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# CHARACTERIZATIONS OF CERTAIN EIGHT GENERAL UNIVARIATE CONTINUOUS DISTRIBUTIONS INTRODUCED BY CHESNEAU RECENTLY

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**Abstract**

This paper deals with various characterizations of eight general univariate continuous distributions proposed by Chesneau [1]. These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) the hazard function; (iii) reverse hazard function. It should be mentioned that for the characterization (i) the cumulative distribution function need not have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation.

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## 1. Introduction

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Generally speaking, the problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. These characterizations have been established in many different directions. The present work deals with certain characterizations of eight general univariate continuous distributions proposed by Chesneau [1]: (1) Variable Power Parametric of the First Kind (VPP1stK); (2) Variable Power Parametric of the Second Kind (VPP3rdK); (3) Variable Power Parametric of the Third Kind (VPP3rdK); (4) Variable Power Parametric of the Fourth Kind (VPP4thK); (5) Variable Power Parametric of the Fifth Kind (VPP5thK); (6) Variable Power Parametric of the Sixth Kind (VPP6thK); (7) Variable Power Parametric of the Seventh Kind (VPP7thK); (8) Variable Power Parametric of the Eighth Kind (VPP8thK). Certain interesting examples as well as further complements of these distributions are given in Chesneau [1] for interested readers.

We list below the cumulative distribution function (cdf) and probability density function (pdf) of each one of these distributions in the same order as listed above. We will be employing the same notation for the parameters as chosen by the original author (Chesneau).

(1) The cdf and pdf of (VPP1stK) are given, respectively, by

$$F(x;\, a,\, b) = e^{-\frac{[-\ln(x)]^{b+1}}{x^{a}}}, \quad 0 \le x \le 1, \tag{1.1}$$

and

$$f(x;\, a,\, b) = x^{-1}[-\ln(x)]^{b} P(x), \quad 0 < x < 1, \tag{1.2}$$

where $a \ge 0,\, b > -1$ are parameters and $P(x) = x^{-a}[-a\ln(x) + b + 1]$ $e^{-\frac{[-\ln(x)]^{b+1}}{x^{a}}}$.

(2) The cdf and pdf of (VPP2ndK) are given, respectively, by

$$F(x;\, a,\, b,\, c) = e^{\ln(x)[a+bx+cx\ln(x)]}, \quad 0 \le x \le 1, \tag{2.1}$$

and

$$f(x;\, a,\, b,\, c) = x^{-1}P(x), \quad 0 < x < 1, \tag{2.2}$$

where $b \le 0$, $c \in [0, 1]$, $a > 0$, $a \ge 1_{\{c \ne 0\}} - b$ are parameters and $P(x) = \left\{a + bx + (b + 2c)x\ln(x) + cx[\ln(x)]^2\right\}e^{\ln(x)[a+bx+cx\ln(x)]}$.

(3) The cdf and pdf of (VPP3rdK) are given, respectively, by

$$F(x;\, a,\, b) = e^{a\ln(1-x)(1-x)^b} - (1 - x), \quad 0 \le x \le 1, \tag{3.1}$$

and

$$f(x;\, a,\, b) = (1 - x)^{b-1}P(x), \quad 0 < x < 1, \tag{3.2}$$

where $a \in [0, 1]$, $b \ge 1$ are parameters and $P(x) = (1 - x)^{1-b} - a[1 + b\ln(1 - x)]e^{a\ln(1-x)(1-x)^b}$.

(4) The cdf and pdf of (VPP4thK) are given, respectively, by

$$F(x;\, a,\, b) = \frac{1}{a}\left\{e^{\ln(x)(1+ax+bx\ln(x))} - e^{x\ln(x)}\right\}, \quad 0 \le x \le 1, \tag{4.1}$$

and

$$f(x;\, a,\, b) = \frac{1}{a}x^{-1}P(x), \quad 0 < x < 1, \tag{4.2}$$

where $b \le 0$, $a \ge 1 - b$ are parameters and $P(x) = \{[1 + ax + bx\ln(x)] + x\ln(x)[a + b + b\ln(x)]\}e^{\ln(x)[1+ax+bx\ln(x)]} - x(1 + \ln(x))e^{x\ln(x)}$.

