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GENERALIZED *k*-JACOBSTHAL AND *k*-JACOBSTHAL-LUCAS NUMBERS

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Abstract

In this note, we consider the following numbers: $\{J_{n,m}(k)\}$− the generalized k-Jacobsthal numbers and $\{j_{n,m}(k)\}$− the generalized k-Jacobsthal-Lucas numbers. Also, we introduce the incomplete numbers $\{J^r_{n,m}(k)\}$ and $\{j^r_{n,m}(k)\}$. Next, we introduce and consider the sequences of numbers: $\{J^l_{n,m}(k)\}$− the l-th convolution of the sequence $\{J_{n,m}(k)\}$ and $\{j^s_{n,m}(k)\}$− the s-th convolution of the sequence $\{j_{n,m}(k)\}$.

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1. Introduction and Definitions

In papers [1], [2], Djordjević considered two classes of polynomials: $J_{n,m}(x)$– the generalized Jacobsthal polynomials and $j_{n,m}(x)$– the generalized Jacobsthal-Lucas polynomials. The particular cases of these polynomials are so-called Jacobsthal polynomials $J_n(x)$ and Jacobsthal-Lucas polynomials $j_n(x)$, which were investigated by Horadam [7].

If $x = 1$ in $J_{n,m}(x)$ and $j_{n,m}(x)$, we get the generalized Jacobsthal numbers $J_{n,m}$ and the generalized Jacobsthal-Lucas numbers $j_{n,m}$ (see [3]). Namely, in [3], authors defined the incomplete generalized Jacobsthal numbers $J_{n,m}^k$ by

$$J_{n,m}^k = \sum_{r=0}^{k} \binom{n-1-(m-1)r}{r} 2^r, \quad 0 \le k \le [(n-1)/m],\ n,\ m \in \mathbb{N}, \tag{1.1}$$

with

$$J_{n,m}^{[(n-1)/m]} = J_{n,m}, \quad J_{n,m}^k = 0, \quad 0 \le n < mk+1, \tag{1.2}$$

$$J_{mk+l,m}^k = J_{mk+l-1,m}, \quad l = 1, \dots, m, \tag{1.3}$$

and the incomplete generalized Jacobsthal-Lucas numbers $j_{n,m}^k$ by

$$j_{n,m}^k = \sum_{r=0}^{k} \frac{n-(m-2)r}{n-(m-1)r} \binom{n-(m-1)r}{r} 2^r, \quad 0 \le k \le [n/m], \tag{1.4}$$

with

$$j_{n,m}^{[n/m]} = j_{n,m}, \quad j_{n,m}^k = 0, \quad 0 \le n < mk, \tag{1.5}$$

$$j_{mk+l,m}^k = j_{mk+l-1,m}, \quad l = 1, \dots, m, \tag{1.6}$$

where $n,\ m \in \mathbb{N}$.

Motivated essentially by the work by Pintér and Srivastava [10] and by the recent works [8] and [9], in this note, we define the generalized k-Jacobsthal numbers $J_{n,m}(k)$ and the generalized k-Jacobsthal-Lucas numbers $j_{n,m}(k)$, respectively, by the following recurrence relations:

$$J_{n,m}(k) = k\,J_{n-1,m}(k) + 2J_{n-m,m}(k), \tag{1.7}$$

with

$$J_{0,m}(k) = 0, \quad J_{n,m}(k) = k^{n-1}, \quad n = 1, \dots, m-1, (n \geq m,\ n,\ m \in \mathbb{N});$$

and

$$j_{n,m}(k) = k\,j_{n-1,m}(k) + 2j_{n-m,m}(k), \tag{1.8}$$

with

$$j_{0,m}(k) = 2, \quad j_{n,m}(k) = k^{n}, \quad n = 1, \dots, m-1, (n \geq m,\ n,\ m \in \mathbb{N}).$$

From the recurrence relations (1.7) and (1.8), we get the following explicit formulas:

$$J_{n,m}(k) = \sum_{i=0}^{[(n-1)/m]} \binom{n-1-(m-1)i}{i} k^{n-1-mi} \cdot 2^{i}, \tag{1.9}$$

and

$$j_{n,m}(k) = \sum_{i=0}^{[n/m]} \frac{n-(m-2)i}{n-(m-1)i} \binom{n-(m-1)i}{i} k^{n-mi} \cdot 2^{i}. \tag{1.10}$$

Let $J_{n,m}(k)$ be the coefficients of a power series, and let's consider the corresponding analytic function $F_m(t)$, defined by

$$F_m(t) = J_{0,m}(k) + J_{1,m}(k) \cdot t + J_{2,m}(k) \cdot t^2 + \cdots. \tag{1.11}$$

The function (1.11) is called the generating function of the k-Jacobsthal numbers (see [8], [9]).

