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A COLLECTION OF NEW VARIABLE-POWER PARAMETRIC CUMULATIVE DISTRIBUTION FUNCTIONS FOR (0, 1)-SUPPORTED DISTRIBUTIONS

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Abstract

In the realm of probability and statistics, the research for accurate and flexible cumulative distribution functions of distributions with support on the unit interval $(0, 1)$ is a prominent challenge. Such distributions are ideal for analyzing proportional or rate-type data, which are ubiquitous in various fields, including biology, finance, environmental science, and reliability analysis. In this article, we introduce a novel collection of eight original cumulative distribution functions that harness the concept of variable-power parametric functions to address this long-standing issue. They may depend on one or several tuning parameters, as well as power, logarithmic, or power-logarithmic functions. Based on them, the collection is enriched by some extended or modified versions with the use of standard transformation schemes (power, type II, transmuted, Topp-Leone, odd Fréchet, etc.). Some graphics illustrate

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the validity of the main findings. We also discuss how new families of distributions can be generated from the proposed cumulative distribution functions. Overall, this article offers a versatile set of tools for modelling and analyzing (0, 1)-supported data and much more with the generated families.

1. Introduction

The study of cumulative distribution functions (CDFs) forms the basis of modern statistics, providing a fundamental framework for modeling and analyzing various random characteristics. These functions allow researchers to quantify the likelihood of observing values within a given range and play a crucial role in hypothesis testing, decision-making, and predictive modelling. However, the need for accurate and adaptable CDFs for distributions with support exclusively within the unit interval $(0, 1)$ has been a persistent challenge, with the aim of efficiently analyzing proportional or rate-type data. This challenge necessitates the development of innovative mathematical tools. The recent advances in this direction include the unit gamma distribution established by Consul and Jain [10], log-Lindley distribution proposed by Gómez-Déniz et al. [13], unit Weibull distribution examined by Mazucheli et al. [26] and later refined by Mazucheli et al. [27], unit Gompertz distribution developed by Mazucheli et al. [25], unit Birnbaum-Saunders distribution studied by Mazucheli et al. [24], log-X gamma distribution established by Altun and Hamedani [3], unit inverse Gaussian distribution introduced by Ghitany et al. [12], unit generalized half-normal distribution proposed by Korkmaz [17], unit Johnson SU distribution established by Gündüz and Korkmaz [14], log-weighted exponential distribution developed by Altun [2], unit Rayleigh distribution studied by Bantan et al. [6], arcsecant hyperbolic normal distribution proposed by Korkmaz et al. [20], unit Burr-XII distribution developed by Korkmaz and Chesneau [19], unit power-logarithmic distribution examined by Chesneau [7], transmuted unit Rayleigh distribution studied by Korkmaz et al. [21], unit half-normal distribution created by Bakouch et al. [5], unit Tessier distribution elaborated by Krishna et al. [22], unit Chen distribution investigated by Korkmaz et al. [18], general unit half-logistic geometric distribution created by Nasiru et al. [28], unit exponential Pareto distribution explored by Haj Ahmad et al. [15], unit-Dagum distribution derived in Condino and Domma [9] and unit upper truncated Weibull distribution explored by Okorie et al. [29]. All these distributions are based on diverse functional schemes and techniques, involving one or several tuning parameters. They are mostly based on modifications of existing distributions with diverse supports.

In this article, we present an innovative solution for this topic: a fresh collection of variable-power parametric (VPP) CDFs designed specifically for $(0, 1)$-supported distributions. The VPP CDFs have the originality to be defined with functions of the following form: $[U(x)]^{V(x)}$, i.e., $U(x)$ power $V(x)$, where $U(x)$ and $V(x)$ denote two functions depending on a variable-parameter x and possibly one or several tuning parameters (the convention $0^0 = 1$ holds, as discussed in Vaughan [37]). The concept of VPP CDFs was briefly discussed in a two-dimensional setting by Chesneau [8], but the univariate case has garnered relatively little attention when $V(x)$ is not reduced to a constant. This lack of consideration is potentially due to the inherent complexity of the generated CDFs. However, we argue that this complexity should be reassessed in light of recent developments in computational capabilities. We anticipate that such an innovative approach will significantly propel the field of statistics forward and enhance our understanding of $(0, 1)$-supported data in a more nuanced and accurate manner. Indeed, by allowing flexibility in the exponent of the power function, we are able to create eight versatile CDFs capable of adapting to a wide range of data patterns. In this article, the validity of the proposed VPP CDFs is rigorously established through detailed proofs and graphical analyses. In addition, we determine some natural extensions or modifications of them by employing standard transformation schemes (power, type II, transmuted, Topp-Leone, odd Fréchet, etc.). We also show how numerous new families of distributions can be generated, and some concrete examples of distributions with various supports are given. The collection is thus completed. The contributions of the article are mainly theoretical, aiming to give the mathematical keys for practical developments in future works.

The remainder of the article contains nine sections, each of which presents a new VPP CDF of a specific kind, except for the final section, which proposes a conclusion.

2. VPP CDF of the First Kind

2.1. Main result

The proposition below presents a VPP CDF, say "of the first kind", of the form $x^{V(x)}$, where $V(x)$ is defined as a ratio of distinct logarithmic and power functions.

Proposition 2.1. *The function defined by*

$$F(x) = x^{\frac{[-\ln(x)]^b}{x^a}}, \quad x \in (0, 1), \tag{1}$$

and completed by $F(x) = 0$ *for* $x \leq 0$ *and* $F(x) = 1$ *for* $x \geq 1$, *is a valid CDF for* $a \geq 0$ *and* $b > -1$.

Proof. The proof is based on the definition of a CDF of a $(0, 1)$-supported distribution, that is, $F(x) \geq 0$ for any $x \in \mathbb{R}$, $\lim_{x \to 0^+} F(x) = 0$, $\lim_{x \to 1^-} F(x) = 1$, and, for any $x \in (0, 1)$, $F'(x) \geq 0$. This standard definition will be employed throughout the article.

From Equation (1), for any $x \in (0, 1)$, $a \geq 0$ and $b > -1$, we have $F(x) = x^{\frac{[-\ln(x)]^b}{x^a}} \geq 0$. For any $x \leq 0$ or $x \geq 1$, it is obvious that $F(x) \geq 0$. Furthermore, since $a \geq 0$ and $b > -1$, we have

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} x^{\frac{[-\ln(x)]^b}{x^a}} = \lim_{x \to 0^+} e^{-\frac{[-\ln(x)]^{b+1}}{x^a}} = \lim_{y \to -\infty} e^y = 0.$$

Also, recalling that $1_{\{b \neq 0\}} = 1$ if $b \neq 0$ and $1_{\{b \neq 0\}} = 0$ otherwise, we have

$$\lim_{x \to 1^-} F(x) = \lim_{x \to 1^-} x^{\frac{[-\ln(x)]^b}{x^a}} = 1^{1_{\{b \neq 0\}}} = 1.$$

Also, for any $x \in (0, 1)$, after a differentiation work, we obtain

$$F'(x) = [-\ln(x)]^b x^{\frac{[-\ln(x)]^b}{x^a} - a - 1} [-a \ln(x) + b + 1].$$

Since $a \geq 0$ and $b > -1$, it is clear that $[-\ln(x)]^b x^{\frac{[-\ln(x)]^b}{x^a} - a - 1} \geq 0$, $-a \ln(x) \geq 0$ and $b + 1 > 0$, implying that $F'(x) \geq 0$. As a result, $F(x)$ is non-decreasing for $x \in (0, 1)$. This ends the proof; $F(x)$ is a valid CDF.

□

Thus, the VPP CDF of the first kind is a two-parameter CDF of a (0, 1)-supported distribution. To the best of our knowledge, it is the first time that it is mentioned in the literature. Eventually, for any $x \in (0, 1)$, we can write it under the following form:

$$F(x) = e^{-\frac{[-\ln(x)]^{b+1}}{x^a}}.$$

When $a = 0$, the denominator term simplifies to 1, leading to the VPP CDF of the first kind becoming analogous to a specific case of the CDF associated with the unit Weibull distribution, as detailed by Mazucheli et al. [26]. In this case, when $b = 0$, we get the CDF of the $(0, 1)$-supported uniform distribution, i.e., $F(x) = x$ for $x \in (0, 1)$, completed by $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 1$. This connection underscores the interrelationship between the corresponding distributions under certain conditions, shedding light on their unified behaviour in this particular scenario.

Figure 1 illustrates Proposition 2.1 by displaying the VPP CDF of the first kind for several values of a and b satisfying $a \geq 0$ and $b > -1$. It is made with the R software (see R Core Team [30]).

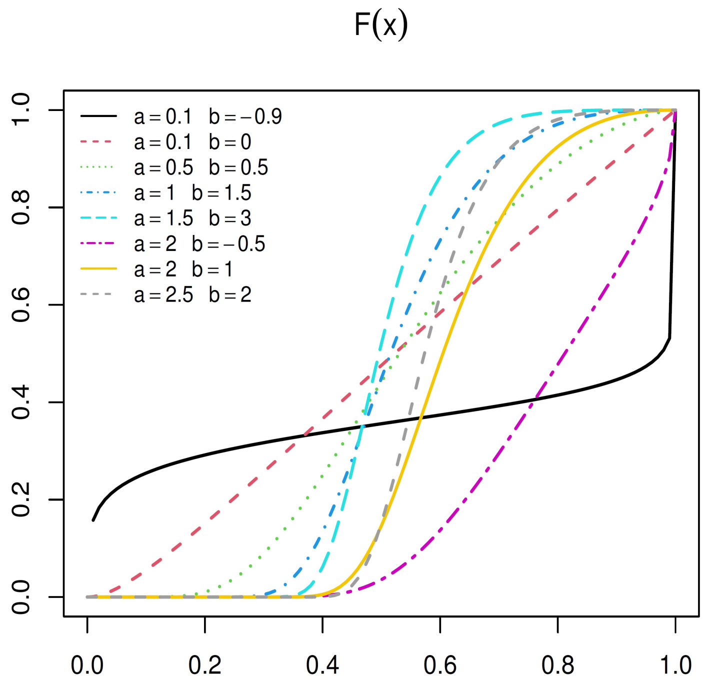

Figure 1. Plots of $F(x)$ in Equation (1) for several values of a and b satisfying $a \geq 0$ and $b > -1$.