(5) The cdf and pdf of (VPP5thK) are given, respectively, by

$$F(x;\, a,\, b,\, c,\, d) = (1+a)e^{-x^{-d}\ln\left[1+\frac{a+b\ln(x)+cx+\ln(x)}{x}\right]},\ 0 \leq x \leq 1, \qquad (5.1)$$

and

$$f(x;\, a,\, b,\, c,\, d) = (1+a)x^{-d-1}P(x), \quad 0 < x < 1, \qquad (5.2)$$

where $a \geq 0,\, b \geq 0,\, c \leq 0$ such that $a - b - c \neq 0,\, d \geq 0$ are parameters and $P(x) = \left\{ \begin{array}{c} d\ln\left[1+[a+b\ln(x)+cx\ln(x)]^{-1}x^{-1}\right] \\ -x\left[\frac{-((a+b\ln(x)+cx\ln(x)))x^{-2}+(bx^{-1}+c\ln(x)+c)x^{-1}}{1+((a+b\ln(x)+cx\ln(x)))x^{-1}}\right] \end{array} \right\}$

$\times\, e^{-x^{-d}\ln\left[1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right]}$.

(6) The cdf and pdf of (VPP6thK) are given, respectively, by

$$F(x;\, a,\, b,\, c,\, d) = \left[1+\frac{a+b\ln(x)+cx+\ln(x)}{x}\right]^{1-x^{-d}}, \quad 0 \leq x \leq 1, \qquad (6.1)$$

and

$$f(x;\, a,\, b,\, c,\, d) = x^{-d-2}P(x), \quad 0 < x < 1, \qquad (6.2)$$

where $a \geq 0,\, b \leq 0,\, c \leq 0$ such that $a - b - c \neq$ are parameters and

$$P(x) = \left[1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right]^{x^{-d}}$$

$$\times \left\{ \begin{array}{c} d[a+b\ln(x)+cx\ln(x)+x]\ln\left[1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right] \\ +(1+x^{d})[a+b\ln(x)-b-cx] \end{array} \right\}.$$

(7) The cdf and pdf of (VPP7thK) are given, respectively, by

$$F(x;\, a,\, b) = e^{\ln(x)x^{-a(1-x^{b})}}, \quad 0 \leq x \leq 1, \qquad (7.1)$$

and

$$f(x;\ a,\ b) = x^{b-1}x^{-a(1-x^{b})}\ln(x)P(x), \quad 0 < x < 1, \tag{7.2}$$

where $a \geq 0,\ b \geq 0$ are parameters and $P(x) = \left[\frac{x^{-b}}{\ln(x)} + ab\ln(x)\right]$ $\times\ e^{\ln(x)x^{-a(1-x^{b})}}$.

(8) The cdf and pdf of (VPP8thK) are given, respectively, by

$$F(x;\ a,\ b) = 1 - e^{\ln(1-x)\left[\frac{x^{a}}{[-\ln(x)]^{b}}\right]}, \quad 0 \leq x \leq 1, \tag{8.1}$$

and

$$f(x;\ a,\ b) = x^{-1}[-\ln(x)]^{b-1}P(x), \quad 0 < x < 1, \tag{8.2}$$

where $a \geq -1,\ b \geq 0$ such that $1 + a + b \neq 0$ are parameters and $P(x) = [-\ln(x)]^{-2b} + \left\{\left[\frac{x^{a+1}}{1-x} - ax^{a-1}\ln(1-x)\right][-\ln(x)] - bx^{a-1}\ln(1-x)\right\}$ $e^{\ln(1-x)\left[\frac{x^{a}}{[-\ln(x)]^{b}}\right]}$.

## 2. Characterization of Distributions

As mentioned in the Introduction, characterizations of distributions is an important area of research which has recently attracted the attention of many researchers. This section deals with various characterizations of the distributions listed in the Introduction. These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) the hazard function; and (iii) the reverse hazard function. It should be mentioned that for the characterization (i) the cdf need not have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation.

**2.1. Characterizations based on two truncated moments**

In this subsection we present characterizations of 8 distributions mentioned in the Introduction, in details, in terms of a simple relationship between two truncated moments. Our first characterization result employs a theorem due to (Glänzel [2]), see Theorem 2.1.1 below. Note that the result holds also when the interval $H$ is not closed. Moreover, as mentioned above, it could be also applied when the cdf $F$ does not have a closed form. As shown in (Glänzel [3]), this characterization is stable in the sense of weak convergence.

