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COLLISION OF TWO PEAKONS IN A GENERAL CAMASSA-HOLM MODEL

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Abstract

We analyze a peakon collision for essentially non-integrable versions of the Camassa-Holm equation. Using the weak asymptotics methods, we construct a two-phase asymptotic solution that satisfies both a one-parameter family of equations and two energy laws. It is shown that the waves with initial amplitudes $A_1 > A_2 > 0$, when interacting, are reflected and exchange their energy: the new amplitudes will be $B_1 = A_2$ instead of A_1 and $B_2 = A_1$ instead of A_2.

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1. Introduction

The "general Degasperis-Procesi-Camassa-Holm" equation is a modern unidirectional approximation of the shallow water system

$$\frac{\partial}{\partial t}\left\{u - \alpha^2\varepsilon_d^2\frac{\partial^2 u}{\partial x^2}\right\} + \frac{\partial}{\partial x}\left\{c_0 u + c_1 u^2 - c_2\left(\varepsilon_d\frac{\partial u}{\partial x}\right)^2 + \varepsilon_d^2(\gamma - c_3 u)\frac{\partial^2 u}{\partial x^2}\right\} = 0, \tag{1}$$

which is a nature generalization of the well-known Camassa-Holm (CH) and Degasperis-Procesi (DP) equations [1]-[3]. Here $x \in \mathbb{R}^1$, $t > 0$; $u = u(x, t)$ parameterizes the elevation of the free water surface relative to the equilibrium state $u = 0$, so the u sign can be arbitrary, c_0 is a constant related to the critical shallow-water wave speed, c_1 characterizes the typical wave amplitude, and ε_d characterizes the dispersion level. The constants $\alpha \geq 0$ and $\gamma \geq 0$ are associated with different characters of the "linear" dispersion manifestation (compare (1) with KdV and Benjamin-Bona-Mahony equations [4]). In the Green-Naghdi approximation the restriction $\alpha + \gamma > 0$ is required [5]. The Equation (1) terms with $c_2 \geq 0$ and $c_3 \geq 0$ can be treated as representations of "nonlinear" dispersion. In the Camassa-Holm approximation $c_2 + c_3 > 0$ [1].

The most important feature of (1) is that, unlike equations with standard "linear" dispersion, this model describes wave breaking phenomena on the water surface. Consequently, classical solutions of the Equation (1) are generally unstable and collapse in a short time. However, a global solvability in terms of distributions for two special cases of the Equation (1) with "nonlinear" dispersion is proved (see [3], [6]-[9] and references therein).

These particular cases are: the CH equation ([1], 1993) if $c_2 = c_3/2$, $c_1 = 3c_3/2\alpha^2$, and $\gamma = 0$; and the DP equation ([2], 1999, see also [3]) if $c_2 = c_3$, $c_1 = 2c_3/\alpha^2$, and $c_0 = \gamma = 0$. It is known that the CH and DP equations have long-living solutions of the travelling wave type, namely, solitons and their continuous analogues: the so-called cuspons (with an unbounded first derivative) and waves with a bounded first derivative,

$$u = A\exp(-|x - At|), \qquad (2)$$

which are called peakon (with $A > 0$) and antipeakon (with $A < 0$), see [1]-[3], [10]-[15]. Moreover, the CH and DP (as well as the KdV) equations are completely integrable, whereas all others particular cases of the model (1) are essentially non-integrable (see, e.g., [6]). Note also that none of them (CH or DP) can't be transformed to another one (DP or CH) [3].

We now turn to discuss non-integrable versions of the model (1). Note that all terms in (1) are well defined for distributions such that $(u'_x)^2$ is an integrable over $\mathbb{R}^1_x$ function [16]. Accordingly, like CH and DP equations, this model admits, under some conditions, not only classical soliton solutions [17], but also non-smooth solutions, see [18] and below. In addition, the solitons of the general version (1), as well as the solitons of perturbed KdV equations, interact almost elastically (see the asymptotic analysis [19] and result of numerical experiments [20]).

