f(u), f(u)) = 0$;
- (iii) $f, g : X \rightarrow X$ satisfy the contractive condition (2.5) with $M \geq 0$.

Suppose also that $\psi_1 : R^+ \rightarrow R^+$ and $\psi_2 : R^+ \rightarrow R^+$ are comparison functions with $\psi_1(0) = 0$.

Then iteration (3.1) is (f, g) -stable.

Proof. For arbitrary $x_0 \in X$, select $x_1 \in X$ such that $f(x_0) = g(x_1)$. Similarly, for $x_1 \in X$, select $x_2 \in X$ such that $f(x_1) = g(x_2)$.

Continuing this process, we select $x_n \in X$ such that $f(x_{n-1}) = g(x_n)$. Hence, iteration (3.1) is well-defined.

Let $\{y_n\}_{n=0}^{\infty} \subset X$ and let $\{\epsilon_n\}_{n=0}^{\infty}$ be a sequence defined by $\epsilon_n = p(y_{n+1}, f(y_n))$.

Suppose that $\{x_n\}_{n=0}^{\infty}$ converges to a common fixed point u of f and g in X , that is, $f(u) = g(u) = u$, where u is in X .

Suppose also that $\lim_{n \rightarrow \infty} \epsilon_n = 0$. Then, we shall prove that $\lim_{n \rightarrow \infty} y_n = u$.

Since X is p -bounded, we assume that $p(f(u), f(y_0)) \leq \delta_p(X)$, $y_0 \in X$, where $\delta_p(X) = \sup \{p(x, y) \mid x, y \in X\} < +\infty$.

Indeed, since $x_n = f(x_{n-1})$, $n = 1, 2, \dots$, then, using the contractive definition (2.5) and the triangle inequality, we obtain

$$\begin{aligned}
p(y_{n+1}, u) &\leq p(y_{n+1}, f(y_n)) + p(f(y_n), u) \\
&= \epsilon_n + p(f(y_n), f(u)) \\
&= \epsilon_n + p(f(u), f(y_n)) \\
&\leq \epsilon_n + \frac{\psi_1(p(u, g(u))) + \psi_2(p(g(u), g(y_n)))}{1 + Mp(u, g(u))} \\
&= \epsilon_n + \frac{\psi_1(p(f(u), f(u))) + \psi_2(p(f(u), f(y_{n-1})))}{1 + Mp(f(u), f(u))} \\
&= \epsilon_n + \frac{\psi_1(0) + \psi_2(p(f(u), f(y_{n-1})))}{1 + M(0)} \\
&= \epsilon_n + \frac{0 + \psi_2(p(f(u), f(y_{n-1})))}{1 + 0} \\
&= \epsilon_n + \psi_2(p(f(u), f(y_{n-1}))) \\
&\leq \epsilon_n + \psi_2\left(\frac{\psi_1(p(u, g(u))) + \psi_2(p(g(u), g(y_{n-1})))}{1 + Mp(u, g(u))}\right) \\
&= \epsilon_n + \psi_2\left(\frac{\psi_1(p(f(u), f(u))) + \psi_2(p(f(u), f(y_{n-2})))}{1 + Mp(f(u), f(u))}\right)
\end{aligned}$$

$$\begin{aligned}
&= \epsilon_n + \psi_2 \left(\frac{\psi_1(0) + \psi_2(p(f(u), f(y_{n-2})))}{1 + M(0)} \right) \\
&= \epsilon_n + \psi_2 \left(\frac{0 + \psi_2(p(f(u), f(y_{n-2})))}{1 + 0} \right) \\
&= \epsilon_n + \psi_2^2(p(f(u), f(y_{n-2}))) \\
&\leq \dots \leq \epsilon_n + \psi_2^n(p(f(u), f(y_0))) \\
&\leq \epsilon_n + \psi_2^n(\delta_p(X)). \tag{3.2}
\end{aligned}$$

But condition (ii) of Definition 2.2 of a comparison function gives

$$\lim_{n \rightarrow \infty} \psi_2^n(\delta_p(X)) = 0.$$

Hence, taking the limit as $n \rightarrow \infty$ of both sides of (3.2) yields

$$\lim_{n \rightarrow \infty} p(y_{n+1}, u) = 0,$$

which implies that

$$\lim_{n \rightarrow \infty} y_n = u.$$

Conversely, let $\lim_{n \rightarrow \infty} y_n = u$. Then,

$$\begin{aligned}
\epsilon_n &= p(y_{n+1}, f(y_n)) \\
&\leq p(y_{n+1}, u) + p(u, f(y_n)) \\
&= p(y_{n+1}, u) + p(f(u), f(y_n)) \\
&\leq p(y_{n+1}, u) + \frac{\psi_1(p(u, g(u))) + \psi_2(p(g(u), g(y_n)))}{1 + Mp(u, g(u))} \\
&= p(y_{n+1}, u) + \frac{\psi_1(p(f(u), f(u))) + \psi_2(p(f(u), f(y_{n-1})))}{1 + Mp(f(u), f(u))} \\
&= p(y_{n+1}, u) + \frac{\psi_1(0) + \psi_2(p(f(u), f(y_{n-1})))}{1 + M(0)}
\end{aligned}$$

$$\begin{aligned}
&= p(y_{n+1}, u) + \frac{0 + \psi_2(p(f(u), f(y_{n-1})))}{1 + 0} \\
&= p(y_{n+1}, u) + \psi_2(p(f(u), f(y_{n-1}))) \\
&\leq p(y_{n+1}, u) + \psi_2\left(\frac{\psi_1(p(u, g(u))) + \psi_2(p(g(u), g(y_{n-1})))}{1 + Mp(u, g(u))}\right) \\
&= p(y_{n+1}, u) + \psi_2\left(\frac{\psi_1(p(f(u), f(u))) + \psi_2(p(f(u), f(y_{n-2})))}{1 + Mp(f(u), f(u))}\right) \\
&= p(y_{n+1}, u) + \psi_2\left(\frac{\psi_1(0) + \psi_2(p(f(u), f(y_{n-2})))}{1 + M(0)}\right) \\
&= p(y_{n+1}, u) + \psi_2\left(\frac{0 + \psi_2(p(f(u), f(y_{n-2})))}{1 + 0}\right) \\
&= p(y_{n+1}, u) + \psi_2^2(p(f(u), f(y_{n-2}))) \\
&\leq \dots \leq p(y_{n+1}, u) + \psi_2^n(p(f(u), f(y_0))) \\
&\leq p(y_{n+1}, u) + \psi_2^n(\delta_p(X)) \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

This completes the proof.

The next theorem is the stability result, where p is an E -distance on a Hausdorff uniform space X .

Theorem 3.2. *Let (X, Φ) , $f, g, u, \psi, \{x_n\}_{n=0}^\infty$ be as defined in Theorem 3.1 above and p be an E -distance on X . Then, iteration (3.1) is (f, g) -stable.*

Proof. We observe that an E -distance function p on X is also an A -distance on X . Therefore, the remaining part of the proof follows the same standard argument as in the proof of Theorem 3.1 above and it is therefore omitted.

Remark 3.1. Theorems 3.1 and 3.2 of this paper are generalizations of those of Berinde [2], Harder and Hicks [10] and many existing stability results in literature.

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