Using the relation (1.11) and the initial conditions in the relation (1.7), we get

$$F_m(t) = (1 - kt - 2t^m)^{-1} = \sum_{n=1}^{\infty} J_{n,m}(k) t^{n-1}. \tag{1.12}$$

Also, in the similar manner, we find that

$$G_m(t) = \frac{2 - kt}{1 - kt - 2t^m} = \sum_{n=0}^{\infty} j_{n,m}(k) t^n, \tag{1.13}$$

is the generating function of the sequence $\{j_{n,m}(k)\}$.

2. Incomplete Generalized *k*-Jacobsthal Numbers

Firstly, we define the incomplete generalized k-Jacobsthal numbers $J_{n,m}^r(k)$ by

$$J_{n,m}^r(k) = \sum_{i=0}^{r} \binom{n-1-(m-1)i}{i} k^{n-1-mi} \cdot 2^i, \quad 0 \le r \le [(n-1)/m], \tag{2.1}$$

with

$$J_{n,m}^{[(n-1)/m]}(k) = J_{n,m}(k), \quad J_{n,m}^r(k) = 0 \ (0 \le n < mr+1), \tag{2.2}$$

$$J_{mr+l,m}^r(k) = J_{mr+l-1,m}(k) \quad (l = 1, \dots, m). \tag{2.3}$$

Also, we define the incomplete generalized k-Jacobsthal-Lucas numbers $j_{n,m}^r(k)$ by

$$j_{n,m}^r(k) = \sum_{i=0}^{r} \frac{n-(m-2)i}{n-(m-1)i} \binom{n-(m-1)i}{i} k^{n-mi} \cdot 2^i, \tag{2.4}$$

with

$$j_{n,m}^{[n/m]}(k) = j_{n,m}(k), \quad j_{n,m}^r(k) = 0 \ (0 \le n < mr), \tag{2.5}$$

$$j^{r}_{mr+l,m}(k) = j_{mr+l-1,m}(k) \quad (l = 1, \ldots, m). \tag{2.6}$$

In this paper, we find the generating functions of the incomplete numbers $J^{r}_{n,m}(k)$ and $j^{r}_{n,m}(k)$.

The following known results ([10], [3]) will be required in our investigation of the generating function of the incomplete generalized k-Jacobsthal numbers $J^{r}_{n,m}(k)$, defined by (2.1). For the theory and applications of the various methods and techniques for deriving generating functions of special functions and polynomials, we may refer the interested reader to a recent treatise on the subject of generating functions by Srivastava and Manocha [11]. It is not difficult to prove the following result (see [3]).

Lemma 2.1. *Let* $\{s_n\}_{n=0}^{\infty}$ *be a complex sequence satisfying the following nonhomogeneous recurrence relation*:

$$s_n = ks_{n-1} + 2s_{n-m} + r_n,\ n \geq m\ (n, m \in \mathbb{N}), \tag{2.7}$$

where $\{r_n\}$ *is a given complex sequence. Then the generating function* $S(t)$ *of the sequence* $\{s_n\}$ *is*

$$S(t) = \left(s_0 - r_0 + \sum_{i=1}^{m-1} t^{i}(s_i - ks_{i-1} - r_i) + G(t)\right)(1 - kt - 2t^{m})^{-1}, \tag{2.8}$$

where $G(t)$ *is the generating function of the sequence* $\{r_n\}$.

Our first result on generating function is contained in Theorem 2.1 below.