In this paper we consider a non-integrable version of (1), which admits the existence of peakons for arbitrary amplitudes [17]. The main subject of the research is the scenario of peakon-peakon interactions. Namely, we assume that

$$\gamma = c_0 = 0,\ \alpha > 0,\ \varepsilon_d = 1,\ c_k > 0,\ k = 1, 2, 3, \text{ and } c_3 = r\alpha^2 c_1, \qquad (3)$$

where the parameter r characterizes the correlation between two types of "nonlinear" dispersion,

$$r = c_3/(c_2 + c_3). \tag{4}$$

To simplify formulas, we rescaling $x' = x / \alpha$, $t' = rc_1 t/\alpha$ and transform the Equation (1) into the divergent form

$$\frac{\partial}{\partial t}(u - u_{xx}) + \frac{1}{r}\frac{\partial}{\partial x}\left\{u^2 + (2r-1)(u_x)^2 - \frac{r}{2}(u^2)_{xx}\right\} = 0. \tag{5}$$

Obviously, Equation (5) coincides with CH and DP equations for $r = 2/3$ and $r = 1/2$, respectively. Moreover, (2) is the exact peakon solution of (5) for any r.

We turn now to consider collisions of peakons more in detail. There is known the explicit formula for such interaction for CH equation (see, e.g., [10]-[14])

$$u = \sum_{i=1}^{2} G_i(t)\exp(-|x - \varphi_i(t)|), \tag{6}$$

$$G_1(t) = (A_1 + A_2 \exp(L(t - t_0)))(1 + \exp(L(t - t_0)))^{-1},$$

$$G_2(t) = (A_2 + A_1 \exp(L(t - t_0)))(1 + \exp(L(t - t_0)))^{-1}, \tag{7}$$

$$\varphi_1(t) = \log(L) + A_1(t - t_0) - \log(A_1 + A_2 \exp(L(t - t_0))),$$

$$\varphi_2(t) = -\log(L) + A_2(t - t_0) + \log(A_1 \exp(L(t - t_0)) + A_2). \tag{8}$$

Here $A_1 > A_2 > 0$ are the amplitudes of the original non-interacting peakons, and $L = A_1 - A_2$. It is easy to see that $G_1(t) \to A_1$ and $G_2(t) \to A_2$ for $t \to -\infty$, whereas $G_1(t) \to A_2$ and $G_2(t) \to A_1$ for $t \to \infty$. Respectively, $\varphi_1(t) \to A_1(t - t_0)$ and $\varphi_2(t) \to A_2(t - t_0)$ for $t \to -\infty$, whereas $\varphi_1(t) \to A_2(t - t_0)$ and $\varphi_2(t) \to A_1(t - t_0)$ for $t \to \infty$. A typical example of the trajectories of interacting CH-peakons is shown in Figure 1. To complete the description of the solution it is suffices to note that the minimal distance s_0 between the trajectories is

$$s_0 = 2\log(A_1 + A_2)/L, \tag{9}$$

which is realized at $t = t_0$ when wave (6) has the form

$$u\big|_{t=t_0} = \frac{1}{2}(A_1 + A_2)\sum_{i=1}^{2} \exp(-|x - \varphi_i(t_0)|). \tag{10}$$

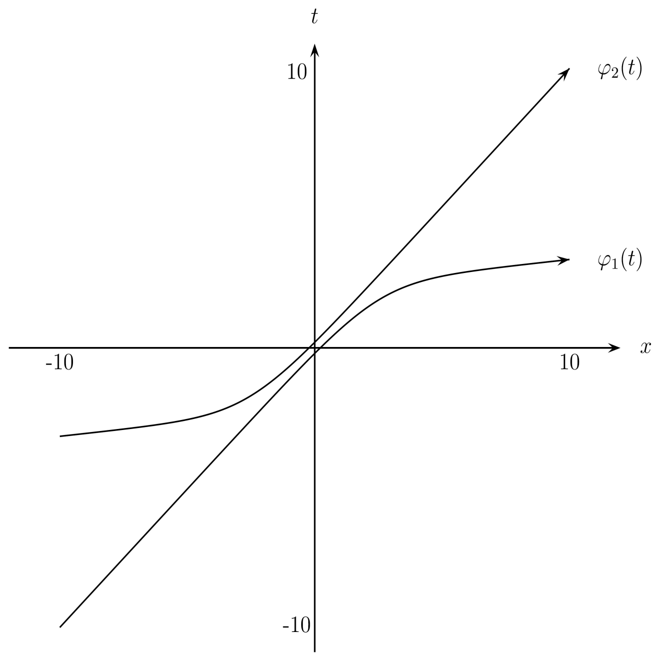

Figure 1. Trajectories $x = \varphi_i(t)$ of CH-peakons with initial amplitudes $A_1 = 1$, $A_2 = 0.1$, and with $t_0 = 0$.