Based on this figure, it is evident that the VPP CDF of the first kind is well-defined for the parameters under consideration. Furthermore, it exhibits a notable level of flexibility, showcasing various concave and convex shapes. A more deep analysis of this flexibility can also be revealed by the study of the corresponding probability density function (PDF) given as

$$f(x) = F'(x) = [-\ln(x)]^b x^{\frac{[-\ln(x)]^b}{x^a} - a - 1}[-a\ln(x) + b + 1], \quad x \in (0, 1),$$

and $f(x) = 0$ for $x \notin (0, 1)$, or the corresponding hazard rate function indicated as

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{1}{1 - x^{\frac{[-\ln(x)]^b}{x^a}}}[-\ln(x)]^b x^{\frac{[-\ln(x)]^b}{x^a} - a - 1}[-a\ln(x) + b + 1],$$

$$x \in (0, 1),$$

and $h(x) = 0$ for $x \notin (0, 1)$.

Some examples of new one-parameter VPP CDFs derived from this CDF are listed below.

- The function specified by

$$F(x) = x^{\frac{1}{x^a}} = x^{x^{-a}}, \quad x \in (0, 1),$$

with the addition of $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $a \geq 0$. It is derived from Equation (1) by choosing $b = 0$.

- The function defined by

$$F(x) = x^{[-\ln(x)]^b}, \quad x \in (0, 1),$$

and completed by $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $b > -1$. It is obtained using Equation (1) by selecting $a = 0$. As mentioned earlier, it is a special case of the unit Weibull distribution.

• The function specified by

$$F(x) = x^{\left[-\frac{\ln(x)}{x}\right]^{a}}, \quad x \in (0, 1),$$

with the addition of $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $a \geq 0$. It is derived from Equation (1) by choosing $b = a$.

The three CDFs presented above possess distinctive VPP expressions, each characterized by the presence of a single parameter. Clearly, these functions are of moderate complexity and may have versatile applicability for diverse modelling and data analysis purposes.

The following section presents further results focused on the VPP CDF of the first kind.

2.2. Complements

By leveraging standard transformation methodologies, it is feasible to develop modified or expanded versions of any CDF. In support of this assertion, Table 1 delineates fundamental schemes for creating novel CDFs derived from an established CDF of a $(0, 1)$-supported distribution, denoted in full generality as $F(x)$. This table also includes pertinent references associated with each scheme.

Table 1. Main schemes to generate new CDFs from an existing CDF $F(x)$ of a $(0, 1)$-supported distribution

Scheme	Reference	Parameter(s)	CDF for $x \in (0, 1)$
Power	(classic)	$\alpha > 0$	$[F(x)]^{\alpha}$
Type II	(classic)	–	$1 - F(1-x)$
Transmuted	Shaw and Buckley [31]	$\lambda \in [-1, 1]$	$(1+\lambda)F(x) - \lambda[F(x)]^2$
Marshall-Olkin	Marshall and Olkin [23]	$\theta > 0$	$\frac{F(x)}{\theta + (1-\theta)F(x)}$
Topp-Leone	Al-Shomrani et al. [1]	$\alpha > 0$	$[F(x)]^{\alpha}[2 - F(x)]^{\alpha}$
Weibull	Alzaatreh et al. [4]	$\alpha > 0, \beta > 0$	$1 - e^{-\left\{-\frac{1}{\beta}\ln[1-F(x)]\right\}^{\alpha}}$
Odd Fréchet	Haq and Elgarhy [16]	$\alpha > 0$	$e^{-\left[\frac{1-F(x)}{F(x)}\right]^{\alpha}}$
Sin	Souza et al. [32]	–	$\sin\left[\frac{\pi}{2}F(x)\right]$
Cos	Souza et al. [33]	–	$1 - \cos\left[\frac{\pi}{2}F(x)\right]$
Tan	Souza et al. [35]	–	$\tan\left[\frac{\pi}{4}F(x)\right]$
Sec	Souza et al. [34]	–	$\sec\left[\frac{\pi}{3}F(x)\right] - 1$

Let us notice that the type II scheme is really connected to the fact that $F(x)$ is a CDF of a $(0, 1)$-supported distribution; this is not the case for the other schemes. Table 1 additionally underscores the significance of trigonometric schemes, namely, the Sin, Cos, Tan and Sec schemes. These schemes introduce novel perspectives on trigonometric distributions, which are particularly relevant for contemporary datasets characterized by intricate structures.

Based on the existing schemes described in Table 1, Table 2 contains extended or modified versions of the VPP CDF of the first kind defined in Equation (1).

Table 2. Extended or modified versions of the CDF in Equation (1) based on the schemes presented in Table 1

Scheme	Parameter(s)	CDF for $x \in (0, 1)$
Power	$\alpha > 0,\ a \geq 0,\ b > -1$	$x^{\alpha \frac{[-\ln(x)]^b}{x^a}}$
Type II	$a \geq 0,\ b > -1$	$1-(1-x)^{\frac{[-\ln(1-x)]^b}{(1-x)^a}}$
Transmuted	$\lambda \in [-1, 1],\ a \geq 0,\ b > -1$	$(1+\lambda)x^{\frac{[-\ln(x)]^b}{x^a}} - \lambda x^{2\frac{[-\ln(x)]^b}{x^a}}$
Marshall-Olkin	$\theta > 0,\ a \geq 0,\ b > -1$	$\dfrac{x^{\frac{[-\ln(x)]^b}{x^a}}}{\theta+(1-\theta)x^{\frac{[-\ln(x)]^b}{x^a}}}$
Topp-Leone	$\alpha > 0,\ a \geq 0,\ b > -1$	$x^{\alpha\frac{[-\ln(x)]^b}{x^a}}\left(2-x^{\frac{[-\ln(x)]^b}{x^a}}\right)^{\alpha}$
Weibull	$\alpha > 0,\ \beta > 0,\ a \geq 0,\ b > -1$	$1-e^{-\left[-\frac{1}{\beta}\ln\left(1-x^{\frac{[-\ln(x)]^b}{x^a}}\right)\right]^{\alpha}}$
Odd Fréchet	$\alpha > 0,\ a \geq 0,\ b > -1$	$e^{-\left(\frac{1-x^{\frac{[-\ln(x)]^b}{x^a}}}{x^{\frac{[-\ln(x)]^b}{x^a}}}\right)^{\alpha}}$
Sin	$a \geq 0,\ b > -1$	$\sin\left(\frac{\pi}{2}x^{\frac{[-\ln(x)]^b}{x^a}}\right)$

Table 2. (Continued)

Scheme	Parameter(s)	CDF for $x \in (0, 1)$
Cos	$a \geq 0,\ b > -1$	$1 - \cos\left(\frac{\pi}{2} x^{\frac{[-\ln(x)]^b}{x^a}}\right)$
Tan	$a \geq 0,\ b > -1$	$\tan\left(\frac{\pi}{4} x^{\frac{[-\ln(x)]^b}{x^a}}\right)$
Sec	$a \geq 0,\ b > -1$	$\sec\left(\frac{\pi}{3} x^{\frac{[-\ln(x)]^b}{x^a}}\right) - 1$

This table expands the scope of the VPP CDF of the first kind by incorporating diverse extensions and modifications, enriching the collection of VPP CDFs.

We now explore various useful techniques to generate continuous distributions using the VPP CDF of the first kind. To begin, as a well-known fact, if $F(x)$ is CDF of a $(0, 1)$-supported distribution and $G(x)$ is any CDF of a continuous distribution (whatever the support), then $J(x) = F[G(x)]$ is a valid CDF. As a result, by selecting $F(x)$ as described in Equation (1), we define a new family of distributions by the following CDF:

$$J(x) = F[G(x)] = [G(x)]^{\frac{\{-\ln[G(x)]\}^b}{[G(x)]^a}}, \quad x \in \mathbb{R}. \tag{2}$$

Based on this, some specific CDFs of distributions with various supports are listed below.

- The function defined by

$$J(x) = x^{\theta^{b+1}\frac{[-\ln(x)]^b}{x^{\theta a}}}, \quad x \in (0, 1),$$

and completed by $J(x) = 0$ for $x \leq 0$ and $J(x) = 1$ for $x \geq 1$, is a valid CDF for $a \geq 0$, $b > -1$ and $\theta > 0$. It is obtained using Equation (2) by choosing the CDF of the power distribution with parameter θ for $G(x)$, i.e., $G(x) = x^{\theta}$ for $x \in (0, 1)$, with the complement of $G(x) = 0$ for $x \leq 0$ and $G(x) = 1$ for $x \geq 1$. A new $(0, 1)$-supported distribution is thus created.

