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LIE SYMMETRIES AND GENERALIZED SOLUTIONS OF THE GARDNER EQUATION

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Abstract

This paper focuses on the symmetries of the Gardner equation or the combined Korteweg-de Vries-modified Korteweg-de Vries (KdV-mKdV) equation. The differential form technique, which is called Harrison method for finding the point symmetry algebra will be explained for the Gardner equation. This will be treated in Section 2, obtaining three dimensional Lie point transformations for the equation. The results obtained from the Harrison approach have been tested and confirmed with the Lie method, in Section 3. Finally, a generalized form of solutions to the Gardner equation is exhibited in Section 4, using the symmetry group of the equation.

1. Introduction

Lie symmetries of differential equations is one of the important concepts in the theory of differential equations and physics. Among others methods, Lie method is a firm one for finding symmetries of differential equations. This method was first applied to determine point symmetries. In 1969-1970, Kent Harrison and Frank Estabrook devised a method to calculate symmetries of differential equations using

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differential forms and Cartan's formulation of differential equations [1]. They were simply trying to understand how the symmetries of Maxwell's equations could be found from the differential form version of those equations. Once they realized that the key to symmetries was the use of the Lie derivative, Kent Harrison applied the method to several others equations such as the one dimensional heat equation, the short wave gas dynamic equation, and the nonlinear Poisson equation (see [2]) and [3]. Here we apply this method to the Gardner equation or the combined KdV-mKdV equation, given as follows:

$$u_{xxx} + (2\delta u - 3\sigma u^2)u_x + u_t = 0, \tag{1.1}$$

where $u(x, t)$ is a function of space x and time variable t; subscripts denoted partial derivatives; δ and σ are real constants, with δ is no wanishing. The Gardner equation is widely used in various branches of physics, such as plasma physics, fluid physics, and quantum field theory. The equation plays a prominent role in ocean wave. This equation has been investigated thoroughly in the literature because it is used to model a variety of nonlinear phenomena [4, 6].

Note that in 2008, Mehdi Nadjafikhah and Seyed-Reza Hejazi applied the differential form approach to the standard KdV equation, with $\delta = \frac{1}{2}$ and $\sigma = 0$ (see [5]). We refer to that paper. Thus, our tools in this method are differential forms and Lie derivatives. In the Section 3, a generalized form of solutions of the KdV-mKdV equation will be given.

2. The Harrison Method Applied to the Gardner Equation

The method proceeds as follow. We consider a set of partial differential equations, defined on a differential manifold $\mathcal{M}$ of n independent variables $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and m dependent variables $u = (u^1, \dots, u^m) \in \mathbb{R}^m$ ($n = 2$ and $m = 1$ in our special case). Let $X = \mathbb{R}^2$, be the space representing the independent variables, and let $U = \mathbb{R}$, representing the space of dependent variable. We define the partial derivatives of the dependent variable as new variables (prolongation) in sufficient number to write the equation as second order equation, thus *prolonging* the manifold $\mathcal{M}$ to a manifold $\mathcal{N} = \mathcal{M}^{(2)}$ of the 2*nd jet-space* $X \times U^{(2)}$ of the manifold $X \times U$. The independent variables $(x, t) \in X$; the dependent variable $u \in U$ and $u^{(2)} = (u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) \in U^{(2)}$, thus $(x, t, u^{(2)}) = (x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) \in X \times U^{(2)}$. The space $\mathcal{M}^{(2)}$ is the corresponding 2*nd prolongation* of the subspace $\mathcal{M} \subset X \times U$. Then we can construct a set of differential forms. We speak of the set of forms, representing the equations, as an ideal *I*. It is to be closed. Then for determining the invariance of the differential equations, we may construct the Lie derivative of the forms in the ideal *I*. Lie derivative of geometrical object, like tensors, are associated with symmetries of those objects. If the Lie derivative vanishes, then the vector V represents the direction of an infinitesimal symmetry transformation in the manifold. Here the Lie derivative will be denoted by $\mathcal{L}$ and the forms are our tensors. It is now simple to treat the invariance of a set of differential equations. A set of equations is invariant if a transformation leaves the equations still satisfied, provided that the original equations are satisfied. In the formalism, we have introduced, this is easily stated: the Lie derivative of forms in the ideal must lie in the ideal: $\mathcal{L}_V I \subset I$. Then if the basis forms in the ideal are annulled, the transformed equations are also annulled. And this should therefore represent symmetries. In practice, this means simply that the Lie derivative of each of the (basis) forms in *I* is a linear combination of the forms in *I*. For further details on the method, see [1, 2, 3, 5].

In the sequel, we utilize this method to find the Lie point symmetries for the Gardner equation of the form (1.1). First, write the Equation (1.1) as a second order equation by defining a new variable $w = u_x$. Thus Equation (1.1) becomes

$$w_{xx} + (2\delta u - 3\sigma u^2)w + u_t = 0. \tag{2.1}$$

Then, we construct a set of 1-forms on the manifold $\mathcal{N}$ as follow.

Lemma 2.1. *For Equation* (1.1), *the required* 1-*forms are*

$$\beta^1 = du - u_t dt - u_x dx, \tag{2.2}$$

$$\beta^2 = du_t - u_{tt} dt - u_{tx} dx, \tag{2.3}$$

$$\beta^3 = du_x - u_{tx} dt - u_{xx} dx.$$

Proof. We consider the 8-dimensional manifold corresponding to the Equation (2.1), with the coordinates $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$. Considering this equation, we have the following contact conditions:

$$du = u_t dt + u_x dx,$$

$$du_t = u_{tt} dt + u_{tx} dx,$$

$$du_x = u_{tx} dt + u_{xx} dx,$$

which lead to the contact 1-forms of Lemma 2.1. □

Then, we have to construct the forms of the ideal I by the following:

Lemma 2.2. *The ideal I consists of the following* 2-*forms*:

$$\gamma^1 = (u_x u_{tt} - u_t u_{tx}) dx \wedge dt + u_{tx} dx \wedge du$$
$$- u_x dx \wedge du_t + u_{tt} dt \wedge du - u_t dt \wedge du_t + du \wedge du_t;$$

$$\gamma^2 = (u_x u_{tx} - u_t u_{xx}) dx \wedge dt + u_{xx} dx \wedge du$$
$$- u_x dx \wedge du_x + u_{tx} dt \wedge du - u_t dt \wedge du_x + du \wedge du_x;$$

$$\gamma^3 = (u_{tx}^2 - u_{tt} u_{xx}) dx \wedge dt + u_{xx} dx \wedge du_t$$
$$- u_{tx} dx \wedge du_x + u_{tx} dt \wedge du_t - u_{tt} dt \wedge du_x + du_t \wedge du_x;$$

$$\gamma^4 = dx \wedge du_x + dt \wedge du_t;$$

$$\gamma^5 = dx \wedge du_{tx} + dt \wedge du_{tt};$$

$$\gamma^6 = dx \wedge du_{xx} + dt \wedge du_{tx};$$

$$\gamma^7 = f(u)dt \wedge du - dx \wedge du + dt \wedge du_{xx}.$$

Proof. The proof of this lemma is straightforward. The forms γ^1, γ^2, γ^3, γ^4, γ^5, and γ^6, are obtained from Lemma 2.1 as follows:

$$\gamma^1 = \beta^1 \wedge \beta^2, \quad \gamma^2 = \beta^1 \wedge \beta^3, \quad \gamma^3 = \beta^2 \wedge \beta^3, \quad \gamma^4 = d\beta^1, \quad \gamma^5 = d\beta^2, \quad \gamma^6 = d\beta^3,$$

where $\wedge$ is the wedge product. And form γ^7 is obtained from Equation (1.1) by noting that

$$u_{xxx} = \frac{du_{xx}}{dx}, \quad u_x = \frac{du}{dx}, \quad u_t = \frac{du}{dt}. \tag{2.4}$$

□

2.1. Lie symmetries of the Gardner equation

Let

$$X = V^t \frac{\partial}{\partial t} + V^x \frac{\partial}{\partial x} + V^u \frac{\partial}{\partial u}, \tag{2.5}$$

be a symmetry generator of the KdV-mKdV equation (1.1), defined on the (t, x, u) space. The second prolongation of X is the vector field

$$V = V^t \frac{\partial}{\partial t} + V^x \frac{\partial}{\partial x} + V^u \frac{\partial}{\partial u} + V^{u_t} \frac{\partial}{\partial u_t} + V^{u_x} \frac{\partial}{\partial u_x} + V^{u_{tt}} \frac{\partial}{\partial u_{tt}} + V^{u_{tx}} \frac{\partial}{\partial u_{tx}} + V^{u_{xx}} \frac{\partial}{\partial u_{xx}}, \tag{2.6}$$

that acts on the manifold $\mathcal{N}$, with the coordinates $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$, where the $V^i (i = 1, 2, \ldots, 8)$ are smooth functions to be determined in $\mathcal{N}$. Write the Lie derivatives of forms in I as linear combinations of themselves as follow:

$$\mathcal{L}_V \gamma^i = \sum_{j=1}^{7} \lambda_j \gamma^j, \quad i = 1, \dots 7, \tag{2.7}$$

where the γ^i are the forms of Lemma 2.2 and the λ_j are 0-forms (functions), for $j = 1, 2, \dots, 7$.

Proposition 2.1. *Corresponding to the* 2-*form* γ^1, *the identity*

$$\mathcal{L}_V \gamma^1 = \sum_{j=1}^{7} \lambda_j \gamma^j, \quad i = 1, 2, \dots 7, \tag{2.8}$$

is equivalent to the following system of partial differential equations:

$$V^{u_x} u_{tt} + u_x V^u_{tt} - u_x V^{u_t}_{,t} - \lambda_1 (u_x u_{tt} - u_t u_{tx})$$

$$- \lambda_2 (u_x u_{tx} - u_t u_{xx}) - \lambda_3 (u^2_{tx} + u_{tt} u_{xx})$$

$$- V^{u_t} u_{tx} - u_t V^{u_{tx}} + (u_x u_{tt} - u_t u_{tx}) V^x_{,x} + (u_x u_{tt} - u_{tx} u_t) V^t_{,t} + u_{tx} V^u_{,t} = 0;$$

$$(u_x u_{tt} - u_t u_{tx}) V^t_{,u}$$

$$+ V^u_{,u_{tx}} + u_{tx} V^x_{,x} + u_{tx} V^u_{,u} - u_x V^{u_t}_{,u} + u_{tt} V^t_{,x} - \lambda_1 u_{tx} - \lambda_2 u_{xx} + \lambda_7 = 0;$$

$$V^u_{tt} - (u_x u_{tt} - u_t u_{tx}) V^x_{,u}$$

$$+ u_{tx} V^x_{,t} + u_{tt} V^t_{,t} + u_{tt} V^u_{,u} - u_t V^{u_t}_{,u} - V^{u_t}_{,t} - \lambda_1 u_{tt} - \lambda_2 u_{tx} - \lambda_7 \delta u = 0;$$

$$(u_x u_{tt} - u_t u_{tx}) V^t_{,u_t}$$

$$- V^{u_x} + u_{tx} V^u_{,u_t} - u_x V^x_{,x} - u_x V^{u_t}_{u_t} - u_t V^t_{,x} + V^u_{,x} + \lambda_1 u_x - \lambda_3 u_{xx} = 0;$$

$$- (u_x u_{tt} - u_t u_{tx}) V^x_{,u_t}$$

$$- u_x V^x_{,t} + u_{tt} V^u_{,u_t} - V^{u_t - u_t} V^t_{,t} - u_t V^t_{,t} - u_t V^{u_t}_{,u_t} + V^u_{,t} \lambda_1 u_t - \lambda_3 u_{tx} - \lambda_4 = 0;$$