An analysis of the solution that arises after collision of peakons in non-integrable versions of (5) is the content of this article. Our basic remark is that the situation with the collision of peakons is analogous to the interaction of shock waves in gas dynamics: the Rankine-Hugoniot conditions describe in detail the waves before the collision, however, the scenarios for further dynamics in the formal approach can be chosen quite arbitrarily. Oleinik [21] and Liu [22] were the first to use smooth regularization to study admissibility criterions for shock wave for scalar equations and systems of hyperbolic equations. Later, this approach, combined with the weak asymptotics method has been successfully used to study collisions and the formation of shock waves [23, 24], to study the stability of waves in problems with non-convex non-linearity [25], as well as in a detailed study of the collision of shock waves for the system of gas dynamics equations, including the process rarefaction wave formation [26, 27].

The main result of the article is the conclusion that the scenario of peakon collision for non-integrable equations is similar to that described above. Moreover, as in the integrable CH and DP cases, the character of the peakon-peakon interaction is much more regular compared to the peakon-antipeakon collision, see [10]-[15].

The content of the article is as follows: Sections 2 and 3 give a detailed construction of the asymptotics of peakons outside the critical time t_0. Section 3 contains also the construction and study of the global solution. In Conclusion we briefly discuss the passage to the limit from smooth regularization to non-smooth solutions.

2. External Asymptotic Solution I

Let us choose a smooth function $h(\eta) \in [0, 1]$ such that

$$h(\eta) + h(-\eta) = 1, \quad \eta \in R, \tag{11}$$

and let there is a constant $c > 0$ such that

$$h(\eta) = O(e^{c\eta}) \to 0 \text{ for } \eta \to -\infty, 1 - h(\eta) = O(e^{-c\eta}) \to 0 \text{ for } \eta \to \infty. \tag{12}$$

Next, we define a small parameter $0 < \varepsilon << 1$ and set $\eta = x/\varepsilon$. Then $h(x/\varepsilon)$ is a regularization of the Heaviside function ($H(x) = 0$ for $x < 0$, $H(x) = 1$ for $x > 0$), that is $h(x/\varepsilon) \to H(x)$ for $\varepsilon \to 0$.

For $t < t_0$ let us represent a similar to (6) ansatz:

$$u(t,\ x,\ \varepsilon) = \sum_{i=1}^{2} G_i(t,\ \varepsilon)\left\{\omega_i^- + [\omega_i]h_i\right\},\quad h_i = h((x - \varphi_i(t,\ \varepsilon))/\varepsilon), \tag{13}$$

where

$$\omega_i^{\mp} = \exp(\pm\,(x - \varphi_i(t,\ \varepsilon))),\quad [\omega_i] = \omega_i^+ - \omega_i^-, \tag{14}$$

$$G_i(t,\ \varepsilon) \to A_i,\quad \varphi_i(t,\ \varepsilon) \to A_i t,\ \text{for}\ t \to -\infty,\quad A_1 > A_2 > 0, \tag{15}$$

and we assume that $\varphi_1(t,\ \varepsilon) < \varphi_2(t,\ \varepsilon)$ for all $t < t_0$.

In order to consider properties of the ansatz, we define the notion of a "smallness in the weak sense".

Definition 1. A function $f(x,\ t,\ \varepsilon)$ is said to be of the value $O_{\mathfrak{D}'}(\varepsilon^{\gamma})$ if the relation

$$(f,\ \psi) \overset{\text{def}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,\ t,\ \varepsilon)\psi(x,\ t)dxdt = O(\varepsilon^{\gamma}) \tag{16}$$

holds for any test function $\psi(x,\ t) \in \mathfrak{D}(\mathbb{R}^2)$.

Let us now state the main properties of the ansatz (13).

