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MULTILINEAR COMMUTATORS OF FRACTIONAL INTEGRALS ASSOCIATE TO OPERATORS IN MORREY SPACES

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Abstract

Let L be the infinitesimal generator of an analytic semigroup on $L^2(\mathbb{R}^n)$ with Gaussian kernel bounds, and $L^{-\alpha/2}$ be the fractional integrals of L for

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$0 < \alpha < n$. In this paper, we obtain the boundedness of multilinear commutators of BMO functions and $L^{-\alpha/2}$ in Morrey spaces.

1. Introduction

Suppose that L is a linear operator on $L^2(\mathbb{R}^n)$, which generates an analytic semigroup e^{-tL} with kernel $p_t(x, y)$ satisfying a Gaussian upper bound, that is,

$$|p_t(x, y)| \leq \frac{C}{t^{n/2}} e^{-c\frac{|x-y|^2}{t}}, \tag{1.1}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$.

For $0 < \alpha < n$, the fractional integrals $L^{-\alpha/2}$ of the operator L is defined by

$$L^{-\alpha/2}f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL}(f) \frac{dt}{t^{-\alpha/2+1}}(x).$$

Let b be a BMO function on $\mathbb{R}^n$. The commutator of b and $L^{-\alpha/2}$ is defined by

$$[b, L^{-\alpha/2}](f)(x) = b(x)L^{-\alpha/2}(f)(x) - L^{-\alpha/2}(bf)(x).$$

Note that if $L = -\Delta$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the classical fractional integrals $\mathcal{I}_\alpha$ (see, for example, [17, Chapter 5]). It is well-known that when $b \in \mathrm{BMO}(\mathbb{R}^n)$, the commutator $[b, \mathcal{I}_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ ([3]). Duong and Yan proved that under condition (1.1), the commutator $[b, L^{-\alpha/2}]$ is still bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ (see [7] and [1]).

Recently, commutators of the classical fractional integrals $\mathcal{I}_\alpha$ in Morrey spaces have been studied by many authors, see [9, 14], and the references therein. The classical Morrey spaces were introduced by Morrey in [11] to investigate the local behaviour of solutions to second order elliptic partial differential equations. The aim of this paper is to continue this line to study the multilinear commutator $[\vec{b}, L^{-\alpha/2}]$ of BMO functions and $L^{-\alpha/2}$ in Morrey spaces. Following [13], we define the multilinear commutator $[\vec{b}, L^{-\alpha/2}]$ by

$$[\vec{b}, L^{-\alpha/2}](f)(x) = \int_{\mathbb{R}^n} \prod_{i=1}^{m} (b_i(x) - b_i(y)) K_\alpha(x, y) f(y) dy, \tag{1.2}$$

holds for each continuous function f with compact support, and for almost all x not in the support of f, where $\vec{b} = \{b_1, \cdots, b_m\}$, b_i's are BMO functions and $K_\alpha(x, y)$ is the kernel of $L^{-\alpha/2}$.

Let $1 \leq p < \infty$ and $0 < \kappa < 1$. The Morrey space is defined by

$$L^{p,\kappa}(\mathbb{R}^n) := \{f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\kappa}} < \infty\}, \tag{1.3}$$

where

$$\|f\|_{L^{p,\kappa}} = \sup_B \left(\frac{1}{|B|^\kappa} \int_B |f|^p dx \right)^{1/p}, \tag{1.4}$$

and the supremum is taken over all balls B in $\mathbb{R}^n$. The following is the main result of this paper:

Theorem 1.1. *Assume condition* (1.1) *and let* $\vec{b} = \{b_1, \cdots, b_m\}$, b_i*'s are BMO functions. Then for* $0 < \alpha < n, 1 < p < \frac{n}{\alpha}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, *and* $0 < \kappa < p / q$, *the multilinear commutator* $[\vec{b}, L^{-\alpha/2}]$ *satisfies*

$$\|[\vec{b}, L^{-\alpha/2}](f)\|_{L^{q,\kappa q/p}} \leq C \left(\prod_{i=1}^{m} \|b_i\|_* \right) \|f\|_{L^{p,\kappa}},$$

where $\|b_i\|_*$ *denotes the BMO norm of* $b_i(x)$.

The paper is organized as follows. In Section 2, we recall some important estimates on BMO functions, maximal functions, and fractional integrals. In Section 3, we will prove the main result. We conclude this paper by giving applications to large classes of differential operators.

Finally, in the sequel, we use C to denote a positive constant, which is independent of the main parameters, but it may vary from line to line.