- The function specified by

$$J(x) = (1 - e^{-\lambda x})^{\frac{[-\ln(1-e^{-\lambda x})]^b}{(1-e^{-\lambda x})^a}}, \quad x > 0,$$

and completed by $J(x) = 0$ for $x \leq 0$, is a valid CDF for $a \geq 0$, $b > -1$ and $\lambda > 0$. It is derived from Equation (2) by selecting the CDF of the exponential distribution with parameter λ for G(x), i.e., $G(x) = 1 - e^{-\lambda x}$ for $x > 0$, with the addition of $G(x) = 0$ for $x \leq 0$. A novel lifetime distribution is thereby established.

- The function defined by

$$J(x) = (1 + e^{-\beta x})^{-(1+e^{-\beta x})^a [\ln(1+e^{-\beta x})]^b}, \quad x \in \mathbb{R},$$

is a valid CDF for $a \geq 0$, $b > -1$ and $\beta > 0$. It is obtained using Equation (2) by choosing the CDF of the logistic distribution with parameter β for $G(x)$, i.e., $G(x) = (1 + e^{-\beta x})^{-1}$ for $x \in \mathbb{R}$. A new whole real line-supported distribution thus emerges.

Beyond the classical composition scheme, a more nuanced result with a focus on a baseline CDF of a $(0, 1)$-supported distribution is presented below.

Proposition 2.2. *Let* $G(x)$ *be a CDF of a* $(0, 1)$*-supported distribution,* $a \in \mathbb{R}$ *and* $b \in \mathbb{R}$. *Let us consider the following function*:

$$F(x) = [G(x)]^{\frac{[-\ln(x)]^b}{x^a}}, \quad x \in (0, 1), \tag{3}$$

which is completed by $F(x) = 0$ *for* $x \leq 0$ *and* $F(x) = 1$ *for* $x \geq 1$. *If we have*

$$\lim_{x \to 0^+} \frac{[-\ln(x)]^b \ln[G(x)]}{x^a} = -\infty, \tag{4}$$

for $a \geq 0$ *and* $b \geq 0$, *then* $F(x)$ *is a valid CDF for such* a *and* b.

Proof. For any $x \in (0, 1)$, $a \geq 0$ and $b \geq 0$, we have $F(x) = [G(x)]^{\frac{[-\ln(x)]^b}{x^a}} \geq 0$. For any $x \leq 0$ or $x \geq 1$, it is obvious that $F(x) \geq 0$. Furthermore, since $a \geq 0$ and $b \geq 0$, by taking into account the limit assumption in Equation (4), we have

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} [G(x)]^{\frac{[-\ln(x)]^b}{x^a}} = \lim_{x \to 0^+} e^{-\frac{[-\ln(x)]^b \ln[G(x)]}{x^a}} = \lim_{y \to -\infty} e^y = 0,$$

and

$$\lim_{x \to 1^-} F(x) = \lim_{x \to 1^-} [G(x)]^{\frac{[-\ln(x)]^b}{x^a}} = 1^{1_{\{b \neq 0\}}} = 1.$$

Also, for any $x \in (0, 1)$, upon differentiation of $F(x)$, by denoting $g(x)$ the PDF associated to $G(x)$, we obtain

$$F'(x) = [G(x)]^{\frac{[-\ln(x)]^b}{x^a}} \times \left[x^{-a}[-\ln(x)]^b \frac{g(x)}{G(x)} \right.$$
$$\left. - \ln[G(x)]\{bx^{-a-1}[-\ln(x)]^{b-1} + ax^{-a-1}[-\ln(x)]^b\} \right].$$

By the definition of a PDF, we have $g(x) \geq 0$. Furthermore, since $G(x) \in [0, 1]$, $a \geq 0$ and $b \geq 0$, it is clear that $-\ln[G(x)] \geq 0$ and all the main terms are positive, implying that $F'(x) \geq 0$. As a result, $F(x)$ is non-decreasing for $x \in (0, 1)$. The proof is completed; $F(x)$ is a valid CDF. □

This result has the merit of being general. However, it is not optimal; for an explicit CDF $G(x)$, we can improve the possible values for a and b, beyond $a \geq 0$ and $b \geq 0$. This claim was illustrated in Proposition 2.1 with the CDF of the $(0, 1)$-supported uniform distribution, which gives $a \geq 0$ and $b > -1$.

Thanks to Proposition 2.2, a plethora of new VPP CDFs can be elaborated. Some examples are given below.

• The function specified by

$$F(x) = [x(2-x)]^{\frac{[-\ln(x)]^b}{x^a}}, \quad x \in (0, 1),$$

completed by $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $a \geq 0$ and $b \geq 0$. It is derived from Equation (3) by selecting the CDF of the triangular distribution for $G(x)$, i.e., $G(x) = x(2-x)$ for $x \in (0, 1)$, and completed by $G(x) = 0$ for $x \leq 0$ and $G(x) = 1$ for $x \geq 1$. Let us mention that the condition in Equation (4) is satisfied for this CDF. A novel $(0, 1)$-supported distribution is thus added to the literature.

• The function defined by

$$F(x) = \left[\sin\left(\frac{\pi}{2}x\right)\right]^{\frac{[-\ln(x)]^b}{x^a}}, \quad x \in (0, 1),$$

completed by $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $a \geq 0$ and $b \geq 0$. It is obtained using Equation (3) by choosing the CDF of the sine distribution for $G(x)$, i.e., $G(x) = \sin(\frac{\pi}{2}x)$ for $x \in (0, 1)$, with the complement of $G(x) = 0$ for $x \leq 0$ and $G(x) = 1$ for $x \geq 1$. Let us mention that the condition in Equation (4) is satisfied for this CDF. To the best of our knowledge, the corresponding distribution is the very first VPP trigonometric distribution.

We end this part by a result on an original polynomial-power extension of the VPP CDF of the first kind. The key change concerns the denominator term.

Proposition 2.3. *Let us consider the following polynomial-power function*:

$$Q(x) = \sum_{k=1}^{m} a_k x^{b_k},$$

where m *denotes a positive integer, and* $a_1, \ldots, a_m$ *and* $b_1, \ldots, b_m$ *are real numbers. Then the function specified by*

$$F(x) = x^{\frac{[-\ln(x)]^c}{Q(x)}}, \quad x \in (0, 1), \tag{5}$$

and completed by $F(x) = 0$ *for* $x \leq 0$ *and* $F(x) = 1$ *for* $x \geq 1$, *is a valid CDF for non-negative real numbers* $a_1, \ldots, a_m$ *such that* $\sum_{k=1}^{m} a_k \neq 0$, *non-negative real numbers* $b_1, \ldots, b_m$ *and* $c > -1$.

Proof. For any $x \in (0, 1)$, non-negative real numbers $a_1, \ldots, a_m$ such that $\sum_{k=1}^{m} a_k \neq 0$, nonnegative real numbers $b_1, \ldots, b_m$ and $c > -1$, we have $F(x) = x^{\frac{[-\ln(x)]^c}{Q(x)}} \geq 0$. For any $x \leq 0$ or $x \geq 1$, it is obvious that $F(x) \geq 0$. Furthermore, thanks to the considered assumptions on the parameters, we have

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} x^{\frac{[-\ln(x)]^c}{Q(x)}} = \lim_{x \to 0^+} e^{-\frac{[-\ln(x)]^{c+1}}{Q(x)}} = \lim_{y \to -\infty} e^y = 0,$$

and

$$\lim_{x \to 1^-} F(x) = \lim_{x \to 1^-} x^{\frac{[-\ln(x)]^c}{Q(x)}} = 1^{\frac{1_{\{c \neq 0\}}}{\sum_{k=1}^{m} a_k}} = 1.$$

Also, for any $x \in (0, 1)$, by applying standard differentiation rules, we get

$$F'(x) = [-\ln(x)]^c x^{\frac{[-\ln(x)]^c}{Q(x)}} \left\{ \frac{c+1}{xQ(x)} - \ln(x) \frac{Q'(x)}{[Q(x)]^2} \right\},$$

with

$$Q'(x) = \sum_{k=1}^{m} a_k b_k x^{b_k - 1}.$$

Owing to the assumptions made on the parameters, it is clear that $[-\ln(x)]^c x^{\frac{[-\ln(x)]^c}{Q(x)}} \geq 0$, $-\ln(x) \geq 0$, $Q(x) \geq 0$, $Q'(x) \geq 0$ and $c + 1 > 0$, implying that $F'(x) \geq 0$. Hence, $F(x)$ is non-decreasing for $x \in (0, 1)$. This ends the proof; $F(x)$ is a valid CDF. □

This extended version of Proposition 2.1 demonstrates the potential for exploring novel modelling horizons through the utilization of the VPP CDF concept.

Other kinds of VPP CDFs are examined in the rest of the study.

3. VPP CDF of the Second Kind

3.1. Main result

The proposition below presents a VPP CDF, say "of the second kind", of the form $x^{V(x)}$, where $V(x)$ is defined as a sum of polynomial and polynomial-logarithmic functions.

Proposition 3.1. *The function defined by*

$$F(x) = x^{a+bx+cx\ln(x)}, \quad x \in (0, 1), \tag{6}$$

with the addition of $F(x) = 0$ *for* $x \leq 0$ *and* $F(x) = 1$ *for* $x \geq 1$, *is a valid CDF for* $b \leq 0$, $a \geq 1_{\{c \neq 0\}} - b$, $a > 0$, *and* $c \in [0, 1]$.