Lemma 1. *There hold the following relations:*

$$h_i^k = H_i + O_{\mathfrak{D}'}(\varepsilon),\quad \frac{d}{dx}h_i^k = \delta(x - \varphi_i(t,\ \varepsilon)) + O_{\mathfrak{D}'}(\varepsilon), \tag{17}$$

$$h_1 h_2 = (1 - \lambda(\sigma))H_1 + \lambda(\sigma)H_2 + O_{\mathfrak{D}'}(\varepsilon), \tag{18}$$

where $k = 1, 2, \dots,$ $\delta(x - a)$ *is the Dirac delta-function,* $(\delta(x - a), \psi(x)) = \psi(a)$, $H_i = H(x - \varphi_i(t,\ \varepsilon))$ *is the Heaviside function* (*see, e.g.*, [16]), *and*

$$\lambda(\sigma) = \int_{-\infty}^{\infty} h'(\eta)h(\sigma - \eta)d\eta,\quad \sigma = s/\varepsilon,\quad s = \varphi_2(t,\ \varepsilon) - \varphi_1(t,\ \varepsilon). \tag{19}$$

Proof. For any test function $\psi(x)$ we get

$$(h_i^k,\ \psi(x)) = \int_{\varphi_i}^{\infty} \psi(x)dx + \int_{-\infty}^{\varphi_i} h_i^k \psi(x)dx - \int_{\varphi_i}^{\infty} (1 - h_i^k)\psi(x)dx. \tag{20}$$

Next,

$$\left| \int_{-\infty}^{\varphi_i} h_i^k \psi(x)dx \right| = \varepsilon \left| \int_{-\infty}^{0} h^k(\eta)\psi(\varphi_i + \varepsilon\eta)d\eta \right| \le c\varepsilon \int_{-\infty}^{0} e^{c\eta}d\eta = O(\varepsilon). \tag{21}$$

Repeating the same estimate for the last term in (20), we obtain the first relation in (17). Furthermore,

$$\int_{-\infty}^{\infty} \frac{dh_i^k}{dx}\psi(x)dx = \int_{-\infty}^{\infty} \frac{dh^k(\eta)}{d\eta}\psi(\varphi_i + \varepsilon\eta)d\eta = \psi(\varphi_i) + O(\varepsilon). \tag{22}$$

Finally, the equality (11) implies

$$h_1 h_2 = h_1 - h_1 h((\varphi_2 - x)/\varepsilon). \tag{23}$$

Next,

$$(h_1 h((\varphi_2 - x)/\varepsilon),\ \psi(x))$$

$$= \left(h_1 h((\varphi_2 - x)/\varepsilon), \frac{d}{dx}\int_{-\infty}^{x} \psi(x')dx' \right)$$

$$= -\frac{1}{\varepsilon}\int_{-\infty}^{\infty} \{h_1' h((\varphi_2 - x)/\varepsilon) - h_1 h'((\varphi_2 - x)/\varepsilon)\} \int_{-\infty}^{x} \psi(x')dx'dx$$

$$= -\int_{-\infty}^{\infty} \left\{ h'(\eta)h(\sigma - \eta)\int_{-\infty}^{\varphi_1} \psi(x')dx' - h'(\eta)h(\sigma - \eta)\int_{-\infty}^{\varphi_2} \psi(x')dx' \right\} d\eta$$

$$+ O(\varepsilon) = \lambda(\sigma)\left\{ \int_{\varphi_1}^{\infty} \psi(x')dx' - \int_{\varphi_2}^{\infty} \psi(x')dx' \right\}$$

$$= \lambda(\sigma)(\{H(x - \varphi_1) - H(x - \varphi_2)\},\ \psi(x)) + O(\varepsilon). \tag{24}$$

Combining (23) and (24) we pass to the equality (18). □

Formulas (17) and (18) allow us to calculate all the terms of Equation (5), namely,

$$u_{xx} = u - 2\sum_{i=1}^{2} G_i \delta(x - \varphi_i) + O_{\mathfrak{D}'}(\varepsilon), \tag{25}$$

$$(u_x)^2 = u^2 + 4G_1G_2\big(e^s - 2\lambda(\sigma)\cosh(s)\big)(H_1 - H_2) + O_{\mathfrak{D}'}(\varepsilon), \tag{26}$$

$$(u^2)_{xx} = 4u^2 - 4\sum_{i=1}^{2} G_i^2\delta_i - 4G_1G_2\big(e^s - 2\lambda\sinh(s)\big)(\delta_1 + \delta_2)$$

$$+ 8G_1G_2\big(e^s - 2\lambda\cosh(s)\big)(H_1 - H_2) + O_{\mathfrak{D}'}(\varepsilon). \tag{27}$$

Substitution of (25)-(27) converts the Equation (5) to the following form:

$$\frac{\partial}{\partial t}\sum_{i=1}^{2} G_i\delta(x - \varphi_i) + \frac{\partial}{\partial x}\Big\{\sum_{i=1}^{2} G_i^2\delta_i + G_1G_2\big(e^s - 2\lambda\sinh(s)\big)(\delta_1 + \delta_2)$$

$$- \varkappa G_1G_2\big(e^s - 2\lambda\cosh(s)\big)(H_1 - H_2)\Big\} = O_{\mathfrak{D}'}(\varepsilon), \tag{28}$$

where

$$\varkappa = 2(1 - r)/r. \tag{29}$$

Since the functions δ_i and δ_i' are linearly independent, relation (28) entails the first meaningful result.