Theorem 2.1. *The generating function of the incomplete generalized k-Jacobsthal numbers* $J^{r}_{n,m}(k)$ *is given by*

$$R^{r}_{m}(t) = t^{mr+1} S^{r}_{m}(t), \tag{2.9}$$

where

$$S_m^r(t) = \left(J_{mr,m}(k) + \sum_{i=1}^{m-1} t^i (J_{mr+i,m}(k) - kJ_{mr+i-1,m}(k)) - \frac{2^{r+1}t^m}{(1-kt)^{r+1}} \right)$$

$$\times (1 - kt - 2t^m)^{-1}.$$

Proof. Using the explicit formula (2.1) and the recurrence relation (1.7), we get

$$J_{n,m}^r(k) - kJ_{n-1,m}^r(k) - 2J_{n-m,m}^r(k)$$

$$= \sum_{i=0}^{r} \binom{n-1-(m-1)i}{i} k^{n-1-mi} \cdot 2^i - k\sum_{i=0}^{r} \binom{n-2-(m-1)i}{i} k^{n-2-mi} \cdot 2^i$$

$$- 2\sum_{i=0}^{r} \binom{n-1-m-(m-1)i}{i} k^{n-1-m-mi} \cdot 2^i$$

$$= \sum_{i=0}^{r} \binom{n-1-(m-1)i}{i} k^{n-1-mi} \cdot 2^i - \sum_{i=0}^{r} \binom{n-2-(m-1)i}{i} k^{n-1-mi} \cdot 2^i$$

$$- \sum_{i=1}^{r+1} \binom{n-2-(m-1)}{i-1} k^{n-1-mi} \cdot 2^i$$

$$= -\frac{(n-1-m-(m-1)r)!}{r!\,(n-1-m-mr)!} k^{n-1-m-mr} \cdot 2^{r+1}.$$

So, having in view (2.2), we get

$$s_0 = J_{mr+1,m}^r(k), \quad s_1 = J_{mr+2,m}^r(k), \ \ldots, \ s_{m-1} = J_{mr+m,m}^r(k),$$

and

$$s_n = J_{mr+n+1,m}^r(k).$$

Suppose also that

$$r_0 = r_1 = \cdots = r_{m-1} = 0 \quad \text{and} \quad r_n = \binom{n-m+r}{n-m} k^{n-m} \cdot 2^{r+1}.$$

Then, for the generating function $G(t)$ of the sequence $\{r_n\}$, we can show that

$$G(t) = \frac{2^{r+1}t^m}{(1-kt)^{r+1}}.$$

Thus, in view of the above lemma, the generating function $S_m^r(t)$ of the sequence $\{s_n\}$ satisfies the following relation:

$$S_m^r(t) \cdot (1 - kt - 2t^m) = J_{mr,m}(k) + \sum_{i=1}^{m-1} t^i (J_{mr+i,m}(k) - kJ_{mr+i-1,m}(k))$$
$$- \frac{2^{r+1}t^m}{(1-kt)^{r+1}}.$$

So

$$R_m^r(t) = t^{mr+1} \cdot S_m^r(t).$$

3. Incomplete Generalized *k*-Jacobsthal-Lucas Numbers

For the incomplete generalized k-Jacobsthal-Lucas numbers $j_{n,m}^r(k)$, defined by (2.4), we can prove the following generating function.

Theorem 3.1. *The generating function of the incomplete generalized* k*-Jacobsthal-Lucas numbers* $j_{n,m}^r(k)$ $(r \in \mathbb{N}_0)$ *is given by*

$$F_m^r(t) \cdot (1 - kt - 2t^m) = \sum_{n=0}^{\infty} j_{n,m}^r(k) t^n = t^{mr} j_{mr-1,m}(k)$$
$$+ t^{mr} \left(\sum_{l=1}^{m-1} t^l (j_{mr+l-1,m}(k) - kj_{mr+l-2,m}(k)) - \frac{2^{r+1}t^m(2-t)}{(1-kt)^{r+1}} \right). \tag{3.1}$$

Proof. Similarly to the proof of Theorem 2.1, from the explicit formula (2.4), it follows that

$$j_{n,m}^{r}(k) - k j_{n-1,m}^{r}(k) - 2 j_{n-m,m}^{r}(k) = -\frac{n-m+2r}{n-m+r}\binom{n-m+r}{n-m} k^{n-m} \cdot 2^{r+1}.$$