**Theorem 2.1.1.** *Let* $(\Omega, \mathcal{F}, \mathbf{P})$ *be a given probability space and let* $H = [d, e]$ *be an interval for some* $d < e(d = -\infty, e = \infty$ *might as well be allowed). Let* $X : \Omega \to H$ *be a continuous random variable with the distribution function* $F$ *and let* $q_1$ *and* $q_2$ *be two real functions defined on* $H$ *such that*

$$\mathbf{E}[q_2(X)|X \geq x] = \mathbf{E}[q_1(X)|X \geq x]\eta(x), \quad x \in H,$$

*is defined with some real function* $\eta$. *Assume that* $q_1, q_2 \in C^1(H)$, $\eta \in C^2(H)$ *and* $F$ *is twice continuously differentiable and strictly monotone function on the set* $H$. *Finally, assume that the equation* $\eta q_1 = q_2$ *has no real solution in the interior of* $H$. *Then* $F$ *is uniquely determined by the functions* $q_1, q_2$ *and* $\eta$, *particularly*

$$F(x) = \int_a^x C\left|\frac{\eta'(u)}{\eta(u)q_1(u) - q_2(u)}\right|\exp(-s(u))du,$$

*where the function* $s$ *is a solution of the differential equation* $s' = \dfrac{\eta' q_1}{\eta q_1 - q_2}$ *and C is the normalization constant, such that* $\int_H dF = 1$.

Here is our first characterization.

**Proposition 2.1.1.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable and let* $q_1(x) = [P(x)]^{-1}$ *and* $q_2(x) = q_1(x)[-\ln(x)]$ *for* $0 < x < 1$. *The random variable* $X$ *has pdf* (1.2) *if and only if the function* $\eta$ *defined in Theorem* 2.1.1 *has the form*

$$\eta(x) = \left(\frac{b+1}{b+2}\right)[-\ln(x)], \quad 0 < x < 1.$$

**Proof.** Let $X$ be a random variable with pdf (1.2), then

$$(1 - F(x))E[q_1(X)|X \geq x] = \int_x^1 u^{-1}[-\ln(u)]^b du$$

$$= \left(\frac{1}{b+1}\right)[-\ln(x)]^{b+1}, \quad 0 < x < 1,$$

and

$$(1 - F(x))E[q_2(X)|X \geq x] = \int_x^1 u^{-1}[-\ln(u)]^{b+1} du$$

$$= \left(\frac{1}{b+2}\right)[-\ln(x)]^{b+2}, \quad 0 < x < 1,$$

and finally

$$\eta(x)q_1(x) - q_2(x) = -\frac{q_1(x)}{b+2}[-\ln(x)] < 0 \text{ for } 0 < x < 1.$$

Conversely, if $\eta$ is given as above, then

$$s'(x) = \frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{(b+1)x^{-1}}{[-\ln(x)]}, \quad 0 < x < 1,$$

and hence

$$s(x) = -(b+1)\ln[-\ln(x)], \quad 0 < x < 1.$$

Now, in view of Theorem 2.1.1, $X$ has density (1.2).

**Corollary 2.1.1.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable and let* $q_1(x)$ *be as in Proposition* 2.1.1. *The pdf of X is* (1.2) *if and only if there exist functions* $q_2$ *and* $\eta$ *defined in Theorem* 2.1.1 *satisfying the differential equation*

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{(b+1)x^{-1}}{[-\ln(x)]}, \quad 0 < x < 1.$$

**Corollary 2.1.2.** *The general solution of the differential equation in Corollary* 2.1.1 *is*

$$\eta(x) = [-\ln(x)]^{-(b+1)} \left[ -\int (b+1)x^{-1} [-\ln(x)]^b (q_1(x))^{-1} q_2(x) dx + D \right],$$

*where* $D$ *is a constant.*

**Proof.** If $X$ has pdf (1.2), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(x) = \left( \frac{(b+1)x^{-1}}{[-\ln(x)]} \right) \eta(x) - \left( \frac{(b+1)x^{-1}}{[-\ln(x)]} \right) (q_1(x))^{-1} q_2(x),$$

or

$$\eta'(x) [-\ln(x)]^{b+1} - (b+1)x^{-1} [-\ln(x)]^b \eta(x)$$
$$= -(b+1)x^{-1} [-\ln(x)]^b (q_1(x))^{-1} q_2(x),$$

or

$$\frac{d}{dx} \left\{ \eta(x) [-\ln(x)]^{b+1} \right\} = -(b+1)x^{-1} [-\ln(x)]^b (q_1(x))^{-1} q_2(x),$$

from which we arrive at

$$\eta(x) = [-\ln(x)]^{-(b+1)} \left[ -\int (b+1)x^{-1} [-\ln(x)]^b (q_1(x))^{-1} q_2(x) dx + D \right],$$

Note that a set of functions satisfying the differential equation in Corollary 2.1.1, is given in Proposition 2.1.1 with $D = 0$. However, it should also be noted that there are other triplets $(q_1, q_2, \eta)$ satisfying the conditions of Theorem 2.1.1.