Lemma 2. *Let* $t < t_0$. *Then function* (13) *satisfies Equation* (5) *with accuracy* $O_{\mathfrak{D}'}(\varepsilon)$ *if and only if the following system of equations is satisfied*:

$$\frac{dG_1}{dt} = \varkappa G_1G_2\big(e^s - 2\lambda(\sigma)\cosh(s)\big),$$

$$\frac{dG_2}{dt} = -\varkappa G_1G_2\big(e^s - 2\lambda(\sigma)\cosh(s)\big), \tag{30}$$

$$\frac{d\varphi_1}{dt} = G_1 + G_2\big(e^s - 2\lambda(\sigma)\sinh(s)\big),$$

$$\frac{d\varphi_2}{dt} = G_2 + G_1\big(e^s - 2\lambda(\sigma)\sinh(s)\big), \tag{31}$$

Obviously, Equations (30) imply the conservation law

$$\frac{d}{dt}(G_1 + G_2) = 0 \quad \Rightarrow G_1 + G_2 = A_1 + A_2. \tag{32}$$

Thus, there is a function $g = g(t, \varepsilon)$ such that

$$G_1 = A_1 - g, \quad G_2 = A_2 + g. \tag{33}$$

Therefore, from (30)-(33), we get

$$\frac{dg}{dt} = -\varkappa(A_1 - g)(A_2 + g)\big(e^s - 2\lambda(\sigma)\cosh(s)\big), \tag{34}$$

$$\frac{ds}{dt} = (A_2 - A_1 + 2g)\big(1 - e^s + 2\lambda(\sigma)\sinh(s)\big). \tag{35}$$

Let us analyze the functions s and g. The equalities (34) and (35) imply the following relation:

$$\frac{d}{ds}\ln\{(A_1 - g)(A_2 + g)\} = \varkappa\frac{e^s - 2\lambda(\sigma)\cosh(s)}{1 - e^s + 2\lambda(\sigma)\sinh(s)}. \tag{36}$$

We choose now constants $c_0 > 0$ and $\mu \in (0, 1)$ and assume

$$s \geq c_0\varepsilon^{1-\mu}. \tag{37}$$

Then $\sigma \geq c_0\varepsilon^{-\mu}$ and $\lambda(\sigma) = 1 + O(\exp(-c'\sigma))$. This allows us to convert the equality (36) to the following form:

$$\frac{d}{ds}\ln\{(A_1 - g)(A_2 + g)\} = -\varkappa\frac{e^{-s}}{1 - e^{-s}} = -\varkappa\frac{d}{ds}\ln(1 - e^{-s}). \tag{38}$$

Integration of (38) and assumption (15) entail the equality

$$(A_1 - g)(A_2 + g) = A_1A_2(1 - e^{-s})^{-\varkappa}. \tag{39}$$

Solving the quadratic equation (39) and again taking into account (15), we get

$$g = \frac{1}{2}\big\{A_1 - A_2 - \sqrt{D}\big\}, \quad D = (A_1 + A_2)^2 - 4A_1A_2(1 - e^{-s})^{-\varkappa}. \tag{40}$$

The most important consequence of this formula is the following:

Lemma 3. *The trajectories* $\varphi_1(t, \varepsilon)$ *and* $\varphi_2(t, \varepsilon)$ *do not intersect for* $t < t_0$.

Indeed, the hypothesis $s \to 0$ entails $D \to -\infty$.