Let

$$s_0 = j_{mr-1}, m(k), \quad s_1 = j_{mr,m}(k), \ldots, s_{m-1} = j_{mr+m,m}(k),$$

and

$$s_n = j_{mr+n+1,m}(k).$$

Suppose also that

$$r_0 = r_1 = \cdots = r_{m-1} = 0 \text{ and } r_n = \frac{n-m+2r}{n-m+r}\binom{n-m+r}{n-m} k^{n-m} \cdot 2^{r+1}.$$

Then, using the known method based upon the above lemma, we find that

$$G(t) = \frac{2^{r+1} t^m (2-t)}{(1-kt)^{r+1}},$$

is the generating function of the sequence $\{r_n\}$. Now, we can easily get (3.1). □

Remark 1. Specially, for $m = 2$, Theorem 3.1 yields the generating function for the incomplete numbers $j_n(k)$:

$$F_2^r \cdot (1 - kt - 2t^2) = t^{2r}\left(j_{2r-1}(k) + t(j_{2r}(k) - k j_{2r-1}(k)) - \frac{2^{r+1} t^2 (2-t)}{(1-kt)^{r+1}}\right).$$

4. Convolutions of the Generalized *k*-Jacobsthal and *k*-Jacobsthal-Lucas Numbers

Here we define the sequences of numbers $\{J^{l}_{n,m}\}$– the l-th convolution of the generalized k- Jacobsthal numbers, and $\{j^{s}_{n,m}\}$– the s-th convolution of the generalized k- Jacobsthal-Lucas numbers, where l and s are some nonnegative integers.

The l-th convolution of the generalized k- Jacobsthal numbers $\{J^{l}_{n,m}(k)\}$ is given by the following generating function:

$$F_l(t) = (1 - kt - 2t^m)^{-(l+1)} = \sum_{n=1}^{\infty} J^{l}_{n,m}(k)t^{n-1}, \quad l \geq 0. \tag{4.1}$$

The s-th convolution of the generalized k-Jacobsthal-Lucas numbers $\{J^{s}_{n,m}(k)\}$ is defined by

$$G_s(t) = \left(\frac{2 - kt}{1 - kt - 2t^m}\right)^{s+1} = \sum_{n=0}^{\infty} j^{s}_{n,m}(k)t^{n}, \quad s \geq 0. \tag{4.2}$$

Next, from (4.1), using the known method, we find the following explicit formula for the sequence $\{J^{l}_{n,m}(k)\}$:

$$J^{l}_{n,m}(k) = \sum_{i=0}^{[(n-1)/m]} \frac{(l+1)_{n-1-(m-1)i}}{i!\,(n-1-mi)!} k^{n-1-mi} \cdot 2^{i}, \tag{4.3}$$

where

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1), \quad \alpha \neq 0, -1, \cdots, -(n-1).$$

So, from (4.2), we get

$$\sum_{n=0}^{\infty} j_{n,m}^{s}(k)t^{n} = (2-kt)^{s+1}(1-kt-2t^{m})^{-(s+1)}$$

$$= \sum_{i=0}^{s+1}\binom{s+1}{i}2^{s+1-i}(-1)^{i}k^{i}\sum_{n=1}^{\infty}J_{n,m}^{s}(k)t^{n+i-1}.$$

Hence, we conclude that

$$J_{n,m}^{s}(k) = \sum_{i=0}^{s+1}\binom{s+1}{i}2^{s+1-i}(-k)^{i}J_{n+1-i,m}^{s}(k).$$

Next, using the known formulas (see [12]):

$$(\alpha)_{n-k} = \frac{(-1)^{k}(\alpha)_{n}}{(1-\alpha-n)_{k}},\ (n-k)! = \frac{(-1)^{k}n!}{(-n)_{k}},\quad (\alpha)_{n+k} = (\alpha)_{n}(\alpha+n)_{k},$$

we find that

$$\frac{(l+1)_{n-1-(m-1)i}}{(n-1-mi)!} = \frac{(-1)^{(m-1)i}(l+1)_{n-1}(1-n)_{mi}}{(1-l-n)_{(m-1)i}(-1)^{mi}(n-1)!}$$

$$= \frac{(-1)(l+1)_{n-1}(1-n)_{mi}}{(1-l-n)_{(m-1)i}(n-1)!},$$

and

$$J_{n,m}^{l}(k) = \sum_{i=0}^{[(n-1)/m]}\frac{(-1)^{i}(l+1)_{n-1}(1-n)_{mi}}{i!\,(n-1)!\,(1-l-n)_{(m-1)i}}\,k^{n-1}\cdot\left(\frac{2}{k^{m}}\right)^{i}$$

$$= \frac{(l+1)_{n-1}}{(n-1)!}\,k^{n-1}\sum_{i=0}^{[(n-1)/m]}\frac{(-1)^{i}(1-n)_{mi}}{i!\,(1-l-n)_{(m-1)i}}\cdot\left(\frac{2}{k^{m}}\right)^{i}.$$

