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GENERALIZED UNCERTAINTY RELATION ASSOCIATED WITH A MONOTONE OR AN ANTI-MONOTONE PAIR SKEW INFORMATION

KENJIRO YANAGI and SATOSHI KAJIHARA

Division of Applied Mathematical Science
Graduate School of Science and Engineering
Yamaguchi University
Ube, 755-8611
Japan
e-mail: yanagi@yamaguchi-u.ac.jp

Graduate School of Science and Engineering
Yamaguchi University
Yamaguchi, 753-8511
Japan
e-mail: p002va@yamaguchi-u.ac.jp

Abstract

We give a trace inequality related to the uncertainty relation based on the monotone or anti-monotone pair skew information, which is one of the generalizations of result given by [5]. The present paper includes the result for generalized Wigner-Yanase-Dyson skew information as a particular case [14].

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1. Introduction

Wigner-Yanase skew information

$$I_\rho(H) = \frac{1}{2} Tr\,[(i[\rho^{1/2}, H])^2] = Tr[\rho H^2] - Tr[\rho^{1/2} H \rho^{1/2} H],$$

was defined in [10]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state ρ and an observable H. Here we denote the commutator by $[X, Y] = XY - YX$. This quantity was generalized by Dyson

$$I_{\rho,\alpha}(H) = \frac{1}{2} Tr[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])] = Tr[\rho H^2] - Tr[\rho^\alpha H \rho^{1-\alpha} H], \quad \alpha \in [0, 1],$$

which is known as the Wigner-Yanase-Dyson skew information. It is famous that the convexity of $I_{\rho,\alpha}(H)$ with respect to ρ was successfully proven by Lieb in [7]. And also this quantity was generalized by Cai and Luo

$$\begin{aligned} I_{\rho,\alpha,\beta}(H) &= \frac{1}{2} Tr[(i[\rho^\alpha, H])(i[\rho^\beta, H])\rho^{1-\alpha-\beta}] \\ &= \frac{1}{2}\{Tr[\rho H^2] + Tr[\rho^{\alpha+\beta} H \rho^{1-\alpha-\beta} H] \\ &\quad - Tr[\rho^\alpha H \rho^{1-\alpha} H] - Tr[\rho^\beta H \rho^{1-\beta} H]\}, \end{aligned}$$

where $\alpha, \beta \geq 0$ and $\alpha + \beta \leq 1$. The convexity of $I_{\rho,\alpha,\beta}(H)$ with respect to ρ was proven by Cai and Luo in [2] under some restrictive condition. In this paper, we let $M_n(\mathbb{C})$ be the set of all $n \times n$ complex matrices, $M_{n,sa}(\mathbb{C})$ be the set of all $n \times n$ self-adjoint matrices, $M_{n,+}(\mathbb{C})$ be the set of strictly positive elements of $M_n(\mathbb{C})$, and $M_{n,+,1}(\mathbb{C})$ be the set of strictly positive density matrices, that is, $M_{n,+,1}(\mathbb{C}) = \{\rho \in M_n(\mathbb{C}) | Tr[\rho] = 1, \rho > 0\}$. If it is not otherwise specified, from now on, we shall treat the case of faithful states, that is, $\rho > 0$. The relation between the Wigner-Yanase skew information and the uncertainty relation was studied in [9]. Moreover, the relation between the Wigner-Yanase-Dyson skew information and the uncertainty relation was studied in [6, 11]. In our paper [11] and [13], we defined a generalized skew information and then derived a kind of an uncertainty relations. And also in [14, 15], we gave an uncertainty relation of two parameter generalized Wigner-Yanase-Dyson skew information. In this paper, we consider three parameter generalized Wigner-Yanase-Dyson skew information and give a kind of generalized uncertainty relations, which is a generalization of the result of Ko and Yoo [5].

2. Trace Inequality of Wigner-Yanase-Dyson Skew Information

We review the relation between the Wigner-Yanase skew information and the uncertainty relation. In quantum mechanical system, the expectation value of an observable H in a quantum state ρ is expressed by $Tr[\rho H]$. It is natural that the variance for a quantum state ρ and an observable H is defined by $V_\rho(H) = Tr[\rho(H - Tr[\rho H]I)^2] = Tr[\rho H^2] - Tr[\rho H]^2$. It is famous that, we have

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2, \tag{2.1}$$

for a quantum state ρ and two observables A and B. The further strong results was given by Schrödinger

$$V_\rho(A)V_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A, B)\}|^2 \geq \frac{1}{4}|Tr[\rho[A, B]]|^2,$$

where the covariance is defined by $\mathrm{Cov}_\rho(A, B) = Tr[\rho(A - Tr[\rho A]I)(B - Tr[\rho B]I)]$. However, the uncertainty relation for the Wigner-Yanase skew information failed (see [9, 6, 11]).