**Proposition 2.1.2.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable and let* $q_{1(x)} = [P(x)]^{-1}$ *and* $q_2(x) = q_1(x)[-\ln(x)]$ *for* $0 < x < 1$. *The random variable* $X$ *has pdf* (2.2) *if and only if the function* $\eta$ *defined in Theorem* 2.1.1 *has the form*

$$\eta(x) = \frac{1}{2}[-\ln(x)], \quad 0 < x < 1.$$

**Proof.** Let $X$ be a random variable with pdf (2.2), then

$$(1 - F(x))E[q_1(X)|X \geq x] = \int_x^1 u^{-1}du = [-\ln(x)], \quad 0 < x < 1,$$

and

$$(1 - F(x))E[q_2(X)|X \geq x] = \int_x^1 u^{-1}[-\ln(u)]du = \frac{1}{2}[-\ln(x)]^2, \quad 0 < x < 1,$$

and finally

$$\eta(x)q_1(x) - q_2(x) = -\frac{q_1(x)}{2}[-\ln(x)] < 0 \text{ for } 0 < x < 1.$$

Conversely, if $\eta$ is given as above, then

$$s'(x) = \frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{x^{-1}}{[-\ln(x)]}, \quad 0 < x < 1,$$

and hence

$$s(x) = -\ln(-\ln(x)), \quad 0 < x < 1.$$

Now, in view of Theorem 2.1.1, $X$ has density (2.2).

**Corollary 2.1.3.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable and let* $q_1(x)$ *be as in Proposition* 2.1.2. *The pdf of* $X$ *is* (2.2) *if and only if there exist functions* $q_2$ *and* $\eta$ *defined in Theorem* 2.1.1 *satisfying the differential equation*

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{x^{-1}}{[-\ln(x)]}, \quad 0 < x < 1.$$

**Corollary 2.1.4.** *The general solution of the differential equation in Corollary* 2.1.3 *is*

$$\eta(x) = [-\ln(x)]^{-1} \left[\int x^{-1}(q_1(x))^{-1} q_2(x) dx + D\right],$$

*where* $D$ *is a constant.*

**Proof.** It is similar to that of Corollary 2.1.2.

Note that a set of functions satisfying the differential equation in Corollary 2.1.3, is given in Proposition 2.1.2 with $D = 0$. However, it should also be noted that there are other triplets $(q_1, q_2, \eta)$ satisfying the conditions of Theorem 2.1.1.

**Proposition 2.1.3.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable and let* $q_1(x) = [P(x)]^{-1}$ *and* $q_2(x) = q_1(x)(1 - x)^b$ *for* $0 < x < 1$. *The random variable X has pdf* (3.2) *if and only if the function* $\eta$ *defined in Theorem* 2.1.1 *has the form*

$$\eta(x) = \frac{1}{2}(1 - x)^b, \quad 0 < x < 1.$$

**Proof.** Let $X$ be a random variable with pdf (3.2), then

$$(1 - F(x))E[q_1(X) | X \geq x] = \int_x^1 (1 - u)^{b-1} du = \frac{1}{b}(1 - x)^b, \quad 0 < x < 1,$$

and

$$(1 - F(x))E[q_2(X) | X \geq x] = \int_x^1 (1 - u)^{2b-1} du = \frac{1}{2b}(1 - x)^{2b}, \quad 0 < x < 1,$$

and finally

$$\eta(x)q_1(x) - q_2(x) = -\frac{q_1(x)}{2}(1-x)^b < 0 \text{ for } 0 < x < 1.$$

Conversely, if $\eta$ is given as above, then

$$s'(x) = \frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = b(1-x)^{-1}, \quad 0 < x < 1,$$

and hence

$$s(x) = -\,b\ln(1-x), \quad 0 < x < 1.$$

Now, in view of Theorem 2.1.1, $X$ has density (3.2).