Now the explicit formula (4.3) can be written in the following form, for $n := n+1$:

$$J_{n+1,m}^{l}(k) = \frac{(l+1)_{n}}{n!}\,k^{n}{}_{m}F_{m-1}\left[\begin{matrix}\frac{-n}{m}, & \frac{1-n}{m}, & \ldots, & \frac{m-1-n}{m}; & \frac{-2}{k^{m}}\\ \frac{-l-n}{m-1}, & \frac{1-l-n}{m-1}, & \ldots, & \frac{m-2-l-n}{m-1} & \end{matrix}\right],$$

where

$$_mF_{m-1}\left[\begin{matrix} a_1,\, a_2,\, \dots,\, a_m;\, z \\ \\ b_1,\, b_2,\, \dots,\, b_{m-1} \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_m)_n}{(b_1)_n (b_2)_2 \dots (b_{m-1})_n} \cdot \frac{z^n}{n!},$$

is the generalized hypergeometric function (see [11]).

5. Partial Derivatives and Convolution Numbers

In this section, we consider again the numbers $J^l_{n,m}(k)$, on the other manner. Namely, we find the connection of the l-th convolution and the partial derivative with respect to k of the sequence $J_{n,m}(k)$.

Let

$$J^{(l)}_{n,m}(k) = \frac{\partial^l \{J_{n,m}(k)\}}{\partial k^l}.$$

Differentiating the relation (1.12) one by one, l-times, with respect to k, we get the following relation:

$$J^{(l)}_{n,m}(k) = l!\, J^l_{n-l,m}(k). \tag{5.1}$$

One result is given by the following statement.

Theorem 5.1. *For* $m \geq 1$ *it holds*

$$J^{(m)}_{n+m,m}(k) + 2(m-1) J^{(m)}_{n,m}(k) = (m-1)!\,(n+m-1) J^{m-1}_{n,m}(k). \tag{5.2}$$

Proof. Using the relation (5.1) in the left side of the relation (5.2), and using (4.3), we get

$$J^{(m)}_{n,m}(k) + 2(m-1) J^{(m)}_{n,m}(k) = m!\, J^m_{n,m}(k) + 2(m-1) m!\, J^m_{n-m,m}(k)$$

$$= m!\,(J^m_{n,m}(k) + 2(m-1) J^m_{n-m,m}(k))$$

$$= m! \sum_{i=0}^{[(n-1)/m]} \frac{(m+1)_{n-1-(m-1)i}}{i!\,(n-1-mi)!} k^{n-1-mi} \cdot 2^i$$

$$+ 2(m-1)m! \sum_{i=0}^{[(n-1-m)/m]} \frac{(m+1)_{n-1-m-(m-1)i}}{i!\,(n-1-m-mi)!} k^{n-1-m-mi} \cdot 2^i$$

$$= m! \sum_{i=0}^{[(n-1)/m]} \frac{(m+1)_{n-1-(m-1)i}}{i!\,(n-1-mi)!} k^{n-1-mi} \cdot 2^i$$

$$+ (m-1)m! \sum_{i=0}^{[(n-1)/m]} \frac{(m+1)_{n-2-(m-1)i}}{(i-1)!\,(n-1-mi)!} k^{n-1-mi} \cdot 2^i$$

$$= (m-1)!\,(m+n-1) \sum_{i=0}^{[(n-1)/m]} \frac{(m)_{n-1-(m-1)i}}{i!\,(n-1-mi)!} k^{n-1-mi} \cdot 2^i$$

$$= (m-1)!\,(m+n-1) J_{n,m}^{m-1}(k).$$

□

The result, more general than the previous one, is given by the next theorem.