$$-(u_x u_{tt} - u_t u_{tx})V^x_{,u_x} + u_{tt}V^u_{,u_x} - u_t V^{u_t}_{,u_x} + \lambda_2 u_t + \lambda_3 u_{tt} = 0;$$

$$-u_{tx}V^x_{,u_t} - u_x V^x_{,u} - u_{tt}V^t_{,u_t} - u_t V^t_{,u} + V^u_{,u} + V^{u_t}_{,u_t} - \lambda_1 = 0;$$

$$-u_{tx}V^x_{,u_x} - u_{tt}V^t_{,u_x} + V^{u_t}_{,u_x} - \lambda_2 = 0;$$

$$(u_x u_{tt} - u_t u_{tx})V^t_{,u_{tt}} + u_{tx}V^u_{,u_{tt}} - u_x V^{u_t}_{,u_{tt}} = 0;$$

$$(u_x u_{tt} - u_t u_{tx})V^t_{,u_x} + u_{tx}V^u_{u_x} - u_x V^{u_t}_{u_x} + \lambda_2 u_x + \lambda_3 u_{tx} - \lambda_4 = 0;$$

$$(u_x u_{tt} - u_t u_{tx})V^t_{,u_{tx}} + u_{tx}V^u_{,u_{tx}} - u_x V^{u_t}_{,u_{tx}} - \lambda_5 = 0;$$

$$(u_x u_{tt} - u_t u_{tx})V^t_{,u_{xx}} + u_{tx}V^u_{,u_{xx}} - u_x V^{u_t}_{,u_{xx}} - \lambda_6 = 0;$$

$$-(u_x u_{tt} - u_t u_{tx})V^x_{,u_{tt}} + u_{tt}V^u_{,u_{tt}} - u_t V^{u_t}_{,u_{tt}} - \lambda_5 = 0;$$

$$-(u_x u_{tt} - u_t u_{tx})V^x_{,u_{tx}} + u_{tt}V^u_{,u_{tx}} - u_t V^{u_t}_{,u_{tx}} - \lambda_6 = 0;$$

$$-(u_x u_{tt} - u_t u_{tx})V^x_{,u_{xx}} + u_{tt}V^u_{,u_{xx}} - u_t V^{u_t}_{,u_{xx}} + \lambda_7 = 0;$$

$$-u_{tx}V^x_{,u_{tt}} - u_{tt}V^t_{,u_{tt}} + V^{u_t}_{,u_{tt}} = 0;$$

$$-u_{tx}V^x_{,u_{tx}} - u_{tt}V^t_{,u_{tx}} + V^{u_t}_{,u_{tx}} = 0;$$

$$-u_{tx}V^x_{,u_{xx}} - u_{tt}V^t_{,u_{xx}} + V^{u_t}_{,u_{xx}} = 0;$$

$$u_x V^x_{,u_x} + u_t V^t_{,u_x} - V^u_{,u_x} - \lambda_3 = 0;$$

$$u_x V^x_{,u_{tx}} + u_t V^t_{,u_{tx}} - V^u_{,u_{tx}} = 0;$$

$$u_t V^t_{,u_{tt}} + u_x V^x_{,u_{tt}} - V^u_{,u_{tt}} = 0;$$

$$_x V^x_{,u_{xx}} + u_t V^t_{,u_{xx}} - V^u_{,u_{xx}} = 0.$$

Proof. First, expand the left-hand-side of (2.8) by using some simple features of Lie derivatives of differential forms:

$$\mathcal{L}_V x^i = V^i,\ \mathcal{L}_V(\omega_1 \wedge \omega_2) = (\mathcal{L}_V\omega_1)\wedge\omega_2 + \omega_1 \wedge (\mathcal{L}_V\omega_2),$$

$$\mathcal{L}_V dx^i = d(\mathcal{L}_V x^i) = dV^i, \tag{2.9}$$

where x^i is a coordinate of $\mathcal{N}$ and V^i is a component of V. Expanding the dV^i in the resulting expression of $\mathcal{L}_V\gamma^1$ by the usual chain rule (since the V^i are functions in $\mathcal{N}$), using all eight variables, some terms drop out. This is due to the fact that $dt \wedge dt = 0$, $dx \wedge dx = 0$, etc., by the antisymmetry of 1-forms and leads to