Now let us analyze the behaviour of $g(s)$ near the critical point $s_0 = s(t_0)$. Setting $ds/dt|_{t=t_0} = 0$, we obtain from the Equation (35)

$$g_0 \overset{\text{def}}{=} g(t_0) = (A_1 - A_2)/2. \tag{41}$$

Combining (40) with (41) yields

$$s_0 = -\log\left\{1 - \left(4A_1A_2(A_1 + A_2)^{-2}\right)^{1/\varkappa}\right\} \geq \text{const} > 0, \tag{42}$$

which justifies the assumption (37). Next we find from (34) and (35)

$$g_0' \overset{\text{def}}{=} \left.\frac{dg}{dt}\right|_{t=t_0} = \frac{\varkappa}{4}(A_1 + A_2)^2 e^{-s_0} > 0,$$

$$s_0'' \overset{\text{def}}{=} \left.\frac{d^2 s}{dt^2}\right|_{t=t_0} = 2g_0'(1 - e^{-s_0}) > 0.$$

Collecting the above, we deduce from (30) and (31):

$$G_1 = (A_1 + A_2)/2 - g_0'(t - t_0) + O(t - t_0)^2,$$

$$G_2 = (A_1 + A_2)/2 + g_0'(t - t_0) + O(t - t_0)^2,$$

$$\varphi_1 = \varphi_1^0 + \frac{(A_1 + A_2)}{2}(1 + e^{-s_0})(t - t_0) - \frac{g_0'}{2}(1 - e^{-s_0})(t - t_0)^2 + O(t - t_0)^3,$$

$$\varphi_2 = \varphi_2^0 + \frac{(A_1 + A_2)}{2}(1 + e^{-s_0})(t - t_0) + \frac{g_0'}{2}(1 - e^{-s_0})(t - t_0)^2 + O(t - t_0)^3, \tag{43}$$

where $\varphi_2^0 - \varphi_1^0 = s_0$. In particular,

$$G_1|_{t=t_0} = G_2|_{t=t_0} = (A_1 + A_2)/2. \tag{44}$$

It remains to consider the implementation of the conservation law

$$\frac{d}{dt}\int_{-\infty}^{\infty} u\,dx = 0, \tag{45}$$

which is a direct consequence of integrating Equation (5); as well as the balance law

$$\frac{d}{dt}\int_{-\infty}^{\infty}\left\{u^2 + (u_x)^2\right\}dx = \frac{3r-2}{r}\int_{-\infty}^{\infty}(u_x)^3\,dx, \tag{46}$$

which is obtained by multiplying (5) by u and integrating. Namely,

Lemma 4. *Let* $t < t_0$. *Then the weak asymptotics* (13) *satisfies the equalities* (45) *and* (46) *with the precision* $O(\varepsilon)$.

Proof. Direct integrating of (13) and accounting (32) imply

$$\int_{-\infty}^{\infty} u\,dx = \sum_{i=1}^{2} G_i(2 + O(\varepsilon)) = (A_1 + A_2)(2 + O(\varepsilon)) = \text{const}. \tag{47}$$

Next, for $t < t_0$ we calculate

$$\int_{-\infty}^{\infty} u^2 dx = \sum_{i=1}^{2} G_i^2 + 2G_1G_2(1+s)e^{-s} + O(\varepsilon). \tag{48}$$

Therefore, (26) and (48) entail

$$\int_{-\infty}^{\infty}\left\{u^2 + (u_x)^2\right\}dx = 2\int_{-\infty}^{\infty}\left\{u^2 - 2G_1G_2e^{-s}(H_1 - H_2)\right\}dx + O(\varepsilon)$$

$$= 2\left\{(A_1 + A_2)^2 + 2G_1G_2(e^{-s} - 1)\right\} + O(\varepsilon). \tag{49}$$

Moreover, applying (26) again we obtain

$$\int_{-\infty}^{\infty}(u_x)^3 dx = -4G_1G_2e^{-s}\int_{-\infty}^{\infty}u_x(H_1 - H_2)dx + O(\varepsilon)$$

$$= -4G_1G_2(G_2 - G_1)(1 - e^{-s})e^{-s} + O(\varepsilon). \tag{50}$$

Finally, calculating the derivatives in accordance with Equations (30) and (35) we conclude

$$\frac{d}{dt}\left\{G_1G_2(e^{-s} - 1)\right\} + \frac{3r - 2}{4r}G_1G_2(G_2 - G_1)(e^{-s} - 1)e^{-s} = 0. \tag{51}$$

Obviously, the last equality implies the fulfilment of the law (46) with precision $O(\varepsilon)$. □