$$I_\rho(A)I_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2.$$

Recently, Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture

$$U_\rho(H) = \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_\rho(H))^2}, \tag{2.2}$$

then he derived the uncertainty relation on $U_\rho(H)$ in [8]

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2. \tag{2.3}$$

Note that we have the following relation:

$$0 \leq I_\rho(H) \leq U_\rho(H) \leq V_\rho(H). \tag{2.4}$$

The inequality (2.3) is a refinement of the inequality (2.1) in the sense of (2.4). In [13], we studied one-parameter extended inequality for the inequality (2.3).

Definition 2.1. For $0 \leq \alpha \leq 1$, a quantum state ρ and an observable H, we define the Wigner-Yanase-Dyson skew information

$$I_{\rho,\alpha}(H) = \frac{1}{2}Tr[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])]$$

$$= Tr[\rho H_0^2] - Tr[\rho^\alpha H_0 \rho^{1-\alpha} H_0],$$

and we also define

$$J_{\rho,\alpha}(H) = \frac{1}{2}Tr[\{\rho^\alpha, H_0\}\{\rho^{1-\alpha}, H_0\}]$$

$$= Tr[\rho H_0^2] + Tr[\rho^\alpha H_0 \rho^{1-\alpha} H_0],$$

where $H_0 = H - Tr[\rho H]I$ and we denote the anti-commutator by $\{X, Y\} = XY + YX$.

Note that we have

$$\frac{1}{2} Tr[(i[\rho^{\alpha}, H_0])(i[\rho^{1-\alpha}, H_0])] = \frac{1}{2} Tr[(i[\rho^{\alpha}, H])(i[\rho^{1-\alpha}, H])],$$

but we have

$$\frac{1}{2} Tr[\{\rho^{\alpha}, H_0\}\{\rho^{1-\alpha}, H_0\}] \neq \frac{1}{2} Tr[\{\rho^{\alpha}, H\}\{\rho^{1-\alpha}, H\}].$$

Then, we have the following inequalities:

$$I_{\rho,\alpha}(H) \leq I_{\rho}(H) \leq J_{\rho}(H) \leq J_{\rho,\alpha}(H), \tag{2.5}$$

since we have $Tr[\rho^{1/2}H\rho^{1/2}H] \leq Tr[\rho^{\alpha}H\rho^{1-\alpha}H]$ (see [1, 3], for example). If we define

$$U_{\rho,\alpha}(H) = \sqrt{V_{\rho}(H)^2 - (V_{\rho}(H) - I_{\rho,\alpha}(H))^2}, \tag{2.6}$$

as a direct generalization of Equation (2.2), then we have

$$0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_{\rho}(H), \tag{2.7}$$

due to the first inequality of (2.5). We also have

$$U_{\rho,\alpha}(H) = \sqrt{I_{\rho,\alpha}(H) J_{\rho,\alpha}(H)}.$$

From the inequalities (2.4), (2.6), and (2.7), our situation is that, we have

$$0 \leq I_{\rho,\alpha}(H) \leq I_{\rho}(H) \leq U_{\rho}(H),$$

and

$$0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_{\rho}(H).$$

We gave the following uncertainty relation with respect to $U_{\rho,\alpha}(H)$ as a direct generalization of the inequality (2.3).

Theorem 2.1 ([13]). *For* $0 \leq \alpha \leq 1$, *a quantum state* ρ *and observables* A, B,

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \alpha(1-\alpha)|Tr[\rho[A, B]]|^2. \tag{2.8}$$

Now, we define the two parameter extensions of Wigner-Yanase skew information and give an uncertainty relation under some conditions.

Definition 2.2. For $\alpha, \beta \geq 0$, a quantum state ρ and an observable H, we define the generalized Wigner-Yanase-Dyson skew information

$$I_{\rho,\alpha,\beta}(H) = \frac{1}{2}Tr[(i[\rho^{\alpha}, H_0])(i[\rho^{\beta}, H_0])\rho^{1-\alpha-\beta}]$$

$$= \frac{1}{2}\{Tr[\rho H_0^2] + Tr[\rho^{\alpha+\beta}H_0\rho^{1-\alpha-\beta}H_0]$$

$$- Tr[\rho^{\alpha}H_0\rho^{1-\alpha}H_0] - Tr[\rho^{\beta}H_0\rho^{1-\beta}H_0]\},$$