$$\begin{aligned}
\mathcal{L}_V\gamma^1 = {} & (V^{u_x}u_{tt} + u_x V^u_{tt} - V^{u_t}u_{tx} - u_t V^{u_{tx}} + (u_x u_{tt} - u_t u_{tx})V^x_{,x} + u_x u_{tt})dx \wedge dt \\
& - u_{tx}u_t V^t_{,t} dx \wedge dt + (u_{tx}V^u_{,t} - u_x V^{u_t}_{,t} - u_{tt}V^u_{,x} + u_t V^{u_t}_{,x})dx \wedge dt \\
& + \left((u_x u_{tt} - u_t u_{tx})V^t_{,u} + V^u_{,u_{tx}}\right)dx \wedge du + \left(u_{tx}V^x_{,x} + u_{tx}V^u_{,u} - u_x V^{u_t}_{,u}\right)dx \wedge du \\
& + \left(u_{tt}V^t_{,x} - V^{u_t}_{,x}\right)dx \wedge du + \left(V^u_{tt} - (u_x u_{tt} - u_t u_{tx})V^x_{,u} + u_{tx}V^x_{,t}\right)dt \wedge du \\
& + \left(u_{tt}V^t_{,t} + u_{tt}V^u_{,u} - u_t V^{u_t}_{,u} - V^{u_t}_{t}\right)dt \wedge du + \left((u_x u_{tt} - u_t u_{tx})V^t_{,u_t}\right)dx \wedge du_t \\
& - \left(V^{u_x} + u_{tx}V^u_{,u_t} - u_x V^x_{,x} - u_x V^{u_t}_{u_t} - u_t V^t_{,x} + V^u_{,x}\right)dx \wedge du_t + u_x u_{tt}V^t_{,u_x}dx \wedge du_x \\
& + \left(-u_t u_{tx}V^t_{,u_x} + u_{tx}V^u_{u_x} - u_x V^{u_t}_{u_x}\right)dx \wedge du_x - \left((u_x u_{tt} - u_t u_{tx})V^x_{,u_t}\right)dt \wedge du_t \\
& + \left(-u_x V^x_{,t} + u_{tt}V^u_{,u_t} - V^{u_t} - u_t V^t_{,t} - u_t V^{u_t}_{,u_t} + V^u_{,t}\right)dt \wedge du_t - u_x u_{tt}V^x_{,u_x}dt \wedge du_x \\
& + (u_t u_{tx}V^x_{,u_x} + u_{tt}V^u_{,u_x} - u_t V^{u_t}_{,u_x})dt \wedge du_x + \left(-u_{tx}V^x_{,u_t} - u_x V^x_{,u} - u_{tt}V^t_{,u_t}\right)du \wedge du_t \\
& + \left(-u_t V^t_{,u} + V^u_{,u} + V^{u_t}_{,u_t}\right)du \wedge du_t + \left(-u_{tx}V^x_{,u_x} - u_{tt}V^t_{,u_x} + V^{u_t}_{,u_x}\right)du \wedge du_x \\
& + \left((u_x u_{tt} - u_t u_{tx})V^t_{,u_{tt}} + u_{tx}V^u_{,u_{tt}} - u_x V^{u_t}_{,u_{tt}}\right)dx \wedge du_{tt} + u_x u_{tt}V^t_{,u_{tx}}dx \wedge du_{tx}
\end{aligned}$$

$$+\left(-u_t u_{tx} V^t_{,u_{tx}} + u_{tx} V^u_{,u_{tx}} - u_x V^{u_t}_{,u_{tx}}\right) dx \wedge du_{tx} + \left((u_x u_{tt} - u_t u_{tx}) V^t_{,u_{xx}}\right) dx \wedge du_{xx}$$

$$+\left(u_{tx} V^x_{,u_{xx}} - u_x V^{u_t}_{,u_{xx}}\right) dx \wedge du_{xx} + \left(-(u_x u_{tt} - u_t u_{tx}) V^x_{,u_{tt}} + u_{tt} V^u_{,u_{tt}}\right) dt \wedge du_{tt}$$

$$-u_t V^{u_t}_{,u_{tt}} dt \wedge du_{tt} + \left(-(u_x u_{tt} - u_t u_{tx}) V^x_{,u_{tx}} + u_{tt} V^u_{,u_{tx}} - u_t V^{u_t}_{,u_{tx}}\right) dt \wedge du_{tx}$$

$$+\left(-(u_x u_{tt} - u_t u_{tx}) V^x_{,u_{xx}} + u_{tt} V^u_{,u_{xx}} - u_t V^{u_t}_{,u_{xx}}\right) dt \wedge du_{xx} - u_{tx} V^x_{,u_{tt}} du \wedge du_{tt}$$

$$+\left(-u_{tt} V^t_{,u_{tt}} + V^{u_t}_{,u_{tt}}\right) du \wedge du_{tt} + \left(-u_{tx} V^x_{,u_{tx}} - u_{tt} V^t_{,u_{tx}} + V^{u_t}_{,u_{tx}}\right) du \wedge du_{tx}$$

$$+\left(-u_{tx} V^x_{,u_{xx}} - u_{tt} V^t_{,u_{xx}} + V^{u_t}_{,u_{xx}}\right) du \wedge du_{xx} + \left(u_x V^x_{,u_x} + u_t V^t_{,u_x} - V^u_{,u_x}\right) du_t \wedge du_x$$

$$+\left(u_x V^x_{,u_{tx}} + u_t V^t_{,u_{tx}} - V^u_{,u_{tx}}\right) du_t \wedge du_{tx} + \left(u_t V^t_{,u_{tt}} + u_x V^x_{,u_{tt}} - V^u_{,u_{tt}}\right) du_t \wedge du_{tt}$$

$$+\left(u_x V^x_{,u_{xx}} + u_t V^t_{,u_{xx}} - V^u_{,u_{xx}}\right) du_t \wedge du_{xx}.$$

And the right-hand-side of (2.8) is of the form

$$\sum_{j=1}^{7} \lambda_j \gamma^j = (\lambda_1(u_x u_{tt} - u_t u_{tx}) + \lambda_2(u_x u_{tx} - u_t u_{xx}) + \lambda_3(u_{tx}^2 - u_{tt} u_{xx})) dx \wedge dt$$

$$+ (\lambda_1 u_{tx} + \lambda_2 u_{xx} - \lambda_7) dx \wedge du + (\lambda_1 u_{tt} + \lambda_2 u_{tx} + \lambda_7 \delta u) dt \wedge du$$

$$+ (-\lambda_1 u_x + \lambda_3 u_{xx}) dx \wedge du_t + (-\lambda_2 u_x - \lambda_3 u_{tx} + \lambda_4) dx \wedge du_x$$

$$+ (-\lambda_1 u_t - \lambda_3 u_{tx} + \lambda_4) dt \wedge du_t$$

$$+ (-\lambda_2 u_t - \lambda_3 u_{tt}) dt \wedge du_x + \lambda_1 du \wedge du_t + \lambda_2 du \wedge du_x$$

$$+ \lambda_5 dx \wedge du_{tx} + \lambda_6 dx \wedge du_{xx}$$

$$+ \lambda_5\, dt \wedge du_{tt} + \lambda_6\, dt \wedge du_{tx} - \lambda_7\, dt \wedge du_{xx} + \lambda_3\, du_t \wedge du_x.$$