By combining all the obtained results, we get the statement

Lemma 5. *Let* $t < t_0$. *Then the function* (13) *is a weak asymptotic* $\operatorname{mod} O_{\mathfrak{D}'}(\varepsilon)$ *solution of the Equation* (5). *Moreover, with exponential accuracy* $\lambda(\sigma) = 1$, $G_i = G_i(t)$, $\varphi_i = \varphi_i(t)$. *For small* $t - t_0$ *the representations* (43) *hold.*

3. External Solution II: Matching

Obviously, the function (13) can be extended to the time t_0,

$$u(t_0, x, \varepsilon) = \frac{1}{2}(A_1 + A_2)\sum_{i=1}^{2}\{\omega_i^- + [\omega_i]h_i\}|_{t=t_0}. \tag{52}$$

However, (13) is not suitable for $t > t_0$, as the property $G_i|_{g\to 0} \to A_i$ for positive time $t - t_0$ contradicts the condition $s > 0$. For this reason, we should consider positive time separately. Let us set the ansatz similar to (13),

$$u_{ext_2}(t, x, \varepsilon) = \sum_{i=1}^{2}\tilde{G}_i(t, \varepsilon)\{\tilde{\omega}_i^- + [\tilde{\omega}_i]h_i\},$$

$$h_i = h((x - \tilde{\varphi}_i(t, \varepsilon))/\varepsilon), \quad \tilde{\omega}_i^{\mp} = \exp(\pm(x - \tilde{\varphi}_i(t, \varepsilon))). \tag{53}$$

In view of Lemma 3 we assume

$$\widetilde{G}_i(t, \varepsilon) \to B_i \ \text{for} \ t \to \infty, \quad 0 < B_1 < B_2, \tag{54}$$

and we assume that $\widetilde{\varphi}_1(t, \varepsilon) < \widetilde{\varphi}_2(t, \varepsilon)$ for all $t > t_0$. Substituting (53) into the Equation (5), we pass to a similar (30), (31) system for the functions $\widetilde{G}_i$ and $\widetilde{\varphi}_i$, $i = 1, 2$. Accordingly, we conclude that

$$\frac{d}{dt}\left(\widetilde{G}_1 + \widetilde{G}_2\right) = 0, \quad \Rightarrow \quad \widetilde{G}_1 = B_1 - \widetilde{g}, \quad \widetilde{G}_2 = B_2 + \widetilde{g}. \tag{55}$$

In turn, instead of (34), (35) we derive

$$\frac{d\widetilde{g}}{dt} = \varkappa(B_1 - \widetilde{g})(B_2 + \widetilde{g})e^{-\widetilde{s}}, \quad \frac{d\widetilde{s}}{dt} = (B_2 - B_1 + 2\widetilde{g})(1 - e^{-\widetilde{s}}). \tag{56}$$

Our assumptions (54) and Equations (56) yield

$$\frac{d\widetilde{g}}{dt}(t, \varepsilon) > 0, \ \frac{d\widetilde{s}}{dt}(t, \varepsilon) > 0 \ \text{for} \ t > t_0, \ \widetilde{g}_0 \overset{\text{def}}{=} \widetilde{g}|_{t=t_0} = (B_1 - B_2)/2, \tag{57}$$

$$\widetilde{g} = \frac{1}{2}\left\{B_1 - B_2 - \sqrt{(B_1 + B_2)^2 - 4B_1B_2(1 - e^{-\widetilde{s}})^{-\varkappa}}\right\}, \tag{58}$$

$$\widetilde{s}_0 \overset{\text{def}}{=} \widetilde{s}|_{t=t_0} = -\log\left\{1 - \left(4B_1B_2(B_1 + B_2)^{-2}\right)^{1/\varkappa}\right\} \geq \text{const} > 0. \tag{59}$$