Proof. For any $x \in (0, 1)$, we immediately have $F(x) = x^{a+bx+cx\ln(x)} \geq 0$. For any $x \leq 0$ or $x \geq 1$, it is clear that $F(x) \geq 0$. Furthermore, since $\lim_{x \to 0^+} x\ln(x) = 0$ and $a > 0$, we have

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} x^{a+bx+cx\ln(x)} = \lim_{x \to 0^+} x^a = 0.$$

On the other hand, it is clear that

$$\lim_{x \to 1^-} F(x) = \lim_{x \to 1^-} x^{a+bx+cx\ln(x)} = \lim_{x \to 1^-} x^{a+b} = 1.$$

Also, for any $x \in (0, 1)$, upon differentiation and a suitable factorizing, we obtain

$$F'(x) = x^{a+bx+cx\ln(x)-1}\{a + bx + bx\ln(x) + 2cx\ln(x) + cx[\ln(x)]^2\}$$

$$= x^{a+bx+cx\ln(x)-1} \times \{a + bx - 1_{\{c \neq 0\}} + bx\ln(x)$$

$$+ cx(1 - cx)[\ln(x)]^2 + [1 + cx\ln(x)]^2 1_{\{c \neq 0\}}\}.$$

It is immediate that $x^{a+bx+cx\ln(x)-1} \geq 0$. Since $b \leq 0$ and $a \geq 1_{\{c \neq 0\}} - b$, we have

$$a + bx - 1_{\{c \neq 0\}} \geq a + b - 1_{\{c \neq 0\}} \geq 0.$$

Furthermore, since $b \leq 0$, we have $bx\ln(x) \geq 0$ and, since $c \in [0, 1]$, we have $cx \geq 0$ and $1 - cx \geq 0$, implying that $cx(1 - cx)[\ln(x)]^2 \geq 0$. It is obvious that $[1 + cx\ln(x)]^2 1_{\{c \neq 0\}} \geq 0$. As a result, we have $F'(x) \geq 0$. Hence, $F(x)$ is non-decreasing for $x \in (0, 1)$. This completes the proof; $F(x)$ is a valid CDF. □

Based on our current understanding, Proposition 3.1 marks the initial reference to the VPP CDF of the second kind in literature. Eventually, for any $x \in (0, 1)$, we can express it as follows:

$$F(x) = e^{\ln(x)[a+bx+cx\ln(x)]}.$$

When $b = 0$ and $c = 0$, the VPP CDF of the second kind corresponds to the one of the power distribution with parameter a. Figure 2 visually represents some specific findings in Proposition 3.1 by showcasing $F(x)$ for various values of a, b, and c. These values adhere to the conditions: $b \leq 0$, $a \geq 1_{c \neq 0} - b$, $a > 0$ and $c \in [0, 1]$.

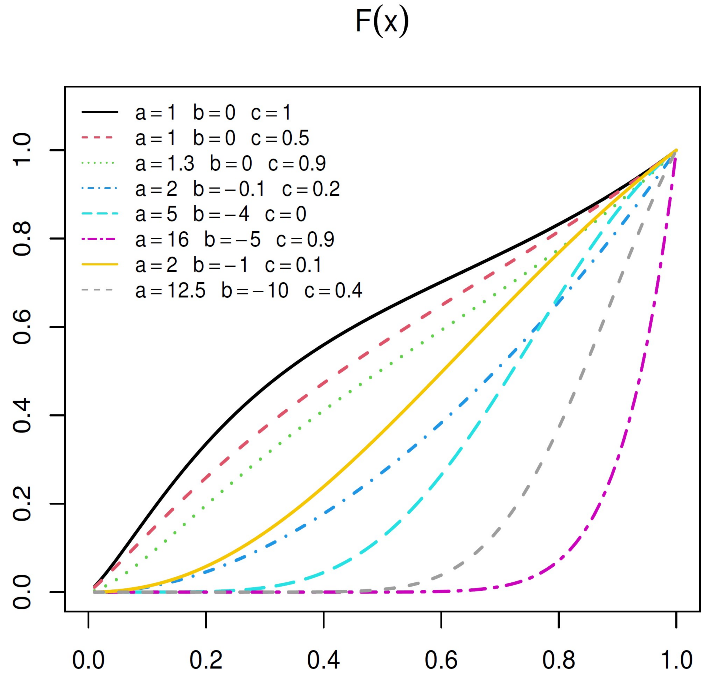

Figure 2. Plots of $F(x)$ in Equation (6) for several values of a, b and c satisfying $b \leq 0$, $a \geq 1_{\{c \neq 0\}} - b$, $a > 0$ and $c \in [0, 1]$.

This figure clearly demonstrates the suitability of the VPP CDF of the second kind for the parameters being considered. Notably, it showcases a remarkable degree of flexibility, displaying a diverse range of concave and convex shapes, mainly in the lower-right triangle. Below, we present a few examples of new and simple one-parameter VPP CDFs that can be derived from this foundational VPP CDF.

• The function specified by

$$F(x) = x^{1+bx}, \quad x \in (0, 1),$$

and completed by $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $b \in [-1, 0]$. It is derived from Equation (6) by selecting $a = 1$ and $c = 0$. It characterizes a new (0, 1)-supported distribution.

• The function defined by

$$F(x) = x^{1+cx\ln(x)}, \quad x \in (0, 1),$$

with the complement of $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $c \in [0, 1]$. It is obtained using Equation (6) by choosing $a = 1$ and $b = 0$. Once more, it characterizes a novel distribution supported on the interval $(0, 1)$.

These two CDFs feature unique VPP expressions, each specified by a single parameter. They offer versatile applicability for a wide range of modeling and data analysis purposes.

3.2. Complements

The VPP CDF of the second kind, as indicated in Equation (6), is extended or modified in Table 3, based on the schemes shown in Table 1.

Table 3. Extended or modified versions of the CDF in Equation (6) based on the schemes presented in Table 1

Scheme	Parameter(s)	CDF for $x \in (0, 1)$
Power	$\alpha > 0, b \le 0, a \ge 1_{\{c \ne 0\}} - b,\ a > 0, c \in [0, 1]$	$x^{\alpha[a+bx+cx\ln(x)]}$
Type II	$b \le 0, a \ge 1_{\{c \ne 0\}} - b,\ a > 0, c \in [0, 1]$	$1-(1-x)^{a+b(1-x)+c(1-x)\ln(1-x)}$
Transmuted	$\lambda \in [-1, 1], b \le 0, a \ge 1_{\{c \ne 0\}} - b,$ $a > 0, c \in [0, 1]$	$(1+\lambda)x^{a+bx+cx\ln(x)}$ $-\lambda x^{2[a+bx+cx\ln(x)]}$
Marshall-Olkin	$\theta > 0, b \le 0, a \ge 1_{\{c \ne 0\}} - b,\ a > 0, c \in [0, 1]$	$\dfrac{x^{a+bx+cx\ln(x)}}{\theta+(1-\theta)x^{a+bx+cx\ln(x)}}$
Topp-Leone	$\alpha > 0, b \le 0, a \ge 1_{\{c \ne 0\}} - b,\ a > 0, c \in [0, 1]$	$x^{\alpha[a+bx+cx\ln(x)]}$ $(2-x^{a+bx+cx\ln(x)})^{\alpha}$
Weibull	$\alpha > 0, \beta > 0, b \le 0, a \ge 1_{\{c \ne 0\}} - b,$ $a > 0, c \in [0, 1]$	$1-e^{-\{-\frac{1}{\beta}\ln[1-x^{a+bx+cx\ln(x)}]\}^{\alpha}}$
Odd Fréchet	$\alpha > 0, b \le 0, a \ge 1_{\{c \ne 0\}} - b,\ a > 0, c \in [0, 1]$	$e^{-\left(\frac{1-x^{a+bx+cx\ln(x)}}{x^{a+bx+cx\ln(x)}}\right)^{\alpha}}$
Sin	$b \le 0, a \ge 1_{\{c \ne 0\}} - b, a > 0, c \in [0, 1]$	$\sin\left(\frac{\pi}{2}x^{a+bx+cx\ln(x)}\right)$
Cos	$b \le 0, a \ge 1_{\{c \ne 0\}} - b, a > 0, c \in [0, 1]$	$1-\cos\left(\frac{\pi}{2}x^{a+bx+cx\ln(x)}\right)$
Tan	$b \le 0, a \ge 1_{\{c \ne 0\}} - b, a > 0, c \in [0, 1]$	$\tan\left(\frac{\pi}{4}x^{a+bx+cx\ln(x)}\right)$
Sec	$b \le 0, a \ge 1_{\{c \ne 0\}} - b, a > 0, c \in [0, 1]$	$\sec\left(\frac{\pi}{3}x^{a+bx+cx\ln(x)}\right)-1$

This table thus presents a selection of CDFs that can be obtained by expanding upon the VPP CDF of the second kind through various extensions and modifications. Additionally, numerous other perspectives for potential variants are conceivable.

We can also use the VPP CDF of the second kind to generate continuous distributions with various supports. Indeed, for any CDF of a continuous distribution (whatever the support), say $G(x)$, based on $F(x)$ as defined in Equation (6), we define a new family of distributions by the following CDF:

$$J(x) = F[G(x)] = [G(x)]^{a+bG(x)+cG(x)\ln[G(x)]}, \quad x \in \mathbb{R}. \tag{7}$$

Some examples are given below.

- The function defined by

$$J(x) = x^{\theta[a+bx^{\theta}+c\theta x^{\theta}\ln(x)]}, \quad x \in (0, 1),$$

and completed by $J(x) = 0$ for $x \leq 0$ and $J(x) = 1$ for $x \geq 1$, is a valid CDF for $b \leq 0$, $a \geq 1_{\{c \neq 0\}} - b$, $a > 0$, $c \in [0, 1]$ and $\theta > 0$. It is derived from Equation (7) by selecting the CDF of the power distribution with parameter θ for $G(x)$.