2. Definitions and Preliminary Results

Denote the Hardy-Littlewood maximal function Mf and its variant $M_{\alpha, r}f$ by

$$Mf(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy, \tag{2.1}$$

and

$$M_{\alpha, r}f(x) = \sup_{x \in B} \left(\frac{1}{|B|^{1-\alpha r/n}} \int_B |f(y)|^r dy \right)^{1/r}, \quad 0 \leq \alpha < n,\ r \geq 1, \tag{2.2}$$

where the sup is taken over all balls B containing x. If $\alpha = 0$, $M_{0, r}f(x)$ will be denoted by $M_r f(x)$. For any $f \in L^p(\mathbb{R}^n)$, $p \geq 1$, the sharp maximal function $M_L^\sharp f$ associated with "generalized approximations to the identity" $\{e^{-tL}, t > 0\}$, is given by

$$M_L^\sharp f(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y) - e^{-t_B L} f(y)| dy, \tag{2.3}$$

where $t_B = r_B^2$ and r_B is the radius of the ball B (see [10]).

A function $b(y) \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to be in $\text{BMO}(\mathbb{R}^n)$, if and only if

$$\sup_B \frac{1}{|B|} \int_B |b(y) - b_B| dy < \infty,$$

where $b_B = \frac{1}{|B|}\int_B b(y)dy$. The BMO norm of $b(y)$ is defined by

$$\|b\|_* = \sup_B \frac{1}{|B|}\int_B |b(y) - b_B|\,dy.$$

Lemma 2.1. (i) *Assume* $b \in BMO$ *and* $M > 1$. *Then for every ball* B, *we have*

$$|b_B - b_{MB}| \le C\|b\|_* \log M.$$

(ii) (*John-Nirenberg lemma*) *Let* $1 \le p < \infty$. *Then* $b \in BMO$, *if and only if*

$$\frac{1}{|B|}\int_B |b(y) - b_B|^p\,dy \le C\|b\|_*^p.$$

Proof. For the proof of this lemma, see [3]. See, also [5] and [8]. □

Lemma 2.2. *For* $1 < p < \infty$ *and* $0 < \kappa < 1$, *we have* $\|Mf\|_{L^{p,\kappa}} \le C\|f\|_{L^{p,\kappa}}$.

For the proof of this lemma, see [4]. Using this lemma, it is easy to known that for $1 < r < p$, we have $\|M_r f\|_{L^{p,\kappa}} = \|M(|f|^r)\|_{L^{p/r,\kappa}}^{1/r} \le C\|f\|_{L^{p,\kappa}}$.

Lemma 2.3. *For all* $0 < \alpha < n$, *we have* $\|M_{\alpha,1}(f)\|_{L^{q,\kappa q/p}} \le C\|f\|_{L^{p,\kappa}}$, *where* $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, *and* $0 < \kappa < p/q$.

For the proof of this lemma, see [9, Theorem 3.5]. From this lemma, for $1 < r < p$, we have $\|M_{\alpha,r}(f)\|_{L^{q,\kappa q/p}} = \|M_{\alpha r,1}(|f|^r)\|_{L^{q/r,\kappa q/p}}^{1/r} \le C\|f\|_{L^{p,\kappa}}$, where α, p, q, and κ satisfy conditions in Lemma 2.3.

Lemma 2.4. *For all* $0 < \alpha < n$, *we have* $\|L^{-\alpha/2}(f)\|_{L^{q,\kappa q/p}} \le C\|f\|_{L^{p,\kappa}}$, *where* $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, *and* $0 < \kappa < p/q$.

Proof. Since the semigroup e^{-tL} has a kernel $p_t(x, y)$, which satisfies the upper bound (1.1), it is easy to check that $|L^{-\alpha/2}(f)(x)| \leq C\mathcal{I}_\alpha(|f|)(x)$, where $\mathcal{I}_\alpha$ is the classical fractional integral defined by

$$\mathcal{I}_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy, \quad 0 < \alpha < n.$$

Note that $\mathcal{I}_\alpha$ is bounded from $L^{p,\kappa}$ to $L^{q,\kappa q/p}$ for $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $0 < \kappa < p/q$, see [9, Theorem 3.6], [4] or [12]. Thus, we have $\|L^{-\alpha/2}(f)\|_{L^{q,\kappa q/p}} \leq C\|f\|_{L^{p,\kappa}}$, where $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $0 < \kappa < p/q$. This completes the proof of this lemma. □

