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THE MULTIPLICATION ALGEBRA OF THE DUPLICATE OF AN ALGEBRA

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Abstract

In this paper, we investigate on the structure of the multiplication algebra of the duplicate of an algebra. For Bernstein algebras, the structure is described using Peirce decomposition.

1. Introduction

In this paper, K is an infinite commutative field of characteristic different from 2 and A is a commutative non-associative K-algebra. We say that (A, ω) is a baric K-algebra if ω, called weight morphism is a nonzero morphism of algebras from A to K.

For any element x of A, the principal powers are defined by $x^1 = x$, $x^{k+1} = xx^k$, $\forall k \geq 1$.

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A nonzero element e of A is an *idempotent* if $e^2 = e$. Whenever the term idempotent is used in this paper, it is nonzero idempotent.

Let $\mathcal{E}nd_k(A)$ be the associative algebra of endomorphisms of A (as a vector space). For any element x of A, we call *right* (respectively, *left multiplication*) by x, the endomorphism R_x (respectively, L_x) of A defined by $R_x(y) = yx$ (respectively, $L_x(y) = xy$), for any $y \in A$. All multiplications of A generates a subalgebra of $\mathcal{E}nd_k(A)$ denoted by $\mathcal{M}_k(A)$ or simply $\mathcal{M}(A)$.

A baric (A, ω) is a Bernstein algebra if $x^2x^2 = \omega(x)^2x^2$, $\forall x \in A$. Several authors have studied the multiplication algebra of a baric algebra. In particular, the multiplication algebra of a Bernstein algebra has been the subject of some publications ([1], [2]). The present paper is devoted to the study of the multiplication algebra of commutative duplicate of a baric algebra.

2. Basic Results

Considering the action defined on A by $\sigma.x = \sigma(x)$, for any σ in $\mathcal{M}(A)$ and any x in A, it is clear that A is an $\mathcal{M}(A)$-left module. In finite dimension, $\dim \mathcal{M}(A) \leq (\dim A)^2$.

The left ideals of A are none other than the $\mathcal{M}(A)$-left module of A. If I is an ideal of A, $(I : A) = \{\sigma \in \mathcal{M}(A) | \sigma(A) \subset I\}$ is an ideal of $\mathcal{M}(A)$. Conversely, if I is an ideal of $\mathcal{M}(A)$, $I(A) = \{\sigma(x) | \sigma \in I, x \in A\}$ is an ideal of A (see [6]). In ([2]), the authors establish the following result.

Proposition 2.1 ([2]). *Let* ω *be a weight morphism and* e *be an idempotent of* A. *We have*:

(i) $\mathcal{M}(A) = KL_e \oplus (N : A)$.

(ii) *The map* $\overline{\omega} : \mathcal{M}(A) \to K$, *defined by* $\overline{\omega}(\alpha L_e + \theta) = \alpha$, *is a weight morphism of* $\mathcal{M}(A)$ *called canonical extension of* ω *to* $\mathcal{M}(A)$.

Notations. Let $\widetilde{N}$ be the subalgebra of $\mathcal{M}(A)$ generated by elements of the form $L_{x_1} \dots L_{x_n}$ such that at least x_i is in N. This subalgebra $\widetilde{N}$ is an ideal of $(N : A) = ker\omega$.

We define in the second symmetric power $S_K^2(A)$ of the K-module A, a multiplication by $(x.y)(x'.y') = xy.x'y'$. This gives a commutative K-algebra called commutative duplicate of the algebra A, denoted $D(A)$. The K-linear map $\mu : D(A) \to A^2$, $x.y \mapsto xy$ is a surjective morphism of K-algebras called *Etherington morphism*. Let $N(A) = ker\omega$. If A is a K-algebra such that A^2 is baric, $D(A)$ is baric. In fact, a weight morphism of $D(A)$ is given by $\omega_d = \omega o \mu$, where $\omega : A^2 \to K$ is a weight morphism of A^2.