Theorem 5.2. *For some* $r \geq 1$, *the following relation holds*:

$$J_{n+m,m}^{(r)}(k) + 2(m-1)J_{n,m}^{(r)}(k) = (r-1)!\,(m+n-1)J_{n+m-r,m}^{r-1}(k). \qquad (5.3)$$

Proof. The proof is similar to the proof of Theorem 5.1. □

Remark 2. If $m = 2$, the formula (5.3) becomes

$$J_{n+2-r,2}^{r-1}(k) = \frac{1}{(r-1)!\,(n+1)} (J_{n+2,2}^{(r)}(k) + 2J_{n,2}^{(r)}(k)). \qquad (5.4)$$

Applying the formula (5.4) for $r = 1, 2, 3,$ we get some members of the sequence $\{J_{n,m}^{r}(k)\}$, which are given in Tables 1 and 2.

Table 1. Numbers $J_{n,2}^{r}(k)$

n	$r = 0$	$r = 1$
0	0	
1	1	1
2	k	$2k$
3	$k^2 + 2$	$3k^2 + 4$
4	$k^3 + 4k$	$4k^3 + 12k$
5	$k^4 + 6k^2 + 4$	$5k^4 + 24k^2 + 12$
6	$k^5 + 6k^3 + 12k$	$6k^5 + 40k^3 + 48k$
7	$k^6 + 10k^4 + 24k^2 + 8$	$7k^6 + 60k^4 + 120k^2 + 32$
8	$k^7 + 12k^5 + 40k^3 + 32k$	$8k^7 + 84k^5 + 240k^3 + 160k$
9	$k^8 + 14k^6 + 60k^4 + 80k^2 + 16$	$9k^8 + 112k^6 + 420k^4 + 480k^2 + 80$

Table 2. Numbers $J_{n,2}^{r}(k)$

n	$r = 2$	$r = 3$
0	0	
1	1	0
2	$3k$	0
3	$6k^2 + 6$	1
4	$10k^3 + 24k$	$4k$
5	$15k^4 + 60k^2 + 24$	$105k^2 + 8$
6	$21k^5 + 120k^3 + 120k$	$20k^3 + 40$
7	$28k^6 + 210k^4 + 360k^2 + 80$	$35k^4 + 120k^2 + 40$
8	$36k^7 + 336k^5 + 840k^3 + 480k$	$56k^5 + 280k^3 + 240k$
9	$45k^8 + 504k^6 + 1680k^4 + 1680k^2 + 240$	$84k^6 + 560k^4 + 840k^2 + 160$

Remark 3. For $m = 3,$ the formula (5.3) becomes

$$J_{n+3,3}^{(r)}(k) + 4J_{n,3}^{(r)}(k) = (r-1)!\,(n+2)J_{n+3-r,3}^{r-1}(k). \tag{5.5}$$

Hence, for $r = 0, 1, 2, 3,$ from (5.5), we get the some initial members of the sequence $\{J_{n,3}^{r}(k)\},$ which are given in Tables 3 and 4.

Table 3. Numbers $J_{n,3}^{r}(k)$

n	$r = 0$	$r = 1$
0	0	
1	1	1
2	k	$2k$
3	$k^2 + 2$	$3k^2$
4	$k^3 + 2$	$4k^3 + 4$
5	$k^4 + 4k$	$5k^4 + 12k$
6	$k^5 + 6k^2$	$6k^5 + 24k^2$
7	$k^6 + 8k^3 + 4$	$7k^6 + 40k^3 + 12$
8	$k^7 + 10k^4 + 12k$	$8k^7 + 60k^4 + 48k$
9	$k^8 + 12k^5 + 24k^2$	$9k^8 + 84k^5 + 120k^2$

Table 4. Numbers $J^r_{n,3}(k)$

n	$r = 2$	$r = 3$
0	0	
1	1	1
2	$3k$	$4k$
3	$6k^2$	$10k^2$
4	$10k^3 + 6$	$20k^3 + 8$
5	$15k^4 + 24k$	$35k^4 + 40k$
6	$21k^5 + 60k^2$	$56k^5 + 120k^2$
7	$28k^6 + 120k^3 + 24$	$84k^6 + 280k^3 + 40$
8	$36k^7 + 210k^4 + 120k$	$120k^7 + 560k^4 + 240k$
9	$45k^8 + 336k^5 + 360k^2$	$165k^8 + 1008k^5 + 840k^2$

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