Lemma 2.5. *Assume that the semigroup* e^{-tL} *has a kernel* $p_t(x, y)$, *which satisfies an upper bound* (1.1), *and let* $\vec{b} = \{b_1, \cdots, b_m\}$, b_i*'s are BMO functions. Then, for every function* $f \in L^p(\mathbb{R}^n)$, $p > 1$, $x \in \mathbb{R}^n$, *and* $1 < r < \infty$, *we have*

$$\sup_{x \in B} \frac{1}{|B|} \int_B \left| e^{-t_B L}\left(\prod_{i=1}^m (b_i - b_{iB})f\right)(y)\right| dy \leq C \prod_{i=1}^m \|b_i\|_* (M(|f|^r))^{\frac{1}{r}}(x),$$

where $t_B = r_B^2$.

Proof. For the proof of this lemma, see [6, Lemma 2.3] and [19, Lemma 2.3]. □

We now state the following lemma, which gives an estimate on the kernel of the difference operator $L^{-\alpha/2} - e^{-tL}L^{-\alpha/2}$. For its proof, see [7, Lemma 3.1].

Lemma 2.6. *Assume that the semigroup* e^{-tL} *has a kernel* $p_t(x, y)$, *which satisfies an upper bound* (1.1). *Then for* $0 < \alpha < 1$, *the difference operator* $L^{-\alpha/2} - e^{-tL}L^{-\alpha/2}$ *has an associated kernel* $K_{\alpha,t}(x, y)$, *which satisfies*

$$|K_{\alpha,t}(x,\ y)| \leq \frac{C}{|x-y|^{n-\alpha}}\frac{t}{|x-y|^{2}}, \tag{2.4}$$

for some constant $C > 0$.

Now, we have the following analogy of the classical Fefferman-Stein inequality [18, Chapter IV] for the sharp maximal function $M_L^{\sharp}f$. For the proof, see [10, Proposition 4.1].

Lemma 2.7. *Assume that the semigroup* e^{-tL} *has a kernel* $p_t(x,\ y)$, *which satisfies an upper bound* (1.1). *Take* $\lambda > 0$, $f \in L^1_{\text{loc}}$ *and a ball* B_0 *such that there exists* $x_0 \in B_0$ *with* $Mf(x_0) < \lambda$. *Then for every* $0 < \eta < 1$, *there exist* $r,\ \gamma > 0$ (*independent of* $\lambda,\ B_0,\ f,\ x_0$) *and constant* $C > 0$ *such that*

$$|\{x \in B_0 : Mf(x) > A\lambda, \quad M_L^{\sharp}f(x) \leq \gamma\lambda\}| \leq C\eta^{r}|B_0|,$$

where $A > 1$ *is a fixed constant, which depends only on n.*

In order to prove our main result, we need the following lemma:

Lemma 2.8. *Let* $0 < \kappa < 1$ *and* $1 < p < \infty$. *Then, for every* $f \in L^1_{\text{loc}}$ *with* $Mf \in L^{p,\kappa}$, *there exists a constant* $C > 0$ *such that*

$$\|Mf\|_{L^{p,\kappa}} \leq C\|M_L^{\sharp}f\|_{L^{p,\kappa}}. \tag{2.5}$$

Proof. Let B be a ball in $\mathbb{R}^n$. Set $E_\lambda = \{x \in B : Mf(x) > \lambda\}$. Then from Whitney decomposition theorem, we know that there exist mutually disjoint cubes Q_k such that $E_\lambda = \cup_k Q_k$ and $10Q_k \cap B \setminus E_\lambda \neq \emptyset$. Denote B_k be the ball with same center as Q_k and $r_B = \frac{1}{2}$ diameter Q_k. Let $\widetilde{B}_k = 10B_k$. Then there exists a $x_k \in \widetilde{B}_k \cap B \setminus E_\lambda$, that is, $Mf(x_k) \leq \lambda$. Let us use Lemma 2.7. There are $C,\ r > 0$ and $A > 1$ such that, if $0 < \eta < 1$ (to be chosen later), we can find $\gamma > 0$ in such a way that

$$|\{x \in \widetilde{B}_k : Mf(x) > A\lambda, \quad M_L^\sharp f(x) \le \gamma\lambda\}| \le C\eta^r |\widetilde{B}_k|.$$