For any x, y, x' and y' in A, we have $L_{x.y}(x'.y') = xy.x'y'$ and $(\mu o L_{x.y})(x'.y') = (xy)(x'y') = (\ell_{xy} o \mu)(x'.y')$, where

$L_{x.y}$ denotes the left multiplication by $x.y$ in $D(A)$ and ℓ_{xy} the right multiplication by xy in A^2. The following diagram is commutative:

$$\begin{array}{ccc} D(A) & \xrightarrow{L_{x.y}} & D(A) \\ \mu \downarrow & & \downarrow \mu \\ A^2 & \xrightarrow{\ell_{xy}} & A^2 \end{array}$$

Proposition 2.2. *The Etherington morphism* $\mu : D(A) \to A^2$ *extends naturally to a morphism of multiplication algebras given by* $\mu_m : \mathcal{M}(D(A)) \to \mathcal{M}(A^2)$, $L_{x.y} \mapsto \ell_{xy}$.

Proof. Indeed, $(L_{x_1.y_1} o L_{x_2.y_2})(x'.y') = x_1x_2.((x_2y_2)(x'y'))$ and $(\mu o L_{x_1.y_1} o L_{x_2.y_2})(x'.y') = (x_1y_1)((x_2y_2)(x'y')) = (\ell_{x_1y_1} o \ell_{x_2y_2} o \mu)(x'.y')$ for all $x_1.y_1$, $x_2.y_2$ and $x'.y'$ in $D(A)$. Hence $\mu_m(L_{x_1.y_1} o L_{x_2.y_2}) = \mu_m(L_{x_1.y_1}) o \mu_m(L_{x_2.y_2})$ and μ_m is a morphism of algebras. □

Since μ is surjective, then μ_m is also. Hence the following result.

Proposition 2.3. *The morphism of K-algebras* $\mu_m : \mathcal{M}(D(A)) \to \mathcal{M}(A^2)$, $L_{x.y} \mapsto \ell_{xy}$ *is surjective. Thus, it has* $\mathcal{M}(D(A)/ker\mu_m \simeq \mathcal{M}(A^2)$ *with* $ker\mu_m = \{\sigma_d \in \mathcal{M}(D(A)) | \mu_m = (\sigma_d) = 0\}$.

Lemma 2.4. *Let* $\sigma_d \in \mathcal{M}(D(A))$ *and* $\sigma = \mu_m(\sigma_d)$. *The following diagram is commutative*:

$$\begin{array}{ccc} D(A) & \xrightarrow{\sigma_d} & D(A) \\ \mu\downarrow & & \downarrow\mu \\ A^2 & \xrightarrow{\sigma} & A^2 \end{array}$$

Proof. Let $\sigma_d = \sum_{finie} L_{x_1.y_1} o L_{x_2.y_2} o \cdots o L_{x_k.y_k}$. We have $\mu_m(\sigma_d) = \sum_{finie} \ell_{x_1y_1} o \ell_{x_2y_2} o \cdots o \ell_{x_ky_k}$ and for any $x.y$ in $D(A)$,

$$\begin{aligned} \mu(\sigma_d(x.y)) &= \mu(\sum_{finie} x_1y_1.(x_2y_2)(((x_3y_3)(\cdots((x_ky_k)(xy)))\cdots)) \\ &= \sum_{finie} (x_1y_1)((x_2y_2)(((x_3y_3)(\cdots((x_ky_k)(xy)))\cdots)) \\ &= \sum_{finie} (\ell_{x_1y_1} o \ell_{x_2y_2} o \cdots o \ell_{x_ky_k})(xy) \\ &= \sum_{finie} (\ell_{x_1y_1} o \ell_{x_2y_2} o \cdots o \ell_{x_ky_k} o \mu)(x.y) \\ &= \sigma(\mu(x.y)), \end{aligned}$$

so $\mu o \sigma_d = \sigma o \mu$ and the diagram is commutative. □

Remark. For any $x.y$ in $D(A)$, $\mu_m(\sigma_d)(xy) = \mu(\sigma_d(x.y))$, thus $\sigma_d \in Ker\mu_m$ is equivalent to $\sigma_d(x.y) \in N(A)$, i.e., $\sigma_d \in (N(A) : D(A))$, so $Ker\mu_m = (N(A) : D(A))$.