and we define

$$J_{\rho,\alpha,\beta}(H) = \frac{1}{2}Tr[\{\rho^{\alpha}, H_0\}\{\rho^{\beta}, H_0\}\rho^{1-\alpha-\beta}]$$

$$= \frac{1}{2}\{Tr[\rho H_0^2] + Tr[\rho^{\alpha+\beta}H_0\rho^{1-\alpha-\beta}H_0]$$

$$+ Tr[\rho^{\alpha}H_0\rho^{1-\alpha}H_0] + Tr[\rho^{\beta}H_0\rho^{1-\beta}H_0]\},$$

where $H_0 = H - Tr[\rho H]I$ and we denote the anti-commutator by $\{X, Y\} = XY + YX$. We remark that $\alpha + \beta = 1$ implies $I_{\rho,\alpha}(H) = I_{\rho,\alpha,1-\alpha}(H)$ and $J_{\rho,\alpha}(H) = J_{\rho,\alpha,1-\alpha}(H)$. We also define

$$U_{\rho,\alpha,\beta}(H) = \sqrt{I_{\rho,\alpha,\beta}(H)J_{\rho,\alpha,\beta}(H)}.$$

In this paper, we assume that $\alpha, \beta \geq 0$ do not necessarily satisfy the condition $\alpha + \beta \leq 1$. We give the following theorem:

Theorem 2.2 ([14]). *For* $\alpha, \beta \geq 0$ *and* $\alpha + \beta \geq 1$ *or* $\alpha + \beta \leq \frac{1}{2}$ *and observables* A, B,

$$U_{\rho,\alpha,\beta}(A)U_{\rho,\alpha,\beta}(B) \geq \alpha\beta|Tr[\rho[A, B]]|^2. \tag{2.9}$$

And we also define the two parameter extensions of Wigner-Yanase skew information, which are different from Definition 2.2.

Definition 2.3. For $\alpha, \beta \geq 0$, a quantum state ρ and an observable H, we define the generalized Wigner-Yanase-Dyson skew information

$$\widetilde{I}_{\rho,\alpha,\beta}(H) = \frac{1}{2}Tr[(i[\rho^{\alpha}, H_0])(i[\rho^{\beta}, H_0])]$$

$$= Tr[\rho^{\alpha+\beta}H_0^2] - Tr[\rho^{\alpha}H_0\rho^{\beta}H_0],$$

and we define

$$\widetilde{J}_{\rho,\alpha,\beta}(H) = \frac{1}{2}Tr[\{\rho^{\alpha}, H_0\}\{\rho^{\beta}, H_0\}]$$

$$= Tr[\rho^{\alpha+\beta}H_0^2] + Tr[\rho^{\alpha}H_0\rho^{\beta}H_0],$$

where $H_0 = H - Tr[\rho H]I$ and we denote the anti-commutator by $\{X, Y\} = XY + YX$. We remark that $\alpha + \beta = 1$ implies $I_{\rho,\alpha}(H) = \widetilde{I}_{\rho,\alpha,1-\alpha}(H)$ and $J_{\rho,\alpha}(H) = \widetilde{J}_{\rho,\alpha,1-\alpha}(H)$. We also define

$$\widetilde{U}_{\rho,\alpha,\beta}(H) = \sqrt{\widetilde{I}_{\rho,\alpha,\beta}(H)\widetilde{J}_{\rho,\alpha,\beta}(H)}.$$

Then, we give the following theorem:

Theorem 2.3 ([15]). *For* $\alpha, \beta \geq 0\,(\alpha\beta \neq 0)$ *and observables* A, B,

$$\widetilde{U}_{\rho,\alpha,\beta}(A)\widetilde{U}_{\rho,\alpha,\beta}(B) \geq \frac{\alpha\beta}{(\alpha+\beta)^2}|Tr[\rho^{\alpha+\beta}[A, B]]|^2.$$

Remark 2.1. We remark that (2.8) is derived by putting $\beta = 1 - \alpha$ in (2.9). Then, Theorem 2.2 is a generalization of Theorem 2.1 given in [13].

3. Trace Inequality of Monotone or Anti-Monotone Pair Skew Information

Definition 3.1. Let $f(x)$, $g(x)$ be nonnegative continuous functions defined on the interval $[0, 1]$. We call the pair (f, g) a compatible in log-increase, monotone pair (CLI monotone pair, in short), if

(a) $(f(x) - f(y))(g(x) - g(y)) \geq 0$ for all $x, y \in [0, 1]$.