Set $U_\lambda = \{x \in B : Mf(x) > A\lambda, M_L^\sharp f(x) \le \gamma\lambda\}$ and so $U_\lambda \subset E_\lambda = \bigcup_k Q_k$ $\subset \bigcup_k \widetilde{B}_k$ since $A > 1$. Then,

$$\begin{aligned}
|U_\lambda| &\le \sum_k |\{x \in \widetilde{B}_k : Mf(x) > A\lambda, \quad M_L^\sharp f(x) \le \gamma\lambda\}| \\
&\le C\eta^r \sum_k |\widetilde{B}_k| \\
&\le C\eta^r \sum_k |Q_k| = C\eta^r |E_\lambda| \\
&= C\eta^r |\{x \in B : Mf(x) > \lambda\}|.
\end{aligned}$$

One can prove that

$$\begin{aligned}
\int_B |Mf|^p \, dx &= A^p \int_0^\infty p\lambda^{p-1} |\{x \in B : Mf(x) > A\lambda\}| d\lambda \\
&\le A^p \int_0^\infty p\lambda^{p-1} \Big(|U_\lambda| + |\{x \in B : M_L^\sharp f(x) > \gamma\lambda\}| \Big) d\lambda \\
&\le CA^p \eta^r \int_B |Mf|^p \, dx + \frac{A^p}{\gamma^p} \int_B |M_L^\sharp f|^p \, dx.
\end{aligned}$$

Let us choose η such that $CA^p \eta^r = 1/2$. The former inequality turns out to be

$$\int_B |Mf|^p \, dx \le 2 \frac{A^p}{\gamma^p} \int_B |M_L^\sharp f|^p \, dx.$$

This implies that

$$\|Mf\|_{L^{p,\kappa}} \le C \|M_L^\sharp f\|_{L^{p,\kappa}}.$$

The proof of this lemma is completed. □

3. Proof of Theorem 1.1

We first prove Theorem 1.1 in the case $0 < \alpha < 1$. For convenience, we use the following notation. Given any positive integer m, for any $i \in \{1, \cdots, m\}$, we denote by C_i^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(i)\}$ of i different elements of $\{1, 2, \cdots, m\}$. For any $\sigma \in C_i^m$, we associate the complementary sequence $\sigma' = \{1, 2, \cdots, m\} \setminus \sigma$. For any $\sigma \in C_i^m$, we define

$$[\vec{b}_\sigma, L^{-\alpha/2}]f(x) = \int_{\mathbb{R}^n} \prod_{j=1}^{i} (b_{\sigma(j)}(x) - b_{\sigma(j)}(y)) K_\alpha(x, y) f(y) dy,$$

for each continuous function f with compact support, and for almost all x not in the support of f. In the case that $\sigma = \{1, 2, \cdots, m\}$, we denote $[\vec{b}_\sigma, L^{-\alpha/2}]$ simply by $[\vec{b}, L^{-\alpha/2}]$.

To prove Theorem 1.1 in the case $0 < \alpha < 1$, we only need to prove the following lemma:

Lemma 3.1. *Assume that the semigroup* e^{-tL} *has a kernel* $p_t(x, y)$, *which satisfies an upper bound* (1.1). *Let* $[\vec{b}, L^{-\alpha/2}]$ *be as in* (1.2) *and* $0 < \alpha < 1$. *Then for any real numbers* r *and* s *greater than* 1 *such that* $rs < n/\alpha$, *there exists a constant* $C > 0$, *such that*

$$M_L^\sharp([\vec{b}, L^{-\alpha/2}]f)(x) \le C\left(\prod_{j=1}^{m} \|b_j\|_*\right) M_{\alpha, rs}(f)(x)$$

$$+ C\sum_{i=1}^{m} \sum_{\sigma \in C_i^m} \left(\prod_{j \in \sigma} \|b_j\|_*\right) M_r([\vec{b}_{\sigma'}, L^{-\alpha/2}]f)(x), \quad (3.1)$$

for every $f \in L_c^\infty(\mathbb{R}^n)$ *and for every* $x \in \mathbb{R}^n$.

Proof. For $m = 1$, Lemma 3.1 was proved in [7]. So, we only need to prove the lemma for $m > 1$. To this end, we make use of induction on m. For any $\vec{\lambda} = (\lambda_1, \cdots, \lambda_m) \in \mathbb{R}^n$, we have

$$[\vec{b}, L^{-\alpha/2}]f(x) = \int_{\mathbb{R}^n} \prod_{i=1}^{m} (b_i(x) - b_i(y)) K_\alpha(x, y) f(y) dy$$

$$= \int_{\mathbb{R}^n} \prod_{i=1}^{m} ((b_i(x) - \lambda_i) - (b_i(y) - \lambda_i)) K_\alpha(x, y) f(y) dy$$