**Corollary 2.1.5.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable and let* $q_1(x)$ *be as in Proposition* 2.1.3. *The pdf of* $X$ *is* (3.2) *if and only if there exist functions* $q_2$ *and* $\eta$ *defined in Theorem* 2.1.1 *satisfying the differential equation*

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = b(1-x)^{-1}, \quad 0 < x < 1.$$

**Corollary 2.1.6.** *The general solution of the differential equation in Corollary* 2.1.5 *is*

$$\eta(x) = [(1-x)]^{-b}\left[-\int b(1-x)^{b-1}(q_1(x))^{-1}q_2(x)dx + D\right],$$

*where* $D$ *is a constant.*

**Proof.** It is similar to that of Corollary 2.1.2.

Note that a set of functions satisfying the differential equation in Corollary 2.1.5, is given in Proposition 2.1.3 with $D = 0$. However, it should also be noted that there are other triplets $(q_1, q_2, \eta)$ satisfying the conditions of Theorem 2.1.1.

**Proposition 2.1.4.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable and let* $q_1(x) = [P(x)]^{-1}$ *and* $q_2(x) = q_1(x) e^{\ln(x)}$ *for* $0 < x < 1$. *The random variable* $X$ *has pdf* (4.2) *if and only if the function* $\eta$ *defined in Theorem* 2.1.1 *has the form*

$$\eta(x) = \frac{1}{2}\left(1 + e^{\ln(x)}\right), \quad 0 < x < 1.$$

**Proof.** Let $X$ be a random variable with pdf (4.2), then

$$(1 - F(x))E[q_1(X)|X \geq x] = \int_x^1 \frac{1}{a} u^{-1} e^{\ln(u)} du$$

$$= \frac{1}{a}\left(1 - e^{\ln(x)}\right), \quad 0 < x < 1,$$

and

$$(1 - F(x))E[q_2(X)|X \geq x] = \int_x^1 \frac{1}{a} u^{-1} e^{2\ln(u)} du$$

$$= \frac{1}{2a}\left(1 - e^{2\ln(x)}\right), \quad 0 < x < 1,$$

and finally

$$\eta(x) q_1(x) - q_2(x) = \frac{q_1(x)}{2}\left(1 - e^{\ln(x)}\right) > 0 \text{ for } 0 < x < 1.$$

Conversely, if $\eta$ is given as above, then

$$s'(x) = \frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{x^{-1} e^{\ln(x)}}{1 - e^{\ln(x)}}, \quad 0 < x < 1,$$

and hence

$$s(x) = -\ln\left(1 - e^{\ln(x)}\right), \quad 0 < x < 1.$$

Now, in view of Theorem 2.1.1, $X$ has density (4.2).

**Corollary 2.1.7.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable and let* $q_1(x)$ *be as in Proposition* 2.1.4. *The pdf of X is* (4.2) *if and only if there exist functions* $q_2$ *and* $\eta$ *defined in Theorem* 2.1.1 *satisfying the differential equation*

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{x^{-1}e^{\ln(x)}}{1 - e^{\ln(x)}}, \quad 0 < x < 1.$$

**Corollary 2.1.8.** *The general solution of the differential equation in Corollary* 2.1.7 *is*

$$\eta(x) = \left(1 - e^{\ln(x)}\right)^{-1}\left[-\int x^{-1}e^{\ln(x)}(q_1(x))^{-1}q_2(x)dx + D\right],$$

*where* $D$ *is a constant.*

**Proof.** It is similar to that of Corollary 2.1.2.

Note that a set of functions satisfying the differential equation in Corollary 2.1.7, is given in Proposition 2.1.4 with $D = \frac{1}{2}$. However, it should also be noted that there are other triplets $(q_1, q_2, \eta)$ satisfying the conditions of Theorem 2.1.1.

**Proposition 2.1.5.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable and let* $q_1(x) = [P(x)]^{-1}$ *and* $q_2(x) = q_1(x)x^{-d}$ *for* $0 < x < 1$. *The random variable* $X$ *has pdf* (5.2) *if and only if the function* $\eta$ *defined in Theorem* 2.1.1 *has the form*

$$\eta(x) = \frac{1}{2}(x^{-d} + 1), \quad 0 < x < 1.$$

**Proof.** Let $X$ be a random variable with pdf (5.2), then

$$(1 - F(x))E[q_1(X)|X \geq x] = \int_x^1 (1 + a)u^{-d-1}du$$

$$= \frac{1 + a}{d}(x^{-d} - 1), \quad 0 < x < 1,$$

and

$$(1 - F(x))E[q_2(X)|X \geq x] = \int_x^1 (1+a)u^{-2d-1}du$$

$$= \frac{1+a}{2d}(x^{-2d} - 1), \quad 0 < x < 1,$$

and finally

$$\eta(x)q_1(x) - q_2(x) = -\frac{q_1(x)}{2}(x^{-d} - 1) < 0 \text{ for } 0 < x < 1.$$

Conversely, if $\eta$ is given as above, then

$$s'(x) = \frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{dx^{-d-1}}{x^{-d} - 1}, \quad 0 < x < 1,$$

and hence

$$s(x) = -\ln(x^{-d} - 1), \quad 0 < x < 1.$$

Now, in view of Theorem 2.1.1, $X$ has density (5.2).