(b) $f(x)$ and $g(x)$ are differentiable on $(0, 1)$ and

$$0 \leq \inf_{0<x<1} \frac{G'(x)}{F'(x)} \leq \sup_{0<x<1} \frac{G'(x)}{F'(x)} < \infty,$$

where $F(x) = \log f(x)$ and $G(x) = \log g(x)$.

Definition 3.2. Let $f(x)$, $g(x)$ be nonnegative continuous functions defined on the interval $[0, 1]$. We call the pair (f, g) a compatible in log-increase, anti-monotone pair (CLI anti-monotone pair, in short), if

(a) $(f(x) - f(y))(g(x) - g(y)) \leq 0$ for all $x, y \in [0, 1]$.

(b) $f(x)$ and $g(x)$ are differentiable on $(0, 1)$ and

$$-\infty < \inf_{0<x<1} \frac{G'(x)}{F'(x)} \leq \sup_{0<x<1} \frac{G'(x)}{F'(x)} \leq 0,$$

where $F(x) = \log f(x)$ and $G(x) = \log g(x)$.

Let $f(x)$, $g(x)$, $h(x)$ be nonnegative continuous functions defined on $[0, 1]$ and be differentiable on $(0, 1)$. We assume that (f, g) is CLI monotone pair and (f, h) is CLI monotone or anti-monotone pair. We introduce the correlation functions in the following way:

Definition 3.3.

$$
\begin{aligned}
I_{\rho,(f,g,h)}(H) &= \frac{1}{2} Tr[(i[f(\rho), H_0])(i[g(\rho), H_0])h(\rho)] \\
&= -\frac{1}{2} Tr[(f(\rho), H_0])([g(\rho), H_0])h(\rho)] \\
&= -\frac{1}{2} Tr[(f(\rho)H_0 - H_0 f(\rho))(g(\rho)H_0 - H_0 g(\rho))h(\rho)] \\
&= -\frac{1}{2} Tr[f(\rho)H_0 g(\rho)H_0 h(\rho) - f(\rho)H_0^2 g(\rho)h(\rho)] \\
&\quad + \frac{1}{2} Tr[H_0 f(\rho)g(\rho)H_0 h(\rho) - H_0 f(\rho)H_0 g(\rho)h(\rho)] \\
&= -\frac{1}{2} Tr[f(\rho)h(\rho)H_0 g(\rho)H_0 - f(\rho)g(\rho)h(\rho)H_0^2] \\
&\quad + \frac{1}{2} Tr[f(\rho)g(\rho)H_0 h(\rho)H_0 - g(\rho)h(\rho)H_0 f(\rho)H_0] \\
&= \frac{1}{2}\{Tr[f(\rho)g(\rho)h(\rho)H_0^2] + Tr[f(\rho)g(\rho)H_0 h(\rho)H_0]\} \\
&\quad - \frac{1}{2}\{Tr[f(\rho)H_0 g(\rho)h(\rho)H_0] + Tr[g(\rho)H_0 f(\rho)h(\rho)H_0]\};
\end{aligned}
$$

$$
\begin{aligned}
J_{\rho,(f,g,h)}(H) &= \frac{1}{2} Tr[\{f(\rho), H_0\}\{g(\rho), H_0\}h(\rho)] \\
&= \frac{1}{2} Tr[(f(\rho)H_0 + H_0 f(\rho))(g(\rho)H_0 + H_0 g(\rho))h(\rho)] \\
&= \frac{1}{2} Tr[f(\rho)H_0 g(\rho)H_0 h(\rho) + f(\rho)H_0^2 g(\rho)h(\rho)] \\
&\quad + \frac{1}{2} Tr[H_0 f(\rho)g(\rho)H_0 h(\rho) + H_0 f(\rho)H_0 g(\rho)h(\rho)] \\
&= \frac{1}{2}\{Tr[f(\rho)g(\rho)h(\rho)H_0^2] + Tr[f(\rho)g(\rho)H_0 h(\rho)H_0]\} \\
&\quad + \frac{1}{2}\{Tr[f(\rho)H_0 g(\rho)h(\rho)H_0] + Tr[g(\rho)H_0 f(\rho)h(\rho)H_0]\};
\end{aligned}
$$

and

$$U_{\rho,(f,g,h)}(H) = \sqrt{I_{\rho,(f,g,h)}(H) J_{\rho,(f,g,h)}(H)}.$$

We are ready to state our main result. For $f,\ g,\ h,$ we let

$$\beta(f,\ g,\ h) = \min\left\{\frac{m}{(1+m+n)^2},\ \frac{m}{(1+m+N)^2},\ \frac{M}{(1+M+n)^2},\ \frac{M}{(1+M+N)^2}\right\}, \tag{3.1}$$

where

$$m = \inf_{0<x<1} \frac{G'(x)}{F'(x)}, \quad M = \sup_{0<x<1} \frac{G'(x)}{F'(x)},$$

$$n = \inf_{0<x<1} \frac{H'(x)}{F'(x)}, \quad N = \sup_{0<x<1} \frac{H'(x)}{F'(x)}.$$