$$= \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} (-1)^{m-i} (\vec{b}(x) - \vec{\lambda})_\sigma \int_{\mathbb{R}^n} (\vec{b}(y) - \vec{\lambda})_{\sigma'} K_\alpha(x, y) f(y) dy$$

$$= \prod_{i=1}^{m} (b_i(x) - \lambda_i) L^{-\alpha/2} f(x) + (-1)^m L^{-\alpha/2} \left(\prod_{i=1}^{m} (b_i - \lambda_i) f \right)(x)$$

$$+ \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} (-1)^{m-i} (\vec{b}(x) - \vec{\lambda})_\sigma$$

$$\times \int_{\mathbb{R}^n} (\vec{b}(y) - \vec{\lambda})_{\sigma'} K_\alpha(x, y) f(y) dy,$$

where for any $\sigma \in C_i^m$, $(\vec{b} - \vec{\lambda})_\sigma = \prod_{j=1}^{i} (b_{\sigma(j)} - \lambda_{\sigma(j)})$. By expanding $(\vec{b}(y) - \vec{\lambda})_{\sigma'} = [(\vec{b}(y) - \vec{b}(x)) + (\vec{b}(x) - \vec{\lambda})]_{\sigma'}$, we obtain

$$[\vec{b}, L^{-\alpha/2}]f(x) = \prod_{i=1}^{m} (b_i(x) - \lambda_i) L^{-\alpha/2} f(x) + (-1)^m L^{-\alpha/2} \left(\prod_{i=1}^{m} (b_i - \lambda_i) f \right)(x)$$

$$+ \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} (-1)^{m-i} (\vec{b}(x) - \vec{\lambda})_\sigma \int_{\mathbb{R}^n} (\vec{b}(y) - \vec{\lambda})_{\sigma'} K_\alpha(x, y) f(y) dy$$

$$= \prod_{i=1}^{m}(b_i(x)-\lambda_i)L^{-\alpha/2}f(x) + (-1)^m L^{-\alpha/2}\left(\prod_{i=1}^{m}(b_i-\lambda_i)f\right)(x)$$

$$+ \sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m} C_{m,i}(\vec{b}(x)-\vec{\lambda})_\sigma[\vec{b}_{\sigma'}, L^{-\alpha/2}]f(x),$$

where $C_{m,i}$ are constants depending only on m and i.

For fixed $x \in \mathbb{R}^n$, B denotes a ball containing x center at x_0 with radius r_B, and $2B$ denotes the ball concentric with B and radius two times the radius of B. Split $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$. Then we can write that

$$[\vec{b}, L^{-\alpha/2}]f(y) = \prod_{i=1}^{m}(b_i(y)-\lambda_i)L^{-\alpha/2}f(y) + (-1)^m L^{-\alpha/2}\left(\prod_{i=1}^{m}(b_i-\lambda_i)f_1\right)(y)$$

$$+ (-1)^m L^{-\alpha/2}\left(\prod_{i=1}^{m}(b_i-\lambda_i)f_2\right)(y)$$

$$+ \sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m} C_{m,i}(\vec{b}(y)-\vec{\lambda})_\sigma[\vec{b}_{\sigma'}, L^{-\alpha/2}]f(y).$$

From this, it follows that

$$e^{-t_BL}([\vec{b}, L^{-\alpha/2}]f)(y) = e^{-t_BL}\left(\prod_{i=1}^{m}(b_i-\lambda_i)L^{-\alpha/2}f\right)(y)$$

$$+ (-1)^m e^{-t_BL}\left(L^{-\alpha/2}\left(\prod_{i=1}^{m}(b_i-\lambda_i)f_1\right)\right)(y)$$

$$+ (-1)^m e^{-t_BL}\left(L^{-\alpha/2}\left(\prod_{i=1}^{m}(b_i-\lambda_i)f_2\right)\right)(y)$$

$$+\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m} C_{m,i}e^{-t_BL}((\vec{b}-\vec{\lambda})_\sigma[\vec{b}_{\sigma'},\ L^{-\alpha/2}]f)(y).$$