Corollary 2.5. *If* A^2 *is a projective K-module, then we have*

$$\mathcal{M}(D(A)) \simeq \mathcal{M}(A^2) \times_{s.d} (N(A) : D(A)).$$

Proof. The sequence

$$0 \to (N(A) : D(A)) \xrightarrow{i_m} \mathcal{M}(D(A)) \xrightarrow{\mu_m} \mathcal{M}(A^2) \to 0$$

being exact, show that it is split. As A^2 is a projective K-module, it exists $\eta : A^2 \to D(A)$ such that $\mu o \eta = 1_{A^2}$. Let $\eta_m : \mathcal{M}(A^2) \to \mathcal{M}\ (D(A))$ be the K-linear map defined by $\eta_m(\sigma)(x.y) = \eta(\sigma(xy))$ for any σ in $\mathcal{M}(A^2)$ and for any $x.y$ in $D(A)$. We have $((\mu_m o \eta_m)(\sigma)(xy) = \mu(\eta_m(\sigma)(x.y)) = \mu(\eta(\sigma(xy))) = \sigma(xy)$, so $\mu_m(\eta_m(\sigma)) = \sigma$, i.e., $\mu_m o \eta_m = 1_{\mathcal{M}(A^2)}$ and the sequence is split. Therefore, $\mathcal{M}(D(A)) \simeq \mathcal{M}(A^2) \underset{s.d}{\times} (N(A) : D(A))$.

□

Theorem 2.6. *If* A^2 *is a projective K-module,* $(N(A) : D(A))$ *is an annihilator of* $\mathcal{M}(D(A))$ *and for any derivation d of* $\mathcal{M}(D(A))$, $d((N(A) : D(A)))$ *is contained in* $(N(A) : D(A))$.

Proof. Let $\sigma_d \in \mathcal{M}(D(A))$ and $\sigma' \in (N(A) : D(A))$. For any $x.y \in D(A)$, $\sigma'(x.y) \in N(A)$ and posing $\sigma = \sum_{finie} \ell_{Z_1} o \ell_{Z_2} o \cdots o \ell_{Z_k}$, $Z_i \in D(A)$, we have $\sigma(\sigma'(x.y)) = 0$ because $D(A)N(A) = 0$, i.e., $\sigma o \sigma' = 0$ and σ' is contained in the right annihilator of $\mathcal{M}(D(A))$.

Conversely, if σ'' is in the right annihilator of $\mathcal{M}(D(A))$, for any $x.y \in D(A)$, we have $0 = L_{e.e}(\sigma''(x.y)) = e.\mu(\sigma''(x.y))$, which implies that $\mu(\sigma''(x.y)) = 0$ because A^2 is a projective K-module. So $\sigma''(x.y) \in N(A)$ and $\sigma'' \in (N(A) : D(A))$. Let now d be a derivation of $\mathcal{M}(D(A))$. For any σ in $\mathcal{M}(D(A))$ and any σ' in $(N(A) : D(A))$, we have $0 = d(\sigma o \sigma') = d\sigma o \sigma' + \sigma o d\sigma' = \sigma o d\sigma'$, so $d\sigma'$ is contained in the right annihilator of $\mathcal{M}(D(A))$. □