We consider the following two assumptions:

(I) $(f,\ g)$ and $(f,\ h)$ are CLI monotone pair satisfying

$$1 + \frac{G(y) - G(x)}{F(y) - F(x)} \le \frac{H(y) - H(x)}{F(y) - F(x)} \quad \text{for } x < y,$$

where $F(x) = \log f(x)$, $G(x) = \log g(x)$, and $H(x) = \log h(x)$.

(II) $(f,\ g)$ is CLI monotone pair and $(f,\ h)$ is CLI anti-monotone pair satisfying

$$1 + \frac{G(y) - G(x)}{F(y) - F(x)} + \frac{H(y) - H(x)}{F(y) - F(x)} \ge 0 \quad \text{for } x < y.$$

Theorem 3.1. *Under the Assumption* (I) *or* (II), *the following inequality holds*:

$$U_{\rho,(f,g,h)}(A) U_{\rho,(f,g,h)}(B) \ge \beta(f,\ g,\ h) |Tr[f(\rho) g(\rho) h(\rho) [A,\ B]]|^2,$$

for $A,\ B \in M_{n,sa}(\mathbb{C})$.

4. Proof of Theorem 3.1

Let $\rho = \sum_{i=1}^{n} \lambda_i |\phi_i\rangle\langle\phi_i| \in M_{n,+,1}(\mathbb{C})$, where $\{|\phi_i\rangle\}_{i=1}^{n}$ is an orthonormal set in $\mathbb{C}^n$. Let (f, g) be a CLI monotone pair and (f, h) be a CLI monotone or anti-monotone pair. By a simple calculation, we have for any $H \in M_{n,sa}(\mathbb{C})$

$$Tr[f(\rho)g(\rho)h(\rho)H_0^2] = \sum_{i,j} \frac{1}{2}\{f(\lambda_i)g(\lambda_i)h(\lambda_i) + f(\lambda_j)g(\lambda_j)h(\lambda_j)\}|a_{ij}|^2; \tag{4.1}$$

$$Tr[f(\rho)g(\rho)H_0 h(\rho)H_0] = \sum_{i,j} \frac{1}{2}\{f(\lambda_i)g(\lambda_i)h(\lambda_j) + f(\lambda_j)g(\lambda_j)h(\lambda_i)\}|a_{ij}|^2; \tag{4.2}$$

$$Tr[f(\rho)H_0 g(\rho)h(\rho)H_0] = \sum_{i,j} \frac{1}{2}\{f(\lambda_i)g(\lambda_j)h(\lambda_j) + f(\lambda_j)g(\lambda_i)h(\lambda_i)\}|a_{ij}|^2; \tag{4.3}$$

$$Tr[g(\rho)H_0 f(\rho)h(\rho)H_0] = \sum_{i,j} \frac{1}{2}\{g(\lambda_i)f(\lambda_j)h(\lambda_j) + g(\lambda_j)f(\lambda_i)h(\lambda_i)\}|a_{ij}|^2, \tag{4.4}$$

where $a_{ij} = \langle\phi_i|H_0|\phi_j\rangle$ and $a_{ij} = \overline{a_{ji}}$. From (4.1)-(4.4), we get

$$I_{\rho,(f,g,h)}(H) = \frac{1}{2}\sum_{i<j}(f(\lambda_i) - f(\lambda_j))(g(\lambda_i) - g(\lambda_j))(h(\lambda_i) + h(\lambda_j))|a_{ij}|^2;$$

$$J_{\rho,(f,g,h)}(H) \geq \frac{1}{2}\sum_{i<j}(f(\lambda_i) + f(\lambda_j))(g(\lambda_i) + g(\lambda_j))(h(\lambda_i) + h(\lambda_j))|a_{ij}|^2.$$

To prove Theorem 3.1, we need to control a lower bound of a functional coming from a CLI monotone or anti-monotone pair. For f, g, h satisfying Assumption (I) or (II), we define a function L on $[0, 1] \times [0, 1]$ by

$$L(x,\ y) = \frac{(f(x)^2 - f(y)^2)(g(x)^2 - g(y)^2)(h(x) + h(y))^2}{(f(x)g(x)h(x) - f(y)g(y)h(y))^2}. \tag{4.5}$$

Proposition 4.1. *Under the Assumption* (I) *or* (II),

$$\min_{x,\ y \in [0,1]} L(x,\ y) \geq 16\beta(f,\ g,\ h),$$

where $\beta(f,\ g,\ h)$ *is defined in* (3.1).