**Corollary 2.1.9.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable and let* $q_1(x)$ *be as in Proposition* 2.1.5. The *pdf of* $X$ *is* (5.2) *if and only if there exist functions* $q_2$ *and* $\eta$ *defined in Theorem* 2.1.1 *satisfying the differential equation*

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{dx^{-d-1}}{x^{-d} - 1}, \quad 0 < x < 1.$$

**Corollary 2.1.10.** *The general solution of the differential equation in Corollary* 2.1.9 *is*

$$\eta(x) = (x^{-d} - 1)^{-1}\left[-\int dx^{-d-1}(q_1(x))^{-1}q_2(x)dx + D\right],$$

*where* $D$ *is a constant.*

**Proof.** It is similar to that of Corollary 2.1.2.

Note that a set of functions satisfying the differential equation in Corollary 2.1.9, is given in Proposition 2.1.5 with $D = \frac{1}{2}$. However, it should also be noted that there are other triplets $(q_1, q_2, \eta)$ satisfying the conditions of Theorem 2.1.1.

**Proposition 2.1.6.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable and let* $q_1(x) = [P(x)]^{-1}$ *and* $q_2(x) = q_1(x)\, x^{-(d+1)}$ *for* $0 < x < 1$. *The random variable* $X$ *has pdf* (6.2) *if and only if the function* $\eta$ *defined in Theorem* 2.1.1 *has the form*

$$\eta(x) = \frac{1}{2}\left(x^{-(d+1)} + 1\right), \quad 0 < x < 1.$$

**Proof.** Let $X$ be a random variable with pdf (6.2), then

$$(1 - F(x))E[q_1(X) | X \geq x] = \int_x^1 u^{-(d+2)} du$$

$$= \frac{1}{d+1}\left(x^{-(d+1)} - 1\right), \quad 0 < x < 1,$$

and

$$(1 - F(x))E[q_2(X) | X \geq x] = \int_x^1 u^{-2d-3} du$$

$$= \frac{1}{2(d+1)}\left(x^{-2(d+1)} - 1\right), \quad 0 < x < 1,$$

and finally

$$\eta(x) q_1(x) - q_2(x) = -\frac{q_1(x)}{2}\left(x^{-(d+1)} - 1\right) < 0 \text{ for } 0 < x < 1.$$

Conversely, if $\eta$ is given as above, then

$$s'(x) = \frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{(d+1)x^{-d-2}}{x^{-(d+1)} - 1}, \quad 0 < x < 1,$$

and hence

$$s(x) = -\ln\left(x^{-(d+1)} - 1\right), \quad 0 < x < 1.$$

Now, in view of Theorem 2.1.1, $X$ has density (6.2).

**Corollary 2.1.11.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable and let* $q_1(x)$ *be as in Proposition* 2.1.6. *The pdf of X is* (6.2) *if and only if there exist functions* $q_2$ *and* $\eta$ *defined in Theorem* 2.1.1 *satisfying the differential equation*

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{(d+1)x^{-d-2}}{x^{-(d+1)} - 1}, \quad 0 < x < 1.$$

**Corollary 2.1.12.** *The general solution of the differential equation in Corollary* 2.1.11 *is*

$$\eta(x) = \left(x^{-(d+1)} - 1\right)^{-1}\left[-\int (d+1)x^{-d-2}(q_1(x))^{-1}q_2(x)dx + D\right],$$

*where* $D$ *is a constant.*

**Proof.** It is similar to that of Corollary 2.1.2.

Note that a set of functions satisfying the differential equation in Corollary 2.1.11, is given in Proposition 2.1.6 with $D = \frac{1}{2}$. However, it should also be noted that there are other triplets $(q_1, q_2, \eta)$ satisfying the conditions of Theorem 2.1.1.