Equating the coefficients of basis 2-forms $(dx \wedge dt,\ dx \wedge du,\ dt \wedge du,\ du \wedge du_t,\ du_t \wedge du_x,\ etc.)$ in both right and left-hand-side of system (2.8), we get the system of Proposition (2.1). □

Now, we define vector fields on the manifold $\mathcal{N}$ by the following:

Lemma 2.3. *For the Gardner equation, the system of partial differential equations*

$$\mathcal{L}_V \gamma^i = \sum_{j=1}^{7} \lambda_j \gamma^j, \quad i = 1, 2, \ldots, 7, \tag{2.10}$$

defines the vector fields of the form

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial t},$$

$$V_3 = \left(\frac{1}{3}x + \frac{2}{3}\frac{\delta^2}{\sigma}t\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \left(\frac{1}{9}\frac{\delta}{\sigma} - \frac{1}{3}u\right)\frac{\partial}{\partial u}$$

$$- \frac{2}{3}u_x \frac{\partial}{\partial u_x} - \left(\frac{4}{3}u_t + \frac{2}{3}\frac{\delta^2}{\sigma}u_x\right)\frac{\partial}{\partial u_t} - \left(\frac{7}{3}u_{tt} + \frac{4}{3}\frac{\delta^2}{\sigma}u_{tx}\right)\frac{\partial}{\partial u_{tt}}$$

$$- \left(\frac{5}{3}u_{tx} + \frac{2}{3}\frac{\delta^2}{\sigma}u_{xx}\right)\frac{\partial}{\partial u_{tx}} - u_{xx}\frac{\partial}{\partial u_{xx}},$$

where the γ^i *are forms of Lemma* 2.2 *and the* λ_j *are* 0*-forms* (*functions*).

Proof. Write the Lie derivatives of all forms $\gamma^i (i = 1, \ldots, 7)$ as linear combinations of themselves, of the form (2.7). After expanding these Lie derivatives as proceed in Lemma 2.8 and eliminating the multipliers λ_i in (2.7), we get a system of PDEs, often called determining equations (see [7], [8]), of the following form:

$$-(2\delta u - 3\sigma u^2)V^t_{,x} + \cdots - V^t_{,t} - V^{u_{xx}}_{u_{xx}} = 0,$$

$$u_t V^u_{,u} - V^{u_t} + \cdots - u_{tx}V^x_{,t} + u_{tx}V^{u_t}_{,u_x} = 0,$$

$$\vdots$$

$$-u_{tx}V^x_{,t} + u_{tt}V^{u_t}_{,u_t} + \cdots - V^{u_{tt}} - u_{tt}V^t_{,t} = 0,$$

$$u_{tx}V^{u_x}_{,u_x} - u_{xx}V^x_{,t} - \cdots - V^{u_{tx}} = 0.$$

After integrating this system, one obtains the following general solution:

$$V^t = C_1 t + C_2, \quad V^x = \left(\frac{1}{3}x + \frac{2}{9}\frac{\delta^2}{\sigma}t\right)C_1 + C_3, \quad V^u = \left(\frac{1}{9}\frac{\delta}{\sigma} - \frac{1}{3}u\right)C_1,$$

$$V^{u_t} = -\left(\frac{4}{3}u_t + \frac{2}{3}\frac{\delta^2}{\sigma}u_x\right)C_1, \quad V^{u_x} = -\frac{2}{3}u_x C_1, \quad V^{u_{xx}} = -C_1 u_{xx},$$

$$V^{u_{tx}} = -\left(\frac{5}{3}u_{tx} + \frac{2}{3}\frac{\delta^2}{\sigma}u_{xx}\right)C_1, \quad V^{u_{tt}} = -\left(\frac{7}{3}u_{tt} + \frac{4}{3}\frac{\delta^2}{\sigma}u_{tx}\right)C_1,$$

where $C_1, C_2,$ and C_3 are arbitrary constants. This yields the three vector fields

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial t},$$

$$V_3 = \left(\frac{1}{3}x + \frac{2}{3}\frac{\delta^2}{\sigma}t\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \left(\frac{1}{9}\frac{\delta}{\sigma} - \frac{1}{3}u\right)\frac{\partial}{\partial u}$$

$$- \frac{2}{3}u_x\frac{\partial}{\partial u_x} - \left(\frac{4}{3}u_t + \frac{2}{3}\frac{\delta^2}{\sigma}u_x\right)\frac{\partial}{\partial u_t} - \left(\frac{7}{3}u_{tt} + \frac{4}{3}\frac{\delta^2}{\sigma}u_{tx}\right)\frac{\partial}{\partial u_{tt}}$$

$$- \left(\frac{5}{3}u_{tx} + \frac{2}{3}\frac{\delta^2}{\sigma}u_{xx}\right)\frac{\partial}{\partial u_{tx}} - u_{xx}\frac{\partial}{\partial u_{xx}}$$

of Lemma 2.3. □

Theorem 2.1. *The KdV-mKdV equation of type* (1.1) *allows a nontrivial symmetry group with the infinitesimal generators*:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t},$$

$$X_3 = \left(\frac{1}{3}x + \frac{2}{3}\frac{\delta^2}{\sigma}t\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \left(\frac{1}{9}\frac{\delta}{\sigma} - \frac{1}{3}u\right)\frac{\partial}{\partial u}.$$

Proof. Theorem 2.1 follows from Lemma 2.3. It is completed when relations

$$[V_i, V_j] = \sum_{k=1}^{j-1} C_{ij}^k V_k \text{ for } i < j, \tag{2.11}$$

are satisfied, where the C_{ij}^k are structures constants.

That is easily verifiable, since the non-vanishing brackets are

$$[V_1, V_3] = \frac{1}{3} V_1, \quad [V_2, V_3] = \frac{2}{3}\frac{\delta^2}{\sigma} V_1 + V_2,$$

for V_1, V_2, V_3 of Lemma 2.3. And then, it is easy to see that the third prolongations of X_1, X_2, X_3 vanish the KdV-mKdV equation (1.1).