For the proof of Proposition 4.1, we need the following lemma:

Lemma 4.1. *If* $a, b, c \geq 0$ *satisfy* $0 < a + b \leq c$ *or if* $a, b \geq 0, c \leq 0$ *satisfy* $a + b + c > 0$, *then the inequality*

$$\frac{(e^{2ar} - 1)(e^{2br} - 1)(e^{cr} + 1)^2}{(e^{(a+b+c)r} - 1)^2} \geq \frac{16ab}{(a + b + c)^2},$$

holds for any real number r.

Proof. We put $e^r = t$. Then, we may prove the following:

$$(t^{2a} - 1)(t^{2b} - 1)(t^c + 1)^2 \geq \frac{16ab}{(a + b + c)^2}(t^{(a+b+c)} - 1)^2, \tag{4.6}$$

for $t > 0$. It is sufficient to prove (4.6) for $t \geq 1$ and $a, b, c \geq 0$, $0 < a + b \leq c$ or $a, b \geq 0, c \leq 0, a + b + c > 0$.

By Lemma 3.3 in [13], we have for $0 \leq p \leq 1$ and $s \geq 1$,

$$(s^{2p} - 1)(s^{2(1-p)} - 1) \geq 4p(1 - p)(s - 1)^2.$$

We assume that $a, b \geq 0$. We put $p = a / (a + b)$ and $s^{1/(a+b)} = t$. Then

$$(t^{2a} - 1)(t^{2b} - 1) \geq \frac{4ab}{(a + b)^2}(t^{a+b} - 1)^2.$$

Then we have

$$(t^{2a} - 1)(t^{2b} - 1)(t^c + 1)^2 \geq \frac{4ab}{(a + b)^2}(t^{a+b} - 1)^2(t^c + 1)^2.$$

In order to show the aimed inequality, we have to prove that

$$(t^{a+b}-1)^2(t^c+1)^2 \geq \frac{4(a+b)^2}{(a+b+c)^2}(t^{a+b+c}-1)^2.$$

Since $a+b+c>0$, it is sufficient to prove the following inequality:

$$(t^{a+b}-1)(t^c+1) \geq \frac{2(a+b)}{a+b+c}(t^{a+b+c}-1), \tag{4.7}$$

for $t \geq 1$ and $a, b, c \geq 0,\ 0 < a+b \leq c$ or $a, b \geq 0,\ c \leq 0,\ a+b+c>0$. We put

$$S(t) = (t^{a+b}-1)(t^c+1) - \frac{2(a+b)}{a+b+c}(t^{a+b+c}-1).$$

Then

$$S'(t) = t^{c-1}\{(c-a-b)t^{a+b} - c + (a+b)t^{a+b-c}\}.$$

Here we put

$$T(t) = (c-a-b)t^{a+b} - c + (a+b)t^{a+b-c}.$$

Then

$$T'(t) = (a+b)(c-a-b)t^{a+b-c-1}(t^c-1).$$

When $a+b \leq c$, $T'(t) \geq 0$. Since $T(1)=0$, $T(t) \geq 0$ for $t \geq 1$. Then $S'(t) \geq 0$. Since $S(1)=0$, $S(t) \geq 0$ for $t \geq 1$. On the other hand, when $c \leq 0$, $T'(t) \geq 0$. Since $T(1)=0$, $T(t) \geq 0$ for $t \geq 1$. Then $S'(t) \geq 0$. Since $S(1)=0$, $S(t) \geq 0$ for $t \geq 1$. Hence we get (4.7). □

Proof of Proposition 4.1. Let $x < y$. In the last line of (4.5), dividing both the numerator and the denominator by $(f(x)g(x)h(x))^2$ and by using $F(x) = \log f(x)$, $G(x) = \log g(x)$, and $H(x) = \log h(x)$, we get

$$L(x, y) = \frac{(e^{2(F(y)-F(x))} - 1)(e^{2(G(y)-G(x))} - 1)(e^{H(y)-H(x)} + 1)^2}{(e^{F(y)-F(x)+G(y)-G(x)+H(y)-H(x)} - 1)^2}.$$