**Proposition 2.1.7.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable and let* $q_1(x) = [P(x)]^{-1}$ *and* $q_2(x) = q_1(x)x^{-a(1-x^b)}$ *for* $0 < x < 1$. *The random variable X has pdf* (7.2) *if and only if the function* $\eta$ *defined in Theorem* 2.1.1 *has the form*

$$\eta(x) = \frac{1}{2}\left(1 + x^{-a(1-x^b)}\right), \quad 0 < x < 1.$$

**Proof.** Let $X$ be a random variable with pdf (7.2), then

$$(1 - F(x))E[q_1(X)|X \geq x] = \int_x^1 u^{b-1}u^{-a(1-u^b)}\ln(u)du$$

$$= \frac{1}{ab}\left(1 - x^{-a(1-x^b)}\right), \quad 0 < x < 1,$$

and

$$(1 - F(x))E[q_2(X)|X \geq x] = \int_x^1 u^{b-1}u^{-2a(1-u^b)}\ln(u)du$$

$$= \frac{1}{2ab}\left(1 - x^{-2a(1-x^b)}\right), \quad 0 < x < 1,$$

and finally

$$\eta(x)q_1(x) - q_2(x) = -\frac{q_1(x)}{2}\left(1 - x^{-a(1-x^b)}\right) > 0 \text{ for } 0 < x < 1.$$

Conversely, if $\eta$ is given as above, then

$$s'(x) = \frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{abx^{b-1}x^{-a(1-x^b)}\ln(x)}{1 - x^{-a(1-x^b)}}, \quad 0 < x < 1,$$

and hence

$$s(x) = -\ln\left(1 - x^{-a(1-x^b)}\right), \quad 0 < x < 1.$$

Now, in view of Theorem 2.1.1, $X$ has density (7.2).

**Corollary 2.1.13.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable and let* $q_1(x)$ *be as in Proposition* 2.1.7. *The pdf of* $X$ *is* (7.2) *if and only if there exist functions* $q_2$ *and* $\eta$ *defined in Theorem* 2.1.1 *satisfying the differential equation*

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{abx^{b-1}x^{-a(1-x^b)}\ln(x)}{1 - x^{-a(1-x^b)}}, \quad 0 < x < 1.$$

**Corollary 2.1.14.** *The general solution of the differential equation in Corollary* 2.1.13 *is*

$$\eta(x) = \left(1 - x^{-a(1-x^b)}\right)^{-1}\left[-\int abx^{b-1}x^{-a(1-x^b)}\ln(x)(q_1(x))^{-1}q_2(x)dx + D\right],$$

*where D is a constant.*

**Proof.** It is similar to that of Corollary 2.1.2.

Note that a set of functions satisfying the differential equation in Corollary 2.1.13, is given in Proposition 2.1.7 with $D = \frac{1}{2}$. However, it should also be noted that there are other triplets $(q_1, q_2, \eta)$ satisfying the conditions of Theorem 2.1.1.

**Proposition 2.1.8.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable and let* $q_1(x) = [P(x)]^{-1}$ *and* $q_2(x) = q_1(x)[-\ln(x)]^b$ *for* $0 < x < 1$. *The random variable X has pdf* (8.2) *if and only if the function* $\eta$ *defined in Theorem* 2.1.1 *has the form*

$$\eta(x) = \frac{1}{2}[-\ln(x)]^b, \quad 0 < x < 1.$$

**Proof.** Let $X$ be a random variable with pdf (8.2), then

$$(1 - F(x))E[q_1(X)|X \geq x] = \int_x^1 u^{-1}[-\ln(u)]^{b-1}du$$

$$= \frac{1}{b}[-\ln(x)]^b, \quad 0 < x < 1,$$

and

$$(1 - F(x))E[q_2(X)|X \geq x] = \int_x^1 u^{-1}[-\ln(u)]^{2b-1}du$$

$$= \frac{1}{2b}[-\ln(x)]^{2b}, \quad 0 < x < 1,$$

and finally

$$\eta(x)q_1(x) - q_2(x) = -\frac{q_1(x)}{2}[-\ln(x)]^b < 0 \text{ for } 0 < x < 1.$$

Conversely, if $\eta$ is given as above, then

$$s'(x) = \frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{b}{x[-\ln(x)]}, \quad 0 < x < 1,$$

and hence

$$s(x) = -\ln\left([-\ln(x)]^b\right), \quad 0 < x < 1.$$

Now, in view of Theorem 2.1.1, $X$ has density (8.2).