Theorem 2.7. *Suppose* A^2 *is a projective K-module and consider the map* $\varphi : D(A) \to \mathcal{M}(D(A))$, $z \mapsto L_z$ *and* $\theta : A^2 \to \mathcal{M}(A^2)$, $x \mapsto \ell_x$. *The following diagram is commutative*:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N(A) & \xrightarrow{i} & D(A) & \xrightarrow{\mu} & A^2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\theta} & & \\
0 & \longrightarrow & (N(A) : D(A)) & \xrightarrow{i_m} & \mathcal{M}(D(A)) & \xrightarrow{\mu_m} & \mathcal{M}(A^2) & \longrightarrow & 0
\end{array}$$

Proof. For any $x.y \in N(A)$, $\varphi(i(x.y)) = \varphi(x.y)$ and $i_m(\varphi(x.y)) = \varphi(x.y)$, so $\varphi o i = i_m o \varphi$. Also, for every $x.y \in D(A)$, $\theta(\mu(x.y)) = \theta(xy) = \ell_{xy}$ and $\mu_m(\varphi(x.y)) = \mu_m(L_{x.y}) = \ell_{xy}$, so $\theta o \mu = \mu_m o \varphi$. It follows that the diagram is commutative. □

The next result concerns the functor $\mathcal{M}$.

Theorem 2.8. *Let* $\mathcal{C}$ *the category of K-algebras and* $\mathcal{D}$ *the category of multiplication K-algebras. Let* $\mathcal{M} : \mathcal{C} \to \mathcal{D}$, $u \in Hom_{\mathcal{C}}(A, B) \mapsto \mathcal{M}(u)$ *defined by* $\mathcal{M}(u)(\sum_{finie} L_{x_1} o L_{x_2} o \cdots L_{x_k}) = \sum_{finie} L_{u(x_1)} o L_{u(x_2)} o \cdots o L_{u(x_k)}$ *for any* $x_i \in A$. *Then* $\mathcal{M}$ *is a covariant functor.*

Proof. Indeed, $\forall u \in Hom_{\mathcal{C}}(A, B)$, $\mathcal{M}(u) \in Hom_{\mathcal{C}}(\mathcal{M}(A), \mathcal{M}(B))$ and $\forall A \in \mathcal{C}$, $\mathcal{M}(1_A) = 1_{\mathcal{M}(A)}$. Furthermore, if $u \in Hom_{\mathcal{C}}(A, B)$ and $v \in Hom_{\mathcal{C}}(B, C)$, we have

$$\mathcal{M}(vou)(\sum_{finie} L_{x_1} o L_{x_2} o \cdots L_{x_k}) = \sum_{finie} L_{v(u(x_1))} o L_{v(u(x_2))} o \cdots o L_{v(u(x_k))}$$

$$= \mathcal{M}(v)(\sum_{finie} L_{u(x_1)} o L_{u(x_2)} o \cdots o L_{u(x_k)})$$

$$= \mathcal{M}(v)(\mathcal{M}(u)(\sum_{finie} L_{x_1} o L_{x_2} o \cdots L_{x_k}))$$

$$= (\mathcal{M}(v) o \mathcal{M}(u))(\sum_{finie} L_{x_1} o L_{x_2} o \cdots L_{x_k}),$$

so $\mathcal{M}(vou) = \mathcal{M}(v) o \mathcal{M}(u)$ and $\mathcal{M}$ is a covariant functor. In particular, if $\mu : D(A) \to A^2$ is Etherington morphism, then $\mathcal{M}(\mu) = \mu_m$. □

Remark. The map $\overline{\omega}_d : \mathcal{M}(D(A)) \to K$ defined by $\overline{\omega}_d(\alpha L_{e.e} + \theta) = \alpha$ for any α in K and for any θ in $(N_d : D(A))$, is a weight morphism of $\mathcal{M}(D(A))$.

We have $\overline{\omega}_d(L_{x.y}) = \omega_d(x.y) = \overline{\omega}(\ell_{x.y})$, so $\overline{\omega}_d(\sigma_d)) = \overline{\omega}(\mu_m(\sigma_d))$, where ω is a weight morphism of A^2. We have also $(N_d : D(A)) = ker\overline{\omega}_d$.