Let $y\in B$. Now we estimate $|[\vec{b},\ L^{-\alpha/2}]f(y)-e^{-t_BL}([\vec{b},\ L^{-\alpha/2}]f)(y)|$ by

$$\begin{aligned}
&|[\vec{b},\ L^{-\alpha/2}]f(y)-e^{-t_BL}([\vec{b},\ L^{-\alpha/2}]f)(y)|\\
&\le\left|\prod_{i=1}^m(b_i(y)-\lambda_i)L^{-\alpha/2}f(y)\right|+\left|L^{-\alpha/2}\left(\prod_{i=1}^m(b_i-\lambda_i)f_1\right)(y)\right|\\
&+\left|e^{-t_BL}\left(\prod_{i=1}^m(b_i-\lambda_i)L^{-\alpha/2}f\right)(y)\right|+\left|e^{-t_BL}\left(L^{-\alpha/2}\left(\prod_{i=1}^m(b_i-\lambda_i)f_1\right)\right)(y)\right|\\
&+\left|L^{-\alpha/2}\left(\prod_{i=1}^m(b_i-\lambda_i)f_2\right)(y)-e^{-t_BL}\left(L^{-\alpha/2}\left(\prod_{i=1}^m(b_i-\lambda_i)f_2\right)\right)(y)\right|\\
&+\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}C\left|(\vec{b}(y)-\vec{\lambda})_\sigma[\vec{b}_{\sigma'},\ L^{-\alpha/2}]f(y)\right|\\
&+\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}C\left|e^{-t_BL}((\vec{b}-\vec{\lambda})_\sigma[\vec{b}_{\sigma'},\ L^{-\alpha/2}]f)(y)\right|\\
&=: F_1(y)+F_2(y)+F_3(y)+F_4(y)+F_5(y)+F_6(y)+F_7(y),
\end{aligned}$$

which yields

$$\begin{aligned}
&\frac{1}{|B|}\int_B\left|[\vec{b},\ L^{-\alpha/2}]f(y)-e^{-t_BL}([\vec{b},\ L^{-\alpha/2}]f)(y)\right|dy\\
&\qquad\le\sum_{j=1}^{7}\frac{1}{|B|}\int_B F_j(y)dy\\
&\qquad=:\sum_{j=1}^{7}I_j(x). \qquad (3.2)
\end{aligned}$$

Let r' be the dual of r such that $1/r + 1/r' = 1$. We first estimate I_1. By the Hölder inequality and Lemma 2.1,

$$I_1(x) = \frac{1}{|B|}\int_B \left|\prod_{i=1}^{m}(b_i(y)-\lambda_i)L^{-\alpha/2}f(y)\right|dy$$

$$\leq \left[\frac{1}{|B|}\int_B \prod_{i=1}^{m}|b_i(y)-\lambda_i|^{r'}dy\right]^{1/r'}\left[\frac{1}{|B|}\int_B |L^{-\alpha/2}f(y)|^{r}dy\right]^{1/r}$$

$$\leq C\prod_{i=1}^{m}\|b_i\|_* M_r(L^{-\alpha/2}f)(x),$$

where $\lambda_i = (b_i)_B$, $i = 1, \cdots, m$. For the term I_2, by Lemmas 2.1 and 2.4 again, we have

$$I_2(x) \leq \left[\frac{1}{|B|}\int_B \left|L^{-\alpha/2}\left(\prod_{i=1}^{m}(b_i-\lambda_i)f_1\right)(y)\right|^{\omega}dy\right]^{1/\omega}$$

$$\leq C\frac{1}{|B|^{\frac{1}{\omega}}}\left[\int_B \left|\left(\prod_{i=1}^{m}(b_i(y)-\lambda_i)f(y)\right)\right|^{s}dy\right]^{1/s}$$

$$\leq C\left[\frac{1}{|B|}\int_B \prod_{i=1}^{m}|b_i(y)-\lambda_i|^{sr'}dy\right]^{1/sr'}\left[\frac{1}{|B|^{1-\frac{\alpha sr}{n}}}\int_B |f(y)|^{sr}dy\right]^{1/sr}$$

$$\leq C\prod_{i=1}^{m}\|b_i\|_* M_{\alpha, rs}(f)(x),$$

where $\frac{1}{\omega} = \frac{1}{s} - \frac{\alpha}{n}$.

Similarly, we obtain by using Lemmas 2.1, 2.4, and 2.5,

$$I_3(x) + I_4(x) \leq C\prod_{i=1}^{m}\|b_i\|_*[M_r(L^{-\alpha/2}f)(x) + M_{\alpha, rs}(f)(x)].$$