**Corollary 2.1.15.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable and let* $q_1(x)$ *be as in Proposition* 2.1.8. *The pdf of X is* (8.2) *if and only if there exist functions* $q_2$ *and* $\eta$ *defined in Theorem* 2.1.1 *satisfying the differential equation*:

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{b}{x[-\ln(x)]}, \quad 0 < x < 1.$$

**Corollary 2.1.16.** *The general solution of the differential equation in Corollary* 2.1.15 *is*

$$\eta(x) = [-\ln(x)]^{-b}\left[-\int bx^{-1}[-\ln(x)]^{b-1}(q_1(x))^{-1}q_2(x)dx + D\right],$$

*where D is a constant.*

**Proof.** It is similar to that of Corollary 2.1.2.

Note that a set of functions satisfying the differential equation in Corollary 2.1.15, is given in Proposition 2.1.8 with $D = 0$. However, it should also be noted that there are other triplets $(q_1, q_2, \eta)$ satisfying the conditions of Theorem 2.1.1.

**2.2. Characterization in terms of hazard function**

The hazard function, $h_F$, of a twice differentiable distribution function, $F$, satisfies the following first order differential equation:

$$\frac{f'(x)}{f(x)} = \frac{h'_F(x)}{h_F(x)} - h_F(x).$$

It should be mentioned that for many univariate continuous distributions, the above equation is the only differential equation available in terms of the hazard function. In this subsection, we present non-trivial characterizations of some of the new distributions in terms of the hazard function, which are not of the above trivial form.

**Proposition 2.2.1.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable. The random variable X has pdf* (8.2) *if and only if its hazard function* $h_F(x)$ *satisfies the following differential equation*:

$$h'_F(x) + x^{-1}h_F(x) = x^{-1}\frac{d}{dx}\left\{[-\ln(x)]^{b-1}\left[\frac{x^{a+1}}{1-x}[-\ln(x)] - b\ln(1-x)x^a\right]\right\},$$

$$0 < x < 1.$$

**Proof.** It is straightforward and hence omitted.

**2.3. Characterization in terms of the reverse (or reversed) hazard function**

The reverse hazard function, $r_F$, of a twice differentiable distribution function, $F$, is defined as

$$r_F(x) = \frac{f(x)}{F(x)},\ x \in \text{support of } F.$$

In this subsection, we present characterizations of some of the new distributions in terms of the reverse hazard function.

**Proposition 2.3.1.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable. The random variable X has pdf* (1.2) *if and only if its reverse hazard function* $r_F(x)$ *satisfies the following differential equation*:

$$r'_F(x) + (a+1)x^{-1}r_F(x) = x^{-a-2}[-\ln(x)]^{b-1}[(b+1)(a\ln(x) - b)],$$

$$0 < x < 1.$$

*with boundary condition* $\lim_{x \to 1} r_F(x) = 0$.

**Proposition 2.3.2.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable. The random variable X has pdf* (2.2) *if and only if its reverse hazard function* $r_F(x)$ *satisfies the following differential equation*:

$$r'_F(x) + x^{-1}r_F(x) = x^{-1}\left[2(b+c) + (b+4c)\ln(x) + c(\ln(x))^2\right], \quad 0 < x < 1,$$

*with boundary condition* $\lim_{x \to 1} r_F(x) = a + b$.

**Proposition 2.3.3.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable. The random variable X has pdf* (3.2) *if and only if its reverse hazard function* $r_F(x)$ *satisfies the following differential equation*:

$$r'_F(x) + (b-1)(1-x)^{-1}r_F(x) = (1-x)\frac{d}{dx}\left\{\frac{P(x)}{F(x)}\right\}, \quad 0 < x < 1,$$

*with boundary condition* $\lim_{x \to 1} r_F(x) = 0$.

**Proposition 2.3.4.** *Let* $X : \Omega \to (0, 1)$ *be a continuous random variable. The random variable X has pdf* (7.2) *if and only if its reverse hazard function* $r_F(x)$ *satisfies the following differential equation*:

$$r'_F(x) - (b-1)x^{-1}r_F(x) = x^{b-1}\frac{d}{dx}$$

$$\times\left\{x^{-a(1-x^b)}\ln(x)\left[\frac{1}{x^b\ln(x)} + ab\ln(x)\right]\right\}, \quad 0 < x < 1,$$

*with boundary condition* $\lim_{x \to 0} r_F(x) = 0$.

## 3. Conclusion

The problem of characterizing a distribution is an important problem in many different applied fields and has attracted the attention of a good number of researchers. Consequently, various characterization results have been reported in the literature. These characterizations have been established in many different directions. The present work deals with the characterizations of certain univariate continuous distribution based on a simple relationship between two truncated moments; in terms of the hazard and reverse hazard functions. We hope that the results of this work will be helpful to the researchers in the applied fields who are interested to know if their chosen underlying distributions are suitable for their data sets.

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