Let $\overline{N}_d$ be the subalgebra of $\mathcal{M}(D(A))$ contained in $(N_d : D(A))$, generated by the elements of the form $L_{z_1} \circ L_{z_2} \circ \cdots \circ L_{z_k}$ such that at least z_i be in N_d. It is clear that $\overline{N}_d$ is an ideal of $\mathcal{M}(D(A))$ included in $(N_d : D(A))$.

Proposition 2.9. *Let* (A, ω) *be a baric K-algebra. We have* $\mu_m((N_d : D(A))) = (N : A^2)$ *and* $\mu_m(\overline{N}_d) = \overline{N}$, *where* $N = ker\omega_{|A^2}$.

Proof. Let σ_d in $(N_d : D(A))$, that is to say $\omega_d \circ \sigma_d = 0$ or $\omega \circ \mu \circ \sigma_d = 0$, so $(\mu \circ \sigma_d)(D(A)) \subset N$ and $\mu_m(\sigma_d)(A^2) \subset N$ because $\mu \circ \sigma_d = \mu_m(\sigma_d) \circ \mu$. Therefore $\mu_m(\sigma_d) \in (N : A^2)$. Let $\sigma \in (N : A^2)$.

Since $\mu_m : \mathcal{M}(D(A)) \to \mathcal{M}(A^2)$ is subjective, it exists $\sigma_d = \alpha L_{e.e} + \theta$ in $\mathcal{M}(D(A))$, α in K, θ in $(N_d : D(A))$ such that $\mu_m(\sigma_d) = \sigma$, that is to say $\alpha \ell_{e^2 + \mu_m(\theta)} = \sigma$, so $\mu_m(\theta) = \sigma$ and $\sigma \in \mu_m((N_d : D(A)))$. Thus $(N : A^2)\sigma \in \mu_m((N_d : D(A)))$ and $\sigma \in \mu_m((N_d : D(A))) = (N : A^2)$. Let $L_{z_1} \circ L_{z_2} \circ \cdots \circ L_{z_k}$ be a generator of $\overline{N}_d$. We have $\mu_m(L_{z_1} \circ L_{z_2} \circ \cdots \circ L_{z_k})$ $= \ell_{\mu(z_1)} \circ \ell_{\mu(z_2)} \circ \cdots \circ \ell_{\mu(z_k)} \in \overline{N}$, so $\mu_m(\overline{N}_d) \subset \overline{N}$. Reciprocal inclusion results from the surjectivity of μ_m and μ. □

The following result is a direct consequence of the previous Proposition 2.9.

Corollary 2.10. *Let* (A, ω) *be a baric K-algebra and* $N = ker\omega_{|A^2}$. *The ideal* $\overline{N}_d$ *is nilpotent if and only if* $\overline{N}$ *is nilpotent.*

Thus we have the following result:

Proposition 2.11. *Let* (A, ω) *be a baric K-algebra and* I_d *an ideal of* $D(A)$. *Then* $\mu_m((I_d : D(A))) \subset (\mu(I_d) : A^2)$.

Proof. Let I_d be an ideal of $D(A)$. Then $\mu(I_d)$ is an ideal of $A^2, (I_d : D(A))$ and $(\mu(I_d) : A^2)$ are, respectively, the ideals of $\mathcal{M}(D(A))$ and $\mathcal{M}(A^2)$. The following commutative diagram gives us $\mu_m(\sigma_d) \in (\mu(I_d) : A^2)$ for any $\sigma_d \in (I_d : D(A))$.

$$\begin{array}{ccc} D(A) & \xrightarrow{\sigma_d} & I_d) \\ \mu \downarrow & & \downarrow \mu \\ A^2 & \xrightarrow{\mu_m(\sigma_d)} & \mu(I_d) \end{array}$$

□

3. Case of Bernstein Algebras

Let (A, ω) be a Bernstein K-algebra. Let $\widetilde{e}_d = 2L^4_{e.e} - L^3_{e.e}$. We have $\widetilde{e}_d(e.e) = e.e$ and $\widetilde{e}_d(x.y) = 0$ for any $x.y$ in N_d. So $\widetilde{e}_d$ is a nonzero idempotent of $\mathcal{M}(D(A))$ not belonging to $(N_d : D(A))$.