Now we turn to estimate the term $I_5(x)$. Using Lemmas 2.1 and 2.6, we have

$$I_5(x) \leq \frac{1}{|B|}\int_B\int_{(2B)^c}|K_{\alpha,\, t_B}(y,\, z)|\prod_{i=1}^{m}|b_i(z)-\lambda_i||f(z)|dzdy$$

$$\leq C\sum_{k=1}^{\infty}\int_{2^k r_B\leq|x_0-z|<2^{k+1}r_B}\frac{1}{|x_0-z|^{n-\alpha}}\frac{r_B}{|x_0-z|}$$

$$\times\prod_{i=1}^{m}|b_i(z)-\lambda_i||f(z)|dz$$

$$\leq C\sum_{k=1}^{\infty}2^{-k}\frac{1}{|2^{k+1}B|^{1-\frac{\alpha}{n}}}\int_{2^{k+1}B}\prod_{i=1}^{m}|b_i(z)-b_{i,2^{k+1}B}$$

$$+\, b_{i,2^{k+1}B}-\lambda_i||f(z)|dz$$

$$\leq C\sum_{k=1}^{\infty}2^{-k}\frac{1}{|2^{k+1}B|^{1-\frac{\alpha}{n}}}\int_{2^{k+1}B}\sum_{i=0}^{m}$$

$$\times\sum_{\sigma\in C_i^m}|(\vec{b}(z)-\vec{b}_{2^{k+1}B})_{\sigma}(\vec{b}_{2^{k+1}B}-\vec{\lambda})_{\sigma'}f(z)|dz$$

$$\leq C\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\sum_{k=1}^{\infty}2^{-k}\frac{1}{|2^{k+1}B|^{1-\frac{\alpha}{n}}}|(\vec{b}_{2^{k+1}B}-\vec{\lambda})_{\sigma'}|$$

$$\times\int_{2^{k+1}B}|(\vec{b}(z)-\vec{b}_{2^{k+1}B})_{\sigma}||f(z)|dz$$

$$\leq C\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\sum_{k=1}^{\infty}2^{-k}(k+1)^{m-i}\prod_{j\in\sigma'}\|b_j\|_*\frac{1}{|2^{k+1}B|^{1-\frac{\alpha}{n}}}$$

$$\times\int_{2^{k+1}B}|(\vec{b}(z)-\vec{b}_{2^{k+1}B})_{\sigma}||f(z)|dz$$

$$\leq C\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\sum_{k=1}^{\infty}2^{-k}(k+1)^{m-i}\prod_{j=1}^{m}\|b_j\|_* M_{\alpha,1}f(x)$$

$$\leq C\prod_{i=1}^{m}\|b_i\|_* M_{\alpha, rs}(f)(x),$$

where $b_{i, 2^{k+1}B} = \frac{1}{|2^{k+1}B|}\int_{2^{k+1}B} b_i(z)dz.$ Finally, by an argument similar to above, we can obtain

$$I_6(x) + I_7(x) \leq C\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}\prod_{j\in\sigma}\|b_j\|_* M_r([\vec{b}_{\sigma'}, L^{-\alpha/2}]f)(x).$$

Combining the estimates for $I_1(x)$ to $I_7(x)$ with (3.2) and then taking supremum over all balls containing x in (3.2) gives us (3.1), which completes the proof of Lemma 3.1. □

Proof of Theorem 1.1. In the case $0 < \alpha < 1$, we can deduce Theorem 1.1 from (3.1) by induction on m. For $m = 1$, from (3.1), we know that there exist two real numbers r and s greater than 1 satisfying $rs < p$ such that

$$M_L^{\sharp}([b, L^{-\alpha/2}]f)(x) \leq C\|b\|_* M_{\alpha, rs}(f)(x) + C\|b\|_* M_r(L^{-\alpha/2}f)(x).$$

This combining with Lemmas 2.2, 2.4, and 2.8 gives

$$\begin{aligned}\|[b, L^{-\alpha/2}]f\|_{L^{q,\kappa q/p}} &\leq C\|M_L^{\sharp}[b, L^{-\alpha/2}]f\|_{L^{q,\kappa q/p}}\\ &\leq C\|b\|_*\|M_{\alpha, rs}(f)\|_{L^{q,\kappa q/p}} + C\|b\|_*\|M_r(L^{-\alpha/2}f)\|_{L^{q,\kappa q/p}}\\ &\leq C\|b\|_*\|f\|_{L^{p,\kappa}}.\end{aligned}$$

For $m > 1$, choose two real numbers r and s greater than 1 such that $rs < p < n/\alpha.$ From (3.1), Lemmas 2.2, 2.3, and 2.8, we have

$$\|[\vec{b}, L^{-\alpha/2}]f\|_{L^{q,\kappa q/p}} \leq C\|M_L^{\sharp}[\vec{b}, L^{-\alpha/2}]f\|_{L^{q,\kappa q/p}}$$

$$\leq C\prod_{i=1}^{m}\|b_i\|_*\|M_{\alpha,rs}(f)\|_{L^{q,\kappa q/p}}$$

$$+ C\sum_{i=1}^{m}\sum_{\sigma\in C_i^m}\prod_{j\in\sigma}\|b_j\|_*\|M_r([\vec{b}_{\sigma'}, L^{-\alpha/2}]f)\|_{L^{q,\kappa q/p}}$$

$$\leq C\prod_{i=1}^{m}\|b_i\|_*\|f\|_{L^{p,\kappa}},$$

where $1/q = 1/p - \alpha/n$, $1 < p < n/\alpha$, and $f \in L^{p,\kappa}(\mathbb{R}^n)$.