Thus,

$$\widetilde{G}_1(t_0) = \widetilde{G}_2(t_0) = (B_1 + B_2)/2. \tag{60}$$

The last step of the construction is the union of the local solutions u_{ext_i},

$$u(x, t, \varepsilon) = u_{ext_1} + \left(u_{ext_2} - u_{ext_1}\right)h\left(\frac{t - t_0}{\varepsilon}\right) + O_{\mathfrak{D}'}(\varepsilon), \tag{61}$$

where $u_{ext_1} = u|_{t \leq t_0}$. It is easy to establish that function (61) is an asymptotic solution of Equation (5) if and only if this equation is satisfied in the main term with respect to ε on the intervals $t < t_0$ and $t > t_0$; and if the equality

$$\left(U_{ext_1} - U_{ext_2}\right)\delta(t - t_0) = O_{\mathfrak{D}'}(\varepsilon), \text{ where } U \overset{\text{def}}{=} u - u_{xx} \tag{62}$$

holds. The first condition is fulfilled due to the construction of the local solutions. To analyze (62) we apply the formula (25) and pass to the equality

$$\sum_{i=1}^{2} G_i \psi(\varphi_i, t)\big|_{t=t_0} - \sum_{i=1}^{2} \widetilde{G}_i \psi(\widetilde{\varphi}_i, t)\big|_{t=t_0} = O(\varepsilon), \tag{63}$$

where $\psi = \psi(x, t)$ is a test function.

We now turn the attention to the energy laws (45) and (46). It is easy to see that the conservation law (45) implies the matching condition

$$G_1 + G_2 = A_1 + A_2 = B_1 + B_2 = \widetilde{G}_1 + \widetilde{G}_2. \tag{64}$$

Furthermore, taking into account Lemma 4, we conclude that the balance law (46) is satisfied with an accuracy $O(\varepsilon)$ on the intervals $t < t_0$ and $t > t_0$. Thus, it remains to analyze the relation

$$\left(E(u_{ext_1}) - E(u_{ext_2})\right)\big|_{t=t_0} \delta(t - t_0) = O_{\mathfrak{D}'(R_t)}(\varepsilon), \tag{65}$$

where

$$E(u) = \int_{-\infty}^{\infty} \{u^2 + (u_x)^2\} dx. \tag{66}$$

Energy $E(u_{ext_1})$ has been calculated in formula (49). This and similar formula for $E(u_{ext_2})$ allow us to transform the relation (65) in the following manner:

$$G_1 G_2 (1 - e^{-s})\psi\big|_{t=t_0} - \widetilde{G}_1 \widetilde{G}_2 (1 - e^{-\widetilde{s}})\psi\big|_{t=t_0} = O(\varepsilon) \Rightarrow s_0 = \widetilde{s}_0, \tag{67}$$

where $\psi = \psi(t)$ is a test function. Summarizing (42), (59), (64), and (67), we deduce

$$A_1 A_2 = B_1 B_2. \tag{68}$$

Furthermore, upon combining (44), (60), (64), and (63), we obtain

$$\sum_{i=1}^{2} \psi(\varphi_i, t)\big|_{t=t_0} = \sum_{i=1}^{2} \psi(\tilde{\varphi}_i, t)\big|_{t=t_0}. \tag{69}$$

In turn, Lemma 3, (64), (68), and (69) imply

$$B_1 = A_2, \quad B_2 = A_1, \quad \varphi_i(t_0) = \tilde{\varphi}_i(t_0), \quad i = 1, 2. \tag{70}$$

Thus, we get the final result

Theorem 1. *Let* $A_1 > A_2 > 0$ *and* $\varphi_1|_{t\to-\infty} < \varphi_2|_{t\to-\infty}$. *Then, for all* $r \in (0, 1)$, *the function* (61), (70) *satisfies the Equation* (5) *and energy laws* (45) *and* (46) *in the weak asymptotic sense.*

4. Conclusion

Smooth regularization (with a small parameter ε) made it possible to describe the collision of peakons for a family of non-integrable (with two exceptions) CH-type equations (5). It has been shown that smoothed peakons, contrary to solitons, interact at a distance: their trajectories do not intersect.

The minimum distance (42) depends on the amplitudes and the parameter $\varkappa = \varkappa(r)$, but remains separated from zero. This property allows passing to the limit $\varepsilon \to 0$ and obtaining the exact two-phase solution for the family of Equation (5). It should be noted that formula (42) in the special cases of the CH equation $(\varkappa = 1)$ and of the DP equation $(\varkappa = 2)$ gives the same distances s_0 as those found using the inverse scattering problem, namely, (9) for $\varkappa = 1$ and

$$s_0 = \log\{(A_1 + A_2)(\sqrt{A_1} + \sqrt{A_2})^2/L^2\} \text{ for } \varkappa = 2.$$

Finally, it remains to note that due to the symmetry $t \to -t$, $u \to -u$ in Equation (5), the results obtained allow us to describe the interaction of two antipeakons.

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