The following section consists of testing the method discussed above with the classical Lie method.

3. Lie Symmetry Approach

The Lie approach is a general and a firm method for finding symmetries of differential equations.

A partial differential equation (PDE) with n independent variables $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and m dependent variables $u = (u^1, \ldots, u^m) \in \mathbb{R}^m$ has a Lie point transformations

$$\tilde{x}_i = x_i + \epsilon\xi_i(x, u) + O(\epsilon^2), \quad \tilde{u}^\alpha = u^\alpha + \epsilon\eta_\alpha(x, u) + O(\epsilon^2),$$

where $\xi_i = \frac{d\tilde{x}_i}{d\epsilon}|_{\epsilon=0}$, for $i = 1, \ldots, n$ and $\eta_\alpha = \frac{d\tilde{u}^\alpha}{d\epsilon}|_{\epsilon=0}$, for $\alpha = 1, \ldots, m$. The associated infinitesimal generator of these transformations can be written as follows:

$$X = \sum_{i=1}^{n} \xi_i(x, u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \eta_\alpha(x, u)\frac{\partial}{\partial u^\alpha}, \tag{3.1}$$

on the space containing independent and dependent variables. Furthermore, the k-th prolongation of the above vector field is given by

$$\mathrm{Pr}^{(k)} X = X + \sum_{\alpha=1}^{m} \sum_{J} \eta_{\alpha}^{J}(x, u^{(k)}) \frac{\partial}{\partial u_{J}^{\alpha}}, \tag{3.2}$$

where

$$\eta_{\alpha}^{J}(x, u^{(k)}) = D_{J}\left(\eta_{\alpha} - \sum_{i=1}^{n} \xi^{i} u_{i}^{\alpha}\right) + \sum_{i=1}^{n} \xi^{i} u_{J,i}^{\alpha}. \tag{3.3}$$

With $u_{i}^{\alpha} = \dfrac{\partial u^{\alpha}}{\partial x_{i}}$ and $u_{J,i}^{\alpha} = \dfrac{\partial u_{J}^{\alpha}}{\partial x_{i}}$; $D_{J} := D_{j_1} \dots D_{j_k}$; the second summation being over all (unordered) multiindices $J = (j_1, \dots, j_l)$, with $1 \leqslant j_l \leqslant n$, for $1 \leqslant l \leqslant k$; $D_{J} := D_{j_1} \dots D_{j_k}$; the D_{j_r}'s represent the operators of total derivative.

The consideration that, under the action of a Lie group of transformations admitted by a differential equation, a solution, which is not invariant with respect to the group, is mapped into a family of solutions, suggests a way of generating new solutions from a known solution. This is especially interesting when one can obtain nontrivial solution from trivial ones.

Let us consider a one-parameter Lie group of transformations, given in coordinates by

$$\tilde{x} = \Phi(x, u; \epsilon), \quad \tilde{u} = \Upsilon(x, u; \epsilon) \tag{3.4}$$

admitted by a system of differential equations $\mathcal{S}$, and let $u = f(x)$ be a solution of the given system $\mathcal{S}$, which is not invariant with respect to the group (3.4).

Theorem 3.1 ([9]). *If* $u = f(x)$ *is not an invariant solution of a system* $\mathcal{S}$ *of differential equations, admitting the group* (3.4), *then*

$$u = \Upsilon(\Phi(x, u; \epsilon), \Upsilon(\Phi(x, u; \epsilon); -\epsilon)) \tag{3.5}$$

implicitly defines a one-parameter family of solutions of the given system.

For specifying the symmetry algebra, we think the algebra generator as follows:

$$X = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial t} + \eta(x, t, u)\frac{\partial}{\partial u}. \tag{3.6}$$

According to the Equation (1.1), we need to know the third prolongation

$$\mathrm{Pr}^{(3)} X = X + \eta^{x} \frac{\partial}{\partial u_x} + \eta^{t} \frac{\partial}{\partial u_t} + \eta^{xx} \frac{\partial}{\partial u_{xx}} + \eta^{xt} \frac{\partial}{\partial u_{xt}} + \eta^{tt} \frac{\partial}{\partial u_{tt}}$$

$$+ \eta^{xxx} \frac{\partial}{\partial u_{xxx}} + \eta^{xxt} \frac{\partial}{\partial u_{xxt}} + \eta^{ttx} \frac{\partial}{\partial u_{ttx}} + \eta^{ttt} \frac{\partial}{\partial u_{ttt}} \tag{3.7}$$

of X. By applying $\mathrm{Pr}^{(3)} X$ on this equation and vanishing, where u is the solution of the Gardner equation, we find the following equation:

$$9\sigma\xi_t = 2\delta^2\tau_t, \ \xi_u = 0, \ 3\xi_x = \tau_t, \ \tau_u = 0, \ \tau_x = 0, \ \tau_{tt} = 0, \ 9\sigma\eta = (-3\sigma u + \delta)\tau_t.$$

(As usual, subscripts indicate derivatives.) These all have the general solution

$$\tau = C_1 t + C_2, \ \xi = \left(\frac{1}{3}x + \frac{2}{9}\frac{\delta^2}{\sigma}t\right)C_1 + C_3, \quad \eta = \left(\frac{1}{9}\frac{\delta}{\sigma} - \frac{1}{3}u\right)C_1; \tag{3.8}$$

where C_1, C_2, C_3 are arbitrary constants. We see that this solution (from (3.8) are exactly the same sets of solutions that the Harrison method gives and so from here on the calculations are identical.

The aim of the next subsection is to give a generalized form of solutions to the Gardner equation. The solutions can be generalized by exponentiating the basis elements $V_i (i = 1, 2, 3)$ of the algebra of the symmetry group of this equation.