- The function specified by

$$J(x) = (1 - e^{-\lambda x})^{a+b(1-e^{-\lambda x})+c(1-e^{-\lambda x})\ln(1-e^{-\lambda x})}, \quad x > 0,$$

with the addition of $J(x) = 0$ for $x \leq 0$, is a valid CDF for $b \leq 0$, $a \geq 1_{\{c \neq 0\}} - b$, $a > 0$, $c \in [0, 1]$ and $\lambda > 0$. It is obtained using Equation (7) by choosing the CDF of the exponential distribution with parameter λ for $G(x)$.

- The function defined by

$$J(x) = (1 + e^{-\beta x})^{-a-b(1+e^{-\beta x})^{-1}+c(1+e^{-\beta x})^{-1}\ln(1+e^{-\beta x})}, \quad x \in \mathbb{R},$$

is a valid CDF for $b \leq 0$, $a \geq 1_{\{c \neq 0\}} - b$, $a > 0$, $c \in [0, 1]$ and $\beta > 0$. It is derived from Equation (7) by selecting the CDF of the logistic distribution with parameter β for $G(x)$.

Based on current knowledge, the three demonstrated CDFs feature distinct VPP expressions, each specified by four tuning parameters and diverse supports. These functions provide extensive versatility for various modelling and data analysis requirements. However, their practical application awaits further exploration in future studies.

4. VPP CDF of the Third Kind

4.1. Main result

The subsequent proposition introduces a new VPP CDF, denoted as “of the third kind”, using the functional form $[U(x)]^{V(x)}$, where $U(x) = 1 - x$ and $V(x)$ is a proportional and exponentiated version of $1 - x$.

Proposition 4.1. *The function defined by*

$$F(x) = (1-x)^{a(1-x)^b} - (1-x), \quad x \in (0, 1), \tag{8}$$

and completed by $F(x) = 0$ *for* $x \leq 0$ *and* $F(x) = 1$ *for* $x \geq 1$, *is a valid CDF for* $a \in [0, 1]$ *and* $b \geq 1$.

Proof. For any $x \in (0, 1)$, $a \in [0, 1]$ and $b \geq 1$, since $a(1-x)^b \in [0, 1]$, we have

$$(1-x) = (1-x)^{a(1-x)^b + [1 - a(1-x)^b]} \leq (1-x)^{a(1-x)^b},$$

which implies that $F(x) = (1-x)^{a(1-x)^b} - (1-x) \geq 0$. For any $x \leq 0$ or $x \geq 1$, it is obvious that $F(x) \geq 0$. Furthermore, we have

$$\lim_{x \to 0^+} F(x) = (1-0)^{a(1-0)^b} - (1-0) = 1^a - 1 = 0,$$

and, since $0^0 = 1$ by convention,

$$\lim_{x \to 1^-} F(x) = (1-1)^{a(1-1)^b} - (1-1) = 0^0 - 0 = 1.$$

Also, for any $x \in (0, 1)$, with standard differentiation rules, we obtain

$$F'(x) = 1 - a(1-x)^{b-1+a(1-x)^b} - ab(1-x)^{b-1+a(1-x)^b} \ln(1-x).$$

Since $a \in [0, 1]$ and $b \geq 1$, we have $(1-x)^{b-1+a(1-x)^b} \in [0, 1]$ and $-ab\ln(1-x) \geq 0$. Therefore, we have

$$F'(x) \geq 1 - a(1-x)^{b-1+a(1-x)} \geq 1 - a \geq 0.$$

As a result, $F(x)$ is non-decreasing for $x \in (0, 1)$. This ends the proof; $F(x)$ is a valid CDF. □

To the best of our knowledge, this is the initial mention of the VPP CDF of the third kind in the literature. For any $x \in (0, 1)$, we may eventually express it via the exponential function as follows:

$$F(x) = e^{a\ln(1-x)(1-x)^b} - (1-x).$$

We can also notice that the CDF of the $(0, 1)$-supported uniform distribution is obtained by selecting $a = 0$.

Figure 3 illustrates Proposition 4.1 by displaying $F(x)$ for various values of a and b satisfying $a \in [0, 1]$ and $b \geq 1$.

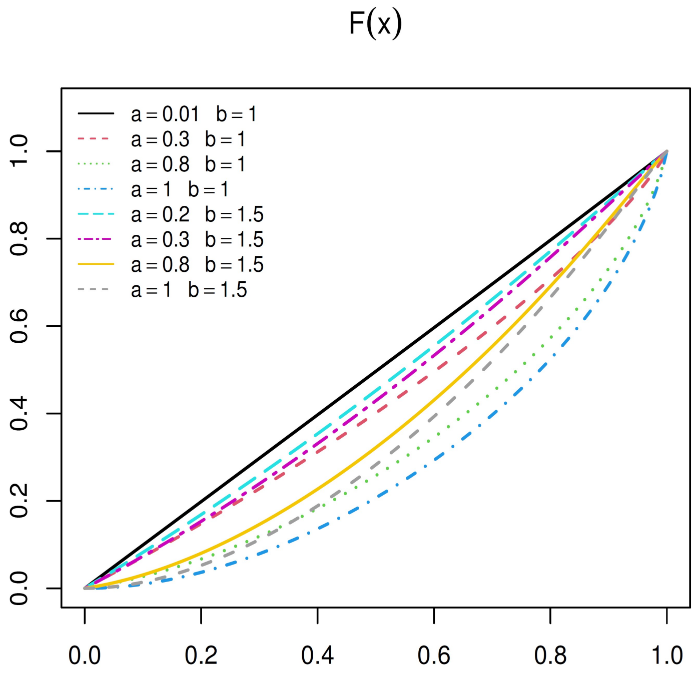

Figure 3. Plots of $F(x)$ in Equation (8) for several values of a and b satisfying $a \in [0, 1]$ and $b \geq 1$.

Upon analyzing this figure, it becomes apparent that the VPP CDF of the third kind is relevant to the parameters being considered. Various convex shapes are noticeable, but all are almost closely aligned with the diagonal represented by the equation $y = x$; the flexibility of the CDF is moderate.

Some examples of new one-parameter VPP CDFs derived from this CDF are listed below.

- The function specified by

$$F(x) = (1 - x)^{a(1-x)} - (1 - x), \quad x \in (0, 1),$$

with the addition of $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $a \in [0, 1]$. It is obtained using Equation (8) by choosing $b = 1$.

• The function defined by

$$F(x) = (1-x)^{(1-x)^b} - (1-x), \quad x \in (0, 1),$$

and completed by $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $b \geq 1$. It is derived from Equation (8) by selecting $a = 1$.

The corresponding $(0, 1)$-supported distributions of these CDFs have not been the object of any study, according to our knowledge.

We complete this part by some extended or modified variants of the VPP CDF of the third kind.

4.2. Complements

Table 4 contains variants of the VPP CDF of the third kind based on the schemes presented in Table 1.

Table 4. Extended or modified versions of the CDF in Equation (8) based on the schemes presented in Table 1

Scheme	Parameter(s)	CDF for $x \in (0, 1)$
Power	$\alpha > 0$, $a \in [0, 1]$, $b \geq 1$	$\left[(1-x)^{a(1-x)^b} - (1-x)\right]^\alpha$
Type II	$a \in [0, 1]$, $b \geq 1$	$x + 1 - x^{ax^b}$
Transmuted	$\lambda \in [-1, 1]$, $a \in [0, 1]$, $b \geq 1$	$(1+\lambda)\left[(1-x)^{a(1-x)^b} - (1-x)\right] - \lambda\left[(1-x)^{a(1-x)^b} - (1-x)\right]^2$
Marshall-Olkin	$\theta > 0$, $a \in [0, 1]$, $b \geq 1$	$\dfrac{(1-x)^{a(1-x)^b} - (1-x)}{\theta + (1-\theta)[(1-x)^{a(1-x)^b} - (1-x)]}$

Table 4. (Continued)

Scheme	Parameter(s)	CDF for $x \in (0, 1)$
Topp-Leone	$\alpha > 0$, $a \in [0, 1]$, $b \geq 1$	$\frac{\left[(1-x)^{a(1-x)^b} - (1-x)\right]^{\alpha}}{\left[2-(1-x)^{a(1-x)^b} + (1-x)\right]^{\alpha}}$
Weibull	$\alpha > 0$, $\beta > 0$, $a \in [0, 1]$, $b \geq 1$	$1 - e^{-\{-\frac{1}{\beta}\ln[1-(1-x)^{a(1-x)^b}+(1-x)]\}^{\alpha}}$
Odd Fréchet	$\alpha > 0$, $a \in [0, 1]$, $b \geq 1$	$e^{-\left[\frac{1-(1-x)^{a(1-x)^b}+(1-x)}{(1-x)^{a(1-x)^b}-(1-x)}\right]^{\alpha}}$
Sin	$a \in [0, 1]$, $b \geq 1$	$\sin\left\{\frac{\pi}{2}\left[(1-x)^{a(1-x)^b} - (1-x)\right]\right\}$
Cos	$a \in [0, 1]$, $b \geq 1$	$1 - \cos\left\{\frac{\pi}{2}\left[(1-x)^{a(1-x)^b} - (1-x)\right]\right\}$
Tan	$a \in [0, 1]$, $b \geq 1$	$\tan\left\{\frac{\pi}{4}\left[(1-x)^{a(1-x)^b} - (1-x)\right]\right\}$
Sec	$a \in [0, 1]$, $b \geq 1$	$\sec\left\{\frac{\pi}{3}\left[(1-x)^{a(1-x)^b} - (1-x)\right]\right\} - 1$

This table enhances and diversifies the collection of available VPP CDFs. It is interesting to remark that, for the type II scheme and $x \in (0, 1)$, the corresponding VPP CDF can be written as

$$F(x) = F_{\star}(x) + P(x),$$

where $F_{\star}(x)$ corresponds to the CDF of the $(0, 1)$-supported uniform distribution and $P(x) = 1 - x^{ax^b}$. Thus, $P(x)$ can be viewed as a new perturbation function of the CDF of the $(0, 1)$-supported uniform distribution. It is the first of the VPP type, to our knowledge. For the general notion of perturbed CDFs, we may refer to de Oliveira et al. [11].