Theorem 3.1. *Let $A = Ke \oplus U \oplus V$ be the Peirce decomposition of a Bernstein K-algebra. Then*

(i) $\mathcal{M}(D(A)) = K\widetilde{e}_d \oplus \widetilde{U}_d \oplus \widetilde{V}_d$, *where* $\widetilde{U}_d = \{\sigma_d \in (N_d : D(A)) | \sigma_d \circ \widetilde{e}_d = \sigma_d\}$ *and* $\widetilde{V}_d = \{\sigma_d \in (N_d : D(A)) | \sigma_d \circ \widetilde{e}_d = 0\}$.

(ii) $\widetilde{U}_d = \{\sigma_d \in (N_d : D(A)) | \sigma_d(N_d) = 0\}$ *and* $\widetilde{V}_d = \{\sigma_d \in (N_d : D(A)) | \sigma_d(e.e) = 0\}$.

(iii) *We have the following relations*: $\widetilde{U}^2_d = 0$, $\widetilde{V}_d\widetilde{U}_d \subset \widetilde{U}_d$ *and* $\widetilde{V}^2_d \subset \widetilde{V}_d$, *particularly* $\widetilde{U}_d$ *is an ideal of* $\mathcal{M}(D(A))$ *and* $\widetilde{V}_d$ *is a left ideal of* $\mathcal{M}(D(A))$.

Proof. The proof is similar to the case of ([2], Theorem 1).

Proposition 3.2. *Let* $\mathcal{M}(D(A)) = K\widetilde{e}_d \oplus \widetilde{U}_d \oplus \widetilde{V}_d$ *be the multiplication algebra of commutative duplicate of a Bernstein algebra A. We have* $\mu_m(\widetilde{U}_d) = \widetilde{U}$ *and* $\mu_m(\widetilde{V}_d) = \widetilde{V}$ *with* $\mathcal{M}(A^2) = K\widetilde{e}_1 \oplus \widetilde{U} \oplus \widetilde{V}$.

Proof. Let $\sigma_d \in \widetilde{U}_d$ and $x \in N = ker\omega_{|A^2}$. We have $\mu_m(\sigma_d)(x) = \mu(\sigma_d(z))$, where $z \in N_d$ such as $\mu(z) = x$, so $\mu_m(\sigma_d)(x) = 0$ and $\mu_m(\sigma_d) \in \widetilde{U}$. Let $\sigma_d \in \widetilde{V}_d$, we have $\mu_m(\sigma_d)(e) = \mu(\sigma_d(e.e)) = \mu(0) = 0$, that is to say $\mu_m(\sigma_d) \in \widetilde{V}$. So $\mu_m(\widetilde{U}_d) \subset \widetilde{U}$ and $\mu_m(\widetilde{V}_d) \subset \widetilde{V}$. Let $\sigma \in \widetilde{U}$. It exists $\sigma_d = \theta + \varphi$ in $\mathcal{M}(D(A))$, with $\theta \in \widetilde{U}_d$ and $\varphi \in \widetilde{V}_d$ such as $\mu_m(\sigma_d) = \sigma$. The equality $\mu_m(\sigma_d) = \sigma$ is equivalent to $\mu_m(\theta) + \mu_m(\varphi) = \sigma$, so $\mu_m(\theta) = \sigma$ because $\mu_m(\varphi) = 0$ due to the direct sum $\mathcal{M}(A^2) = K\widetilde{e}_1 \oplus \widetilde{U} \oplus \widetilde{V}$. Therefore $\mu_m(\theta) = \sigma$ with $\theta \in \widetilde{U}_d$, that is to say $\widetilde{U} \subset \mu_m(\widetilde{U}_d)$. It similarly shows that $\widetilde{V} \subset \mu_m(\widetilde{V}_d)$. □