3.1. Determination of groups G_i associated with generators X_i

The determination of the group of transformations corresponding to each generator is equivalent to solving the following first order system of differential equations:

$$\frac{d\tilde{t}}{d\epsilon} = \tau(\tilde{x}, \tilde{t}, \tilde{u}), \quad \frac{d\tilde{x}}{d\epsilon} = \xi(\tilde{x}, \tilde{t}, \tilde{u}), \quad \frac{d\tilde{u}}{d\epsilon} = \eta(\tilde{x}, \tilde{t}, \tilde{u}), \tag{3.9}$$

with the initial conditions

$$\tilde{t}(0) = t, \quad \tilde{x}(0) = x, \quad \tilde{u}(0) = u. \tag{3.10}$$

3.1.1. Application to the generators of the symmetry group of the Gardner equation

(1) For the generators $X_1 = \frac{\partial}{\partial x}$ and $X_2 = \frac{\partial}{\partial t}$, we have to solve the systems

$$\frac{d\tilde{t}}{d\epsilon_1} = \tau(\tilde{x}, \tilde{t}, \tilde{u}) = 0, \quad \frac{d\tilde{x}}{d\epsilon_1} = \xi(\tilde{x}, \tilde{t}, \tilde{u}) = 1, \quad \frac{d\tilde{u}}{d\epsilon_1} = \eta(\tilde{x}, \tilde{t}, \tilde{u}) = 0, \tag{3.11}$$

and

$$\frac{d\tilde{t}}{d\epsilon_2} = \tau(\tilde{x}, \tilde{t}, \tilde{u}) = 1, \quad \frac{d\tilde{x}}{d\epsilon_2} = \xi(\tilde{x}, \tilde{t}, \tilde{u}) = 0, \quad \frac{d\tilde{u}}{d\epsilon_2} = \eta(\tilde{x}, \tilde{t}, \tilde{u}) = 0. \tag{3.12}$$

From the system (3.12), we obtain

$$(\tilde{t} = \alpha_1, \tilde{x} = \epsilon_1 + \alpha_2, \tilde{u} = \alpha_3), \tag{3.13}$$

and the system (3.13) yields

$$(\tilde{t} = \epsilon_2 + \beta_1, \tilde{x} = \beta_2, \tilde{u} = \beta_3), \tag{3.14}$$

where $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ are arbitrary constants. By utilizing the initial conditions (3.10), the solutions (3.13) and (3.14) become

$$(\tilde{x} = \epsilon_1 + x,\ \tilde{t} = t,\ \tilde{u} = u) \text{ and } (\tilde{x} = x,\ \tilde{t} = \epsilon_2 + t,\ \tilde{u} = u), \tag{3.15}$$

which are space and time translations, respectively.

(2) For the generator $X_3 = \left(\frac{1}{3}x + \frac{2}{3}\frac{\delta^2}{\sigma}t\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \left(\frac{1}{9}\frac{\delta}{\sigma} - \frac{1}{3}u\right)\frac{\partial}{\partial u}$,

the system (3.10) reads

$$\frac{d\tilde{t}}{d\epsilon_3} = \tau(\tilde{x},\ \tilde{t},\ \tilde{u}) = \tilde{t}, \tag{3.16}$$

$$\frac{d\tilde{x}}{d\epsilon_3} = \xi(\tilde{x},\ \tilde{t},\ \tilde{u}) = \frac{1}{3}\tilde{x} + \frac{2\delta^2}{3\sigma}\tilde{t}, \tag{3.17}$$

$$\frac{d\tilde{u}}{d\epsilon_3} = \eta(\tilde{x},\ \tilde{t},\ \tilde{u}) = \frac{\delta}{9\sigma} - \frac{1}{3}\tilde{u}. \tag{3.18}$$

(a) From the Equation (3.17), we get $\tilde{t} = C_1 \exp(\epsilon_3)$ and the initial condition $\tilde{t}(0) = t$ gives

$$\tilde{t} = t\exp(\epsilon_3).$$

(b) The homogeneous equation corresponding to the Equation (3.17) is $\frac{d\tilde{x}}{d\epsilon_3} = \frac{1}{3}\tilde{x}$. The solution of the latter is $\tilde{x} = C\exp(\frac{1}{3}\epsilon_3)$. By variation of the parameter C, we set $\tilde{x} = p(\epsilon_3)\exp(\frac{1}{3}\epsilon_3)$, substitute it in (3.17) and obtain

$$\frac{\partial p}{\partial \epsilon_3}\exp(\frac{1}{3}\epsilon_3) = \frac{2\delta^2}{3\sigma}\tilde{t} = \frac{2\delta^2}{3\sigma}t\exp(\epsilon_3),$$

whence $p(\epsilon_3) = C_2 + \frac{\delta^2}{\sigma}t\exp(\frac{2\epsilon_3}{3})$. Therefore, $\tilde{x} = C_2\exp(\frac{\epsilon_3}{3}) + \frac{\delta^2}{\sigma}t\exp(\epsilon_3)$.

In accordance with the initial condition $\tilde{x}(0) = x$, we have

$$\tilde{x} = \frac{\delta^2}{\sigma}t\exp(\epsilon_3) + x\exp\left(\frac{\epsilon_3}{3}\right) - \frac{\delta^2}{\sigma}t\exp\left(\frac{4\epsilon_3}{3}\right).$$

(c) The homogeneous equation $\frac{d\tilde{u}}{d\epsilon_3} = -\frac{1}{3}\tilde{u}$ associated to the Equation (3.18) yields $\tilde{u} = k\exp(\frac{-\epsilon_3}{3})$. By variation of the parameter k, we set $\tilde{u} = q(\epsilon_3)\exp(\frac{-\epsilon_3}{3})$, substitute it in Equation (3.18) and integrating the resulting equation we obtain: $q(\epsilon_3) = C_3 + \frac{\delta}{3\sigma}\exp(\frac{\epsilon_3}{3})$. Thus $\tilde{u} = C_3\exp(\frac{-\epsilon_3}{3}) + \frac{\delta}{3\sigma}$, where C_3 is an arbitrary constant. By considering the initial condition $\tilde{u}(0) = u$, we have the solution of the non-homogeneous equation (3.18) as follows

$$\tilde{u} = u\exp\left(\frac{-\epsilon_3}{3}\right) + \frac{\delta}{3\sigma}.$$