By the generalized mean value theorem, there exist $z(x < z < y)$, $w(x < w < y)$ such that

$$\frac{G(y) - G(x)}{F(y) - F(x)} = \frac{G'(z)}{F'(z)} = k(z), \quad \frac{H(y) - H(x)}{F(y) - F(x)} = \frac{H'(w)}{F'(w)} = \ell(w).$$

Thus, we have

$$L(x, y) = \frac{(e^{2(F(y)-F(x))} - 1)(e^{2k(z)(F(y)-F(x))} - 1)(e^{\ell(w)(F(y)-F(x))} + 1)^2}{(e^{(1+k(z)+\ell(w))(F(y)-F(x))} - 1)^2}.$$

It follows from Lemma 4.1 that for any $R > 0$, the function

$$(k, \ell) \to A(k, \ell) = \frac{(R^2 - 1)(R^{2k} - 1)(R^{\ell} + 1)^2}{(R^{(1+k+\ell)} - 1)^2},$$

defined in $k \in [m, M]$, $\ell \in [n, N]$ is bounded from below by $\min_{m \le k \le M, n \le \ell \le N}\{A(k, \ell)\}$. It is easy to obtain

$$\min_{m \le k \le M, n \le \ell \le N}\{A(k, \ell)\} \ge 16\beta(f, g, h).$$

We complete the proof. □

Proof of Theorem 3.1. Since

$$\begin{aligned} Tr[f(\rho)g(\rho)h(\rho)[A, B]] &= Tr[f(\rho)g(\rho)h(\rho)[A_0, B_0]] \\ &= 2i \operatorname{Im}\{Tr[f(\rho)g(\rho)h(\rho)A_0B_0]\} \\ &= 2i \operatorname{Im} \sum_{\ell < m} (f(\lambda_\ell)g(\lambda_\ell)h(\lambda_\ell) \\ &\quad - f(\lambda_m)g(\lambda_m)h(\lambda_m))a_{m\ell}b_{\ell m} \\ &= 2i \sum_{\ell < m} (f(\lambda_\ell)g(\lambda_\ell)h(\lambda_\ell) \\ &\quad - f(\lambda_m)g(\lambda_m)h(\lambda_m)) \operatorname{Im}(a_{m\ell}b_{\ell m}), \end{aligned}$$

for any $A, B \in M_{n,sa}(\mathbb{C})$, where $a_{m\ell} = \langle \phi_m | A_0 | \phi_\ell \rangle$ and $b_{\ell m} = \langle \phi_\ell | B_0 | \phi_m \rangle$, we have

$$|Tr[f(\rho)g(\rho)h(\rho)[A, B]]|$$

$$\leq 2\sum_{\ell<m} |f(\lambda_\ell)g(\lambda_\ell)h(\lambda_\ell) - f(\lambda_m)g(\lambda_m)h(\lambda_m)| \, |\operatorname{Im}(a_{m\ell}b_{\ell m})|$$

$$\leq 2\sum_{\ell<m} |f(\lambda_\ell)g(\lambda_\ell)h(\lambda_\ell) - f(\lambda_m)g(\lambda_m)h(\lambda_m)| \, |a_{m\ell}| \, |b_{\ell m}|.$$

By Proposition 4.1, we have

$$\beta(f, g, h)|Tr[f(\rho)g(\rho)h(\rho)[A, B]]|^2$$

$$\leq 4\beta(f, g, h)(\sum_{\ell<m} |f(\lambda_\ell)g(\lambda_\ell)h(\lambda_\ell) - f(\lambda_m)g(\lambda_m)h(\lambda_m)| \, |a_{m\ell}| \, |b_{\ell m}|)^2$$

$$\leq \frac{1}{4}(\sum_{\ell<m} \sqrt{(f(\lambda_\ell)^2 - f(\lambda_m)^2)(g(\lambda_\ell)^2 - g(\lambda_m)^2)(h(\lambda_\ell) + h(\lambda_m))^2} |a_{m\ell}| \|b_{\ell m}|)^2$$

$$= \frac{1}{4}(\sum_{\ell<m} \sqrt{\Delta_f(\ell, m)\Delta_g(\ell, m)\Gamma_h(\ell, m)} |a_{m\ell}| \sqrt{\Gamma_f(\ell, m)\Gamma_g(\ell, m)\Gamma_h(\ell, m)} |b_{\ell m}|)^2,$$