Corollary 3.3. *Let* $\mathcal{M}(D(A)) = K\widetilde{e}_d \oplus \widetilde{U}_d \oplus \widetilde{V}_d$ *be the multiplication algebra of commutative duplicate of a Bernstein algebra A. We have* $\widetilde{U}_d / \widetilde{U} \cap (N(A) : D(A)) \simeq \widetilde{U}$ *and* $\widetilde{V}_d / \widetilde{V} \cap (N(A) : D(A)) \simeq \widetilde{V}$ *with* $\mathcal{M}(A^2) = K\widetilde{e}_1 \oplus \widetilde{U} \oplus \widetilde{V}$.

Proof. Indeed, $\widetilde{U}_d / ker(\mu_{m \setminus \widetilde{U}_d}) \simeq \widetilde{U}$ and $\widetilde{V}_d / ker(\mu_{m \setminus \widetilde{V}_d}) \simeq \widetilde{V}$ by Proposition 2.2. Furthermore $ker(\mu_{m \setminus \widetilde{U}_d}) = \widetilde{U} \cap (N(A) : D(A))$ and $ker(\mu_{m \setminus \widetilde{V}_d}) = \widetilde{V} \cap (N(A) : D(A))$, hence the corollary holds. □

We end this paragraph by giving examples in the case of specific Bernstein algebra.

Example 1. Let $A = Ke \oplus V$ be the Peirce decomposition of a constant Bernstein algebra (i.e., a baric algebra such that $x^2 = \omega(x)e$ for any x in A). Then we have $A^2 = Ke$, $D(A) = Ke.e \oplus N(A)$. We show that then $\mathcal{M}(A) = KL_e$, $\mathcal{M}(A^2) = K1_{A^2}$ and $\mathcal{M}(D(A)) = KL_{e.e}$, so $(N(A) : D(A)) = ker\mu_m = 0$ and $\mathcal{M}(A) \simeq \mathcal{M}(A^2) \simeq \mathcal{M}(D(A))$.

Example 2. Let $A = Ke \oplus U$ be the Peirce decomposition of an elementary Bernstein algebra (i.e., a baric algebra such that $x^2 = \omega(x)x$ for any x in A). Then we have $A^2 = A$, $\mathcal{M}(A) = \mathcal{M}(A^2) = K\widetilde{e}_1 \oplus \{\psi_u, u \in U\} \oplus K\widetilde{e}_2$ and $D(A) = Ke.e \oplus Ke.U \oplus U.U$. It is shown that $\mathcal{M}(D(A)) = K\widetilde{e}_{1d} \oplus \widetilde{U}_d \oplus \widetilde{V}_d$ with $\widetilde{V}_d = K\widetilde{e}_{2d} \oplus \{L_{e.u} - 2L_{e.e}L_{e.u}, u \in U\}$, where $\widetilde{e}_{1d} = 2L_{e.e}^2 - L_{e.e}$ and $\widetilde{e}_{2d} = 4L_{e.e} - 4L_{e.e}^2$. We also show that $(N(A) :$

$D(A)) = \{\psi_v,\, v \in N(A)\} \oplus \{L_{e.u} - 2L_{e.e}L_{e.u},\, u \in U\}$, $\widetilde{U}_d \cap (N(A) : D(A)) =$ $\{\psi_v,\, v \in N(A)\}$ and $\widetilde{V}_d \cap (N(A) : D(A)) = \{L_{e.u} - 2L_{e.e}L_{e.u},\, u \in U\}$, so $\widetilde{U}_d \,/\, \{\psi_v,\, v \in N(A)\} \simeq \{\psi_u,\, u \in U\}$ and $\widetilde{V}_d / \{L_{e.u} - 2L_{e.e}L_{e.u},\, u \in U\} \simeq K\widetilde{e}_2$.

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