We can also use the VPP CDF of the third kind to generate continuous distributions with various supports. For any CDF of a continuous distribution (whatever the support), say $G(x)$, based on $F(x)$ as defined in Equation (8), we define a new family of distributions by the following CDF:

$$J(x) = F[G(x)] = [1 - G(x)]^{a[1-G(x)]} - [1 - G(x)], \quad x \in \mathbb{R}. \tag{9}$$

Some concrete examples of such CDFs are given below.

- The function specified by

$$J(x) = (1 - x^{\theta})^{a(1-x^{\theta})} - (1 - x^{\theta}), \quad x \in (0, 1),$$

with the complement of $J(x) = 0$ for $x \leq 0$ and $J(x) = 1$ for $x \geq 1$, is a valid CDF for $a \in [0, 1]$, $b \geq 1$ and $\theta > 0$. It is obtained using Equation (9) by choosing the CDF of the power distribution with parameter θ for $G(x)$.

- The function defined by

$$J(x) = e^{-\lambda a x e^{-\lambda b x}} - e^{-\lambda x}, \quad x > 0,$$

and completed by $J(x) = 0$ for $x \leq 0$, is a valid CDF for $a \in [0, 1]$, $b \geq 1$ and $\lambda > 0$. It is derived from Equation (9) by selecting the CDF of the exponential distribution with parameter λ for $G(x)$.

- The function specified by

$$J(x) = [1 - (1 + e^{-\beta x})^{-1}]^{a[1-(1+e^{-\beta x})^{-1}]^{b}} - [1 - (1 + e^{-\beta x})^{-1}], \quad x \in \mathbb{R},$$

is a valid CDF for $a \in [0, 1]$, $b \geq 1$ and $\beta > 0$. It is obtained using Equation (9) by choosing the CDF of the logistic distribution with parameter β for $G(x)$.

These CDFs are new and present distinct VPP expressions, with various supports. They provide extensive applicability for various modelling and data analysis requirements. Another VPP CDF option is exhibited in the next section.

5. VPP CDF of the Fourth Kind

5.1. Main result

The result below focuses on a new VPP CDF, denoted as “of the fourth kind” using the functional form $[U(x)]^x$, where $U(x)$ is a sum of polynomial and polynomial-logarithmic functions.

Proposition 5.1. *The function defined by*

$$F(x) = \frac{1}{a}\{[1 + ax + bx\ln(x)]^x - x^x\}, \quad x \in (0, 1), \tag{10}$$

with the addition of $F(x) = 0$ *for* $x \leq 0$ *and* $F(x) = 1$ *for* $x \geq 1$, *is a valid CDF for* $b \leq 0$ *and* $a \geq 1 - b$.

Proof. For any $x \in (0, 1)$, $b \leq 0$ and $a \geq 1 - b$, it is clear that $1 + ax + bx\ln(x) \geq 1 \geq x$, so $[1 + ax + bx\ln(x)]^x \geq x^x$, implying that $F(x) \geq 0$. For any $x \leq 0$ or $x \geq 1$, it is obvious that $F(x) \geq 0$. Furthermore, since $\lim_{x \to 0^+} x\ln(x) = 0$ and $0^0 = 1$, we have

$$\lim_{x \to 0^+} F(x) = \frac{1}{a}[(1 + a \times 0 + b \times 0)^0 - 0^0] = \frac{1}{a}(1 - 1) = 0,$$

and

$$\lim_{x \to 1^-} F(x) = \frac{1}{a}[(1 + a \times 1 + b \times 0)^1 - 1^1] = \frac{1}{a}[(a + 1) - 1] = 1.$$

Also, for any $x \in (0, 1)$, by employing standard differentiation rules and factorizing in a suitable way, we get

$$F'(x) = \frac{1}{a}\left[[1 + ax + bx\ln(x)]^x \left\{\frac{x[a + b + b\ln(x)}{1 + ax + bx\ln(x)} + \ln[1 + ax + bx\ln(x)]\right\} - x^x[\ln(x) + 1]\right].$$

By using the following logarithmic inequalities: $\ln(1+u) \geq \frac{u}{1+u}$ and $\ln(1+u) \leq u$ for $u > -1$, we obtain

$$F'(x) \geq \frac{1}{a}\left[[1 + ax + bx\ln(x)]^{x}\left\{\frac{x[a + b + b\ln(x)]}{1 + ax + bx\ln(x)} + \frac{ax + bx\ln(x)}{1 + ax + bx\ln(x)}\right\} - x^{x}[(x-1)+1]\right].$$

$$= \frac{x}{a}\left\{[2a + b + 2b\ln(x)][1 + ax + bx\ln(x)]^{x-1} - x^{x}\right\}.$$

Now, since $a + b \geq 1$, $a \geq ax$, $2 \geq x$ and $b \leq 0$, we have $2a + b + 2b \ln(x) = (a+b) + a + 2b\ln(x) \geq 1 + ax + bx\ln(x)$, which implies that

$$F'(x) \geq \frac{x}{a}\{[1 + ax + bx\ln(x)]^{x} - x^{x}\} = xF(x) \geq 0.$$

Therefore, $F(x)$ is non-decreasing for $x \in (0, 1)$. The proof is ended; $F(x)$ is a valid CDF. □

To our knowledge, this result marks the first mention in the literature of the VPP CDF of the fourth kind. We can also write it with exponential terms as follows:

$$F(x) = \frac{1}{a}\left\{e^{\ln(x)[1+ax+bx\ln(x)]} - e^{x\ln(x)}\right\}.$$

Unlike the previous VPP CDFs, the VPP CDF of the fourth kind dont recover the CDF of the $(0, 1)$-supported distribution; it really possesses a singular expression. As a notable property discovered in the proof of Proposition 5.1, the following inequality is satisfied: for any $x \in (0, 1)$, we have

$$f(x) \geq xF(x),$$

where $f(x)$ denotes the PDF associated with $F(x)$.

Figure 4 illustrates Proposition 5.1 by showing $F(x)$ for several values of a and b satisfying $b \leq 0$ and $a \geq 1 - b$.

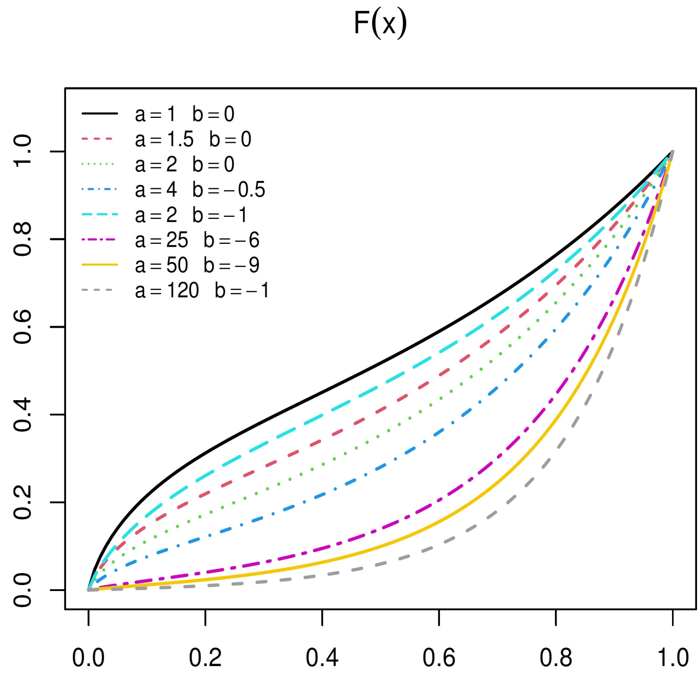

Figure 4. Plots of $F(x)$ in Equation (10) for several values of a and b satisfying $b \leq 0$ and $a \geq 1 - b$.

Upon analysis of this figure, it is evident that the VPP CDF of the fourth kind remains valid for the parameters under consideration. Several shapes, exhibiting both convex and "concave then convex" characteristics, are clearly discernible.

Some examples of new one-parameter VPP CDFs derived from this CDF are listed below.

- The function specified by

$$F(x) = \frac{1}{a}\left[(1 + ax)^{x} - x^{x}\right], \quad x \in (0, 1),$$

and completed by $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $a \geq 1$. It is derived from Equation (10) by selecting $b = 0$.

• The function defined by

$$F(x) = \frac{1}{1-b}\{[1 + (1-b)x + bx\ln(x)]^{x} - x^{x}\}, \quad x \in (0, 1),$$

with the complement of $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $b \leq 0$. It is obtained using Equation (10) by choosing $a = 1 - b$.

Since they are of moderate complexity, both of them can be the object of a statistical study for data analysis of the proportional or rate-type data.

Some variants of the VPP CDF of the fourth kind are described in the next part.

5.2. Complements

Table 5 presents extended or modified versions of the VPP CDF of the fourth kind as indicated in Equation (10). These modifications are derived from the schemes outlined in Table 1.