Then, we find that the group of transformations generated by X_3 is given by

$$(\tilde{x}, \tilde{t}, \tilde{u}) = \left(\frac{\delta^2}{\sigma}t\exp(\epsilon_3) + x\exp\left(\frac{\epsilon_3}{3}\right) - \frac{\delta^2}{\sigma}t\exp\left(\frac{4\epsilon_3}{3}\right), t\exp(\epsilon_3), u\exp\left(\frac{-\epsilon_3}{3}\right) + \frac{\delta}{3\sigma}\right),$$

which is also the symmetry group of the Gardner equation (1.1).

3.2. Generalized form of solutions to the Gardner equation

In summary, solving the flow equations associated to the basis elements $V_i (i = 1, 2, 3)$ of the symmetry algebra of Equation (1.1), we have found

$$\exp(\epsilon_1 V_1)(x, t, u) = (\epsilon_1 + x, t, u),$$

$$\exp(\epsilon_2 V_2)(x, t, u) = (x, \epsilon_2 + t, u),$$

$$\exp(\epsilon_3 V_3)(x, t, u) = (\frac{\delta^2}{\sigma}t\exp(\epsilon_3) + x\exp\left(\frac{\epsilon_3}{3}\right) - \frac{\delta^2}{\sigma}t\exp\left(\frac{4\epsilon_3}{3}\right), t\exp(\epsilon_3),$$

$$u\exp\left(\frac{-\epsilon_3}{3}\right) + \frac{\delta}{3\sigma}).$$

Then, the most general transformation leaving invariant the solution manifold of the Equation (1.1) depends on three parameters: $\epsilon_1, \epsilon_2, \epsilon_3$ through the composition

$$g := \exp(\epsilon_1 V_1)\exp(\epsilon_2 V_2)\exp(\epsilon_3 V_3) \tag{3.19}$$

of the flows $\exp(\epsilon_i V_i)$, $i = 1, 2, 3$, $\epsilon_i \in \mathbb{R}$, generated by the basis elements V_i of the symmetry algebra of Equation (1.1). Thus, the most general transformation g operates according to

$$g(x, t, u) = \{\epsilon_1 + \frac{\delta^2}{\sigma} t \exp(\epsilon_3) + x \exp\left(\frac{\epsilon_3}{3}\right) - \frac{\delta^2}{\sigma} t \exp\left(\frac{4\epsilon_3}{3}\right), \epsilon_2 + t \exp(\epsilon_3),$$

$$\exp\left(\frac{-\epsilon_3}{3}\right)u + \frac{\delta}{3\sigma}\}. \tag{3.20}$$

Proposition 3.1. *The most general solution of the Gardner equation* (1.1) *obtained from a given solution* $u = f(x, t)$, *is of the form*

$$u(x, t) = \exp\left(\frac{-\epsilon_3}{3}\right) f\{x \exp\left(\frac{-\epsilon_3}{3}\right) + \left(1 - \exp\left(\frac{-\epsilon_3}{3}\right)\right)\frac{\delta^2}{\sigma} t + \frac{\delta^2}{\sigma}\epsilon_2 \exp\left(\frac{-\epsilon_3}{3}\right)$$

$$- \epsilon_2 \frac{\delta^2}{\sigma} - \epsilon_1 \exp\left(\frac{-\epsilon_3}{3}\right), t\exp(-\epsilon_3) - \epsilon_2 \exp(-\epsilon_3)\} + \frac{\delta}{3\sigma}. \tag{3.21}$$

Proof. The proof of this theorem is provided by utilizing the general transformation (3.20) and applying the Theorem 3.1.

4. Conclusion

We see that differential forms offer, in some ways, a more natural way of calculating symmetries of differential equations. With this technique, we need only calculate prolongation coefficients up to second order and hence, obtain the prolongation of the generator V. This leads naturally to the point symmetry algebra in study.

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References

[1] B. Kent Harrison and F. B. Estabrook, Geometric approach to invariance groups and solution of partial differential systems, J. Math. Phys. 12 (1971), 653-666.

[2] B. Kent Harrison, The differential form method for finding symmetries, SIGMA 1 (2005), Paper 001, 12 pages.

[3] B. Kent Harrison, Differential form symmetry analysis of two equations cited by Fushchych, Department of Physics and Astronomy, Brigham Young University, Provo, Utah 84602 U.S.A. 1 (1997), 21-33.

[4] Abdul-Majid Wazwaz, Partial Differential Equations and Solitary Waves Theory, Nonlinear Physical Sciences, Springer, USA, 2009.

[5] Mehdi Nadjafikhah and Seyed-Reza Hejazi, Lie symmetries and solutions of KdV equation, Department of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran, International Mathematical Forum 4(4) (2009), 165-176.

[6] C. A. G. Ìomez Sierra, M. Molati and M. P. Ramollo, Exact solutions of a generalized KdV-mKdV equation, International Journal of Nonlinear Science 13(1) (2012), 94-98.

[7] G. W. Bluman and S. Kumei, Symmetries and Differential Equations, Springer Verlag, New York, 1989.

[8] P. J. Olver, Applications of Lie Groups to Differential Equations, Second Edition, GTM, Vol. 107, Springer Verlag, New York, 1993.

[9] Francesco Oliveri, Lie symmetries of differential equations: Classical results and recent contributions, Department of Mathematics, University of Messina, Contrada Papardo, Viale Ferdinando Stagno d'Alcontres 31, I-98166 Messina, Italy, Symmetry 2 (2010), 658-706; doi:10.3390/sym2020658.

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