where $\Delta_f(\ell, m) = f(\lambda_\ell) - f(\lambda_m)$, $\Delta_g(\ell, m) = g(\lambda_\ell) - g(\lambda_m)$, and $\Gamma_f(\ell, m) = f(\lambda_\ell) + f(\lambda_m)$, $\Gamma_g(\ell, m) = g(\lambda_\ell) + g(\lambda_m)$, $\Gamma_h(\ell, m) = h(\lambda_\ell) + h(\lambda_m)$. By Schwartz inequality, we have

$$\beta(f, g, h)|Tr[f(\rho)g(\rho)h(\rho)[A, B]]|^2$$

$$\leq \frac{1}{2}\sum_{\ell<m} \Delta_f(\ell, m)\Delta_g(\ell, m)\Gamma_h(\ell, m)|a_{m\ell}|^2$$

$$\times \frac{1}{2}\sum_{\ell<m} \Gamma_f(\ell, m)\Gamma_g(\ell, m)\Gamma_h(\ell, m)|b_{\ell m}|^2$$

$$\leq I_{\rho,(f,g,h)}(A) J_{\rho,(f,g,h)}(B).$$

Similarly, we have

$$\beta(f,\ g,\ h)|Tr[f(\rho)g(\rho)h(\rho)[A,\ B]]|^2 \le I_{\rho,\ (f,g,h)}(B)J_{\rho,\ (f,g,h)}(A).$$

Hence by multiplying the above two inequalities, we have

$$\beta(f,\ g,\ h)|Tr[f(\rho)g(\rho)h(\rho)[A,\ B]]|^2 \le U_{\rho,\ (f,g,h)}(A)U_{\rho,\ (f,g,h)}(B).$$

□

When $h(x) = 1,$ we obtain the result given by Ko and Yoo [5].

Corollary 4.1 ([5])**.** *If* $(f,\ g)$ *is CLI monotone pair*, *then the following inequality holds*:

$$U_{\rho,\ (f,g)}(A)U_{\rho,\ (f,g)}(B) \ge \beta(f,\ g)|Tr[f(\rho)g(\rho)[A,\ B]]|^2,$$

for $A,\ B \in M_{n,sa}(\mathbb{C}),$ *where*

$$I_{\rho,\ (f,g)}(A) = \frac{1}{2}Tr[(i[f(\rho),\ A_0])(i[g(\rho),\ A_0])],$$

$$J_{\rho,\ (f,g)}(A) = \frac{1}{2}Tr[\{f(\rho),\ A_0\}\{g(\rho),\ A_0\}],$$

$$U_{\rho,\ (f,g)}(A) = \sqrt{I_{\rho,\ (f,g)}J_{\rho,\ (f,g)}},$$

$$\beta(f,\ g) = \min\left\{\frac{m}{(m+M)^2},\ \frac{M}{(m+M)^2}\right\}.$$

We also have the following corollary:

Corollary 4.2. *Let* $f(x) = x^{\alpha}$ $(\alpha \ge 0),$ $g(x) = x^{\beta}$ $(\beta \ge 0),$ *and* $h(x) = x^{\gamma}$ $(\gamma \ge 0$ *or* $\gamma \le 0).$

(1) *If* $\alpha,\ \beta,\ \gamma \ge 0$ *satisfy* $0 < \alpha + \beta \le \gamma,$ *then*

$$\beta(f,\ g,\ h) = \frac{\alpha\beta}{(\alpha+\beta+\gamma)^2}.$$

(2) *If* $\alpha, \beta \geq 0, \gamma \leq 0$ *satisfy* $\alpha + \beta + \gamma > 0,$ *then*

$$\beta(f, g, h) = \frac{\alpha\beta}{(\alpha + \beta + \gamma)^2}.$$

Remark 4.1. When $\alpha, \beta \geq 0, \gamma < 0$ satisfy $\alpha + \beta + \gamma > 0,$ we remark that $h(x)$ is not continuous function on $[0, 1]$ because

$$\lim_{x \to +0} h(x) = +\infty.$$

Then in this case by putting $\epsilon > 0$ such that ϵ is smaller than the minimal eigenvalue of $\rho,$ we can assume that $h(x)$ is continuous on $[\epsilon, 1].$ Hence we obtain the same result as Corollary 4.2.

Remark 4.2. When $\gamma = 0$ in (2) of Corollary 4.2, we have the result in [15] (Theorem 2.3). And when $\alpha + \beta + \gamma = 1$ in Corollary 4.2, we have the result in [14] (Theorem 2.2). That is (1) implies $\alpha, \beta \geq 0, \alpha + \beta \leq \frac{1}{2}$ and (2) implies $\alpha, \beta \geq 0, \alpha + \beta \geq 1.$

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