Now we turn to prove Theorem 1.1 in the general case $0 < \alpha < n$. For any $j = 0, 1, \cdots, n-1$, we denote $p_{1,j}$, $p_{2,j}$, $p_{3,j}$ by

$$\frac{1}{p_{1,j}} = \frac{1}{q} + \frac{\alpha j}{n^2}, \qquad \frac{1}{p_{2,j}} = \frac{1}{p_{1,j}} + \frac{\alpha}{n^2},$$

and

$$\frac{1}{p_{3,j}} = \frac{1}{p_{2,j}} + \frac{\alpha(n-1-j)}{n^2}.$$

Note that

$$[\vec{b}, L^{-\alpha/2}]f = [\vec{b}, (L^{-\alpha/2n})^n]f = \sum_{j=0}^{n-1} L^{-\alpha j/2n}[\vec{b}, L^{-\alpha/2n}]L^{-\alpha(n-1-j)/2n}f.$$

Then, using Lemma 2.4 and Theorem 1.1 in the case $0 < \alpha < 1$, we have

$$\|[\vec{b}, L^{-\alpha/2}]f\|_{L^{q,\kappa q/p}} \leq \sum_{j=0}^{n-1}\|L^{-\alpha j/2n}[\vec{b}, L^{-\alpha/2n}]L^{-\alpha(n-1-j)/2n}f\|_{L^{q,\kappa q/p}}$$

$$\leq C\sum_{j=0}^{n-1}\|[\vec{b}, L^{-\alpha/2n}]L^{-\alpha(n-1-j)/2n}f\|_{L^{p_{1,j},\kappa p_{1,j}/p}}$$

$$\leq C\prod_{i=1}^{m}\|b_i\|_*\sum_{j=0}^{n-1}\|L^{-\alpha(n-1-j)/2n}f\|_{L^{p_{2,j},\kappa p_{2,j}/p}}$$

$$\leq C\prod_{i=1}^{m}\|b_i\|_*\sum_{j=0}^{n-1}\|f\|_{L^{p_{3,j},\kappa p_{3,j}/p}}$$

$$\leq C\prod_{i=1}^{m}\|b_i\|_*\|f\|_{L^{p,\kappa}},$$

since for any $j = 0, 1, \cdots, n-1$, $p_{3,j} = p$ follows from

$$1/p_{3,j} = 1/q + \alpha j/n^2 + \alpha/n^2 + \alpha(n-1-j)/n^2 = 1/q + \alpha/n = 1/p.$$

Hence, the proof of Theorem 1.1 is completed. □

4. Applications

As in Theorem 1.1, the heat kernel upper bound (1.1) implies boundedness of the commutator $[\vec{b}, L^{-\alpha/2}]$. This property (1.1) is satisfied by large classes of differential operators (see [7]). We will list some of them:

(a) The operator A is called the *magnetic Schrödinger operator*, which is given by

$$A = -(\nabla - i\vec{a})^2 + V(x),$$

where $\vec{a} = (a_1, a_2, \cdots, a_n)$, $a_k \in L^2_{\text{loc}}$, and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$. The semigroup e^{-tA} has a kernel $p_t(x, y)$, which satisfies the upper bound (1.1) (see [15] and [16]).

(b) Let $A = (a_{ij}(x))_{1\leq i,j\leq n}$ be an $n \times n$ matrix of complex with entries $a_{ij} \in L^\infty(\mathbb{R}^n)$ satisfying $\text{Re}\sum a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2$ for all $x \in \mathbb{R}^n$, $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{C}^n$ and some $\lambda > 0$. We define divergence form operator

$$Lf \equiv -\operatorname{div}(A\nabla f),$$

which we interpret in the usual weak sense via a sesquilinear form.

It is known that the Gaussian bound (1.1) on the heat kernel e^{-tL} is true when A has real entries, or when $n = 1, 2$ in the case of complex entries, see [2, Chapter 1].

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