Table 5. Extended or modified versions of the CDF in Equation (10) based on the schemes presented in Table 1

Scheme	Parameter(s)	CDF for $x \in (0, 1)$
Power	$\alpha > 0, b \leq 0, a \geq 1-b$	$\left[\frac{1}{a}\{[1+ax+bx\ln(x)]^x - x^x\}\right]^\alpha$
Type II	$b \leq 0, a \geq 1-b$	$1-\frac{1}{a}\{[1+a(1-x)+b(1-x)\ln(1-x)]^{1-x}$ $-(1-x)^{1-x}\}$
Transmuted	$\lambda \in [-1, 1], b \leq 0, a \geq 1-b$	$\frac{1+\lambda}{a}\{[1+ax+bx\ln(x)]^x - x^x\}$ $-\frac{\lambda}{a^2}\{[1+ax+bx\ln(x)]^x - x^x\}^2$
Marshall-Olkin	$\theta > 0, b \leq 0, a \geq 1-b$	$\frac{[1+ax+bx\ln(x)]^x - x^x}{a\theta+(1-\theta)\{[1+ax+bx\ln(x)]^x - x^x\}}$
Topp-Leone	$\alpha > 0, b \leq 0, a \geq 1-b$	$\frac{1}{a^\alpha}\{[1+ax+bx\ln(x)]^x - x^x\}^\alpha$ $\left[2-\frac{1}{a}\{[1+ax+bx\ln(x)]^x - x^x\}\right]^\alpha$
Weibull	$\alpha > 0, \beta > 0, b \leq 0, a \geq 1-b$	$1-e^{-\left\{-\frac{1}{\beta}\ln\left[1-\frac{1}{\alpha}\{[1+ax+bx\ln(x)]^x - x^x\}\right]\right\}^\alpha}$
Odd Fréchet	$\alpha > 0, b \leq 0, a \geq 1-b$	$e^{-\left\{\frac{a-[1+ax+bx\ln(x)]^x + x^x}{[1+ax+bx\ln(x)]^x - x^x}\right\}^\alpha}$
sin	$b \leq 0, a \geq 1-b$	$\sin\left[\frac{\pi}{2a}\{[1+ax+bx\ln(x)]^x - x^x\}\right]$
Cos	$b \leq 0, a \geq 1-b$	$1-\cos\left[\frac{\pi}{2a}\{[1+ax+bx\ln(x)]^x - x^x\}\right]$
Tan	$b \leq 0, a \geq 1-b$	$\tan\left[\frac{\pi}{4a}\{[1+ax+bx\ln(x)]^x - x^x\}\right]$
Sec	$b \leq 0, a \geq 1-b$	$\sec\left[\frac{\pi}{3a}\{[1+ax+bx\ln(x)]^x - x^x\}\right]-1$

This table significantly broadens the scope of the VPP CDF of the fourth kind by integrating various modifications and extensions.

This CDF can also be harnessed for creating a diverse array of continuous distributions across different ranges of support. Indeed, for any CDF of a continuous distribution (whatever the support), say $G(x)$, based on $F(x)$ as defined in Equation (10), we define a new family of distributions by the following CDF:

$$J(x) = F[G(x)] = \frac{1}{a}\left[\{1 + aG(x) + bG(x)\ln[G(x)]\}^{G(x)} - [G(x)]^{G(x)}\right], \quad x \in \mathbb{R}. \tag{11}$$

Some examples are given below.

- The function specified by

$$J(x) = \frac{1}{a}\left[\{1 + ax^{\theta} + b\theta x^{\theta}\ln(x)\}^{x^{\theta}} - x^{\theta x^{\theta}}\right], \quad x \in (0, 1),$$

and completed by $J(x) = 0$ for $x \leq 0$ and $J(x) = 1$ for $x \geq 1$, is a valid CDF for $b \leq 0$, $a \geq 1 - b$ and $\theta > 0$. It is derived from Equation (11) by selecting the CDF of the power distribution with parameter θ for $G(x)$.

- The function defined by

$$J(x) = \frac{1}{a}\left[\{1 + a(1 - e^{-\lambda x}) + b(1 - e^{-\lambda x})\ln(1 - e^{-\lambda x})\}^{1-e^{-\lambda x}} - (1 - e^{-\lambda x})^{1-e^{-\lambda x}}\right], \ x > 0,$$

with the addition of $J(x) = 0$ for $x \leq 0$, is a valid CDF for $b \leq 0, a \geq 1 - b$ and $\lambda > 0$. It is obtained using Equation (11) by choosing the CDF of the exponential distribution with parameter λ for $G(x)$.

• The function specified by

$$J(x) = \frac{1}{a}\left[\{1 + a(1 + e^{-\beta x})^{-1} - b(1 + e^{-\beta x})^{-1}\ln(1 + e^{-\beta x})\}^{(1+e^{-\beta x})^{-1}} - (1 + e^{-\beta x})^{-(1+e^{-\beta x})^{-1}}\right], \quad x \in \mathbb{R},$$

is a valid CDF for $b \leq 0$, $a \geq 1 - b$ and $\beta > 0$. It is derived from Equation (11) by selecting the CDF of the logistic distribution with parameter β for $G(x)$.

These three CDFs each feature distinct VPP expressions and supports, and they depend on third parameters. These functions provide versatile utility for a wide range of modelling and data analysis requirements.

6. VPP CDF of the Fifth Kind

6.1. Main result

The following proposition introduces a new VPP CDF, say "of the fifth kind", using the functional form $[U(x)]^{-\frac{1}{x^d}}$, where $U(x)$ depends on a ratio of polynomial and polynomial-logarithmic functions.

Proposition 6.1. *The function defined by*

$$F(x) = (1 + a)\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{-\frac{1}{x^d}}, \; x \in (0, 1), \tag{12}$$

and completed by $F(x) = 0$ *for* $x \leq 0$ *and* $F(x) = 1$ *for* $x \geq 1$, *is a valid CDF for* $a \geq 0$, $b \leq 0$, $c \leq 0$ *such that* $a - b - c \neq 0$ *and* $d \geq 0$.

Proof. For any $x \in (0, 1)$, $a \geq 0$, $b \leq 0$, $c \leq 0$ and $d \geq 0$, we have $F(x) = (1 + a)\left\{1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right\}^{-\frac{1}{x^d}} \geq 0$. For any $x \leq 0$ or

$x \geq 1$, it is obvious that $F(x) \geq 0$. Furthermore, with the additional assumption $a - b - c \neq 0$, since

$$\lim_{x \to 0^+} \frac{a + b\ln(x) + cx\ln(x)}{x} = b \lim_{x \to 0^+} \frac{\ln(x)}{x} = \infty,$$

and

$$\lim_{x \to 0^+} \frac{1}{x^d} \ln\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right] = \infty,$$

we have

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} (1 + a)\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{-\frac{1}{x^d}}$$

$$= (1 + a) \lim_{x \to 0^+} e^{-\frac{1}{x^d}\ln\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]}$$

$$= (1 + a) \lim_{y \to -\infty} e^y = 0,$$

and

$$\lim_{x \to 1^-} F(x) = \lim_{x \to 1^-} (1 + a)\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{-\frac{1}{x^d}} = \frac{a + 1}{a + 1} = 1.$$

Also, for any $x \in (0, 1)$, by differentiating $F(x)$, we get

$$F'(x) = (1 + a)x^{-d-2}\left\{1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right\}^{-1-\frac{1}{x^d}}$$

$$\times \left\{d[a + b\ln(x) + cx\ln(x) + x]\ln\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]\right.$$

$$\left. + \, a - b + b\ln(x) - cx\right\}.$$

Since $b \leq 0$ and $c \leq 0$, it is clear that $-b \geq 0$, $b\ln(x) \geq 0$, $-cx \geq 0$ and $cx\ln(x) \geq 0$. By taking into account that $a \geq 0$ and $d \geq 0$, all the main terms are non-negative. As a result, $F(x)$ is non-decreasing for $x \in (0, 1)$. This completes the proof; $F(x)$ is a valid CDF. □

Let us notice that, since $a \geq 0, b \leq 0, c \leq 0,$ the condition $a - b - c = |a| + |b| + |c| \neq 0$ means that we can not have $a = b = c = 0$ simultaneously.

To the best of our understanding, Proposition 6.1 represents the initial reference in the literature to the VPP CDF of the fifth kind. Let us notice that, for $x \in (0, 1),$ it can be also expressed as

$$F(x) = (1+a)\left[1 + c\ln(x) + \frac{a + b\ln(x)}{x}\right]^{-\frac{1}{x^d}},$$

and the following exponential expression holds:

$$F(x) = (1+a)\, e^{-\frac{1}{x^d}\ln\left[1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right]}.$$

Figure 5 illustrates Proposition 6.1 by displaying $F(x)$ for several values of a, b and c satisfying $a \geq 0, b \leq 0, c \leq 0$ such that $a - b - c \neq 0,$ and by fixing $d = 1$ to simplify the situation.

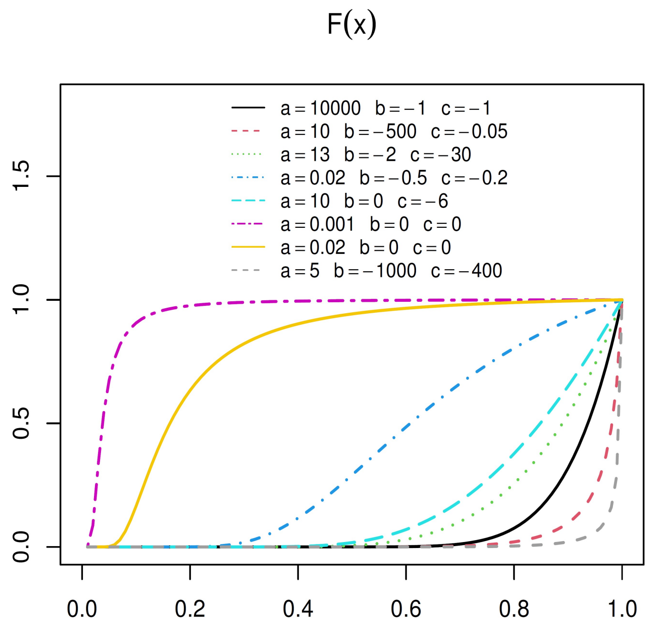

Figure 5. Plots of $F(x)$ in Equation (12) for several values of a, b and c satisfying $a \geq 0, b \leq 0, c \leq 0$ such that $a - b - c \neq 0,$ and $d = 1.$

After analyzing this figure, it becomes clear that the VPP CDF of the fifth kind is valid for the considered parameters. There are various versatile shapes, with different levels of convexity and concavity and more or less pronounced rounded corners.

Below, we present a few examples of new one-parameter VPP CDFs that are derived from this foundational VPP CDF.

• The function specified by

$$F(x) = (1+a)\left[1 + a\frac{1-\ln(x)}{x}\right]^{-1}, \quad x \in (0, 1),$$

with the complement of $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $a > 0$. It is obtained using Equation (12) by choosing $b = -a$, $c = 0$ and $d = 0$.

• The function defined by

$$F(x) = 2\left[1 + \frac{1}{x} - \ln(x)\right]^{-\frac{1}{x^d}}, \quad x \in (0, 1),$$

and completed by $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $d \geq 0$. It is derived from Equation (12) by selecting $a = 1$, $b = 0$ and $c = -1$.

The two CDFs showcased here feature unique VPP expressions, each specified by a single parameter. They offer versatile applicability for a wide range of modelling and data analysis purposes.

6.2. Complements

Table 6 lists some extended or modified versions of the CDF in Equation (12) based on the schemes presented in Table 1.

Table 6. Extended or modified versions of the CDF in Equation (12) based on the schemes presented in Table 1

Scheme	Parameter(s)	CDF for $x \in (0, 1)$
Power	$\alpha > 0, a \geq 0, b \leq 0, c \leq 0,$ $a - b - c \neq 0, d \geq 0$	$(1+a)^{\alpha}\left[1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right]^{-\frac{\alpha}{x^d}}$
Type II	$a \geq 0, b \leq 0, c \leq 0, a - b - c \neq 0, d \geq 0$	$1-(1+a)\left[1+\frac{a+b\ln(1-x)+c(1-x)\ln(1-x)}{1-x}\right]^{-\frac{1}{(1-x)^d}}$
Transmuted	$\lambda \in [-1, 1], a \geq 0, b \leq 0, c \leq 0,$ $a - b - c \neq 0, d \geq 0$	$(1+\lambda)(1+a)\left[1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right]^{-\frac{1}{x^d}}$ $-\lambda(1+a)^2\left\{1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right\}^{-\frac{2}{x^d}}$
Marshall-Olkin	$\theta > 0, a \geq 0, b \leq 0, c \leq 0,$ $a - b - c \neq 0, d \geq 0$	$\frac{(1+a)\left\{1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right\}^{-\frac{1}{x^d}}}{\theta+(1-\theta)(1+a)\left\{1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right\}^{-\frac{1}{x^d}}}$
Topp-Leone	$\alpha > 0, a \geq 0, b \leq 0, c \leq 0,$ $a - b - c \neq 0, d \geq 0$	$(1+a)^{\alpha}\left[1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right]^{-\frac{\alpha}{x}}$ $\times\left\{2-(1+a)\left[1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right]^{-\frac{1}{x^d}}\right\}^{\alpha}$
Weibull	$\alpha > 0, \beta > 0, a \geq 0, b \leq 0,$ $c \leq 0, a - b - c \neq 0, d \geq 0$	$1-e^{-\left[-\frac{1}{\beta}\ln\left\{1-(1+a)\left[1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right]^{-\frac{1}{x^d}}\right\}\right]^{\alpha}}$
Odd Fréchet	$\alpha > 0, a \geq 0, b \leq 0, c \leq 0,$ $a - b - c \neq 0, d \geq 0$	$e^{-\left\{\frac{1-(1+a)\left[1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right]^{-\frac{1}{x^d}}}{(1+a)\left[1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right]^{-\frac{1}{x^d}}}\right\}^{\alpha}}$
Sin	$a \geq 0, b \leq 0, c \leq 0, a - b - c \neq 0, d \geq 0$	$\sin\left\{\frac{\pi}{2}(1+a)\left[1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right]^{-\frac{1}{x^d}}\right\}$
Cos	$a \geq 0, b \leq 0, c \leq 0, a - b - c \neq 0, d \geq 0$	$1-\cos\left\{\frac{\pi}{2}(1+a)\left[1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right]^{-\frac{1}{x^d}}\right\}$

Table 6. (Continued)

Scheme	Parameter(s)	CDF for $x \in (0, 1)$
Tan	$a \geq 0, b \leq 0, c \leq 0, a - b - c \neq 0, d \geq 0$	$\tan\left\{\frac{\pi}{4}(1+a)\left[1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right]^{-\frac{1}{x^d}}\right\}$
Sec	$a \geq 0, b \leq 0, c \leq 0, a - b - c \neq 0, d \geq 0$	$\sec\left\{\frac{\pi}{3}(1+a)\left[1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right]^{-\frac{1}{x^d}}\right\}-1$

This table expands the VPP CDF of the fifth kind by incorporating diverse extensions and modifications, thereby enriching the collection of available VPP CDFs.

On the other hand, we can leverage this CDF to create a spectrum of continuous distributions across different ranges. In fact, for any CDF representing a continuous distribution, regardless of its support, denoted as $G(x)$ and from $F(x)$ as outlined in Equation (12), we can establish a new family of distributions by considering the following CDF:

$$J(x) = F[G(x)] = (1+a)\left[1+\frac{a+b\ln[G(x)]+cx\ln[G(x)]}{G(x)}\right]^{-\frac{1}{[G(x)]^d}}, \quad x \in \mathbb{R}. \tag{13}$$

Some examples of CDFs of original distributions are determined below.

- The function defined by

$$J(x) = (1+a)\left[1+\frac{a+b\theta\ln(x)+c\theta x^{\theta}\ln(x)}{x^{\theta}}\right]^{-\frac{1}{x^{\theta d}}}, \quad x \in (0, 1),$$

with the addition of $J(x) = 0$ for $x \leq 0$ and $J(x) = 1$ for $x \geq 1$, is a valid CDF for $a \geq 0$, $b \leq 0$, $c \leq 0$ such that $a - b - c \neq 0$, $d \geq 0$ and $\theta > 0$. It is obtained using Equation (13) by choosing the CDF of the power distribution with parameter θ for $G(x)$.

• The function specified by

$$J(x) = (1+a)\left[1+\frac{a+b\ln(1-e^{-\lambda x})+c(1-e^{-\lambda x})\ln(1-e^{-\lambda x})}{1-e^{-\lambda x}}\right]^{-\frac{1}{(1-e^{-\lambda x})^{d}}}, \quad x > 0,$$

and completed by $J(x) = 0$ for $x \leq 0$, is a valid CDF for $a \geq 0, b \leq 0, c \leq 0$ such that $a - b - c \neq 0, d \geq 0$ and $\lambda > 0$. It is derived from Equation (13) by selecting the CDF of the exponential distribution with parameter λ for $G(x)$.

• The function defined by

$$J(x) = (1+a)\{1+(1+e^{-\beta x})[a-b\ln(1+e^{-\beta x}) - c(1+e^{-\beta x})^{-1}\ln(1+e^{-\beta x})]\}^{-(1+e^{-\beta x})^{d}}, \quad x \in \mathbb{R},$$

is a valid CDF for $a \geq 0, b \leq 0, c \leq 0$ such that $a - b - c \neq 0, d \geq 0$ and $\beta > 0$. It is obtained using Equation (13) by choosing the CDF of the logistic distribution with parameter β for $G(x)$.

In addition to having unique VPP expressions that are dependent on five parameters, the three presented CDFs depend on different supports. Of course, the number of parameters can be reduced by putting some of them to a natural constant, like -1, 0 or 1.

7. VPP CDF of the Sixth Kind

7.1. Main result

The subsequent proposition presents a novel VPP CDF referred to as "of the sixth kind." This is formulated through the functional expression $[U(x)]^{1-\frac{1}{x^{d}}}$, where $U(x)$ depends on a polynomial-logarithmic function in ratio form. It really differs from the VPP CDF of the fifth kind, despite a certain similarity in the functions used and parameter assumptions.

Proposition 7.1. *The function specified by*

$$F(x) = \left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{1-\frac{1}{x^d}}, \quad x \in (0, 1), \tag{14}$$

with the addition of $F(x) = 0$ *for* $x \leq 0$ *and* $F(x) = 1$ *for* $x \geq 1$, *is a valid CDF for* $a \geq 0$, $b \leq 0$, $c \leq 0$ *such that* $a - b - c \neq 0$ *and* $d > 0$.

Proof. For any $x \in (0, 1)$ and $a \geq 0$, $b \leq 0$, $c \leq 0$ and $d > 0$, we have $F(x) = \left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{1-\frac{1}{x^d}} \geq 0$. For any $x \leq 0$ or $x \geq 1$, it is obvious that $F(x) \geq 0$. Furthermore, with the additional assumption $a - b - c \neq 0$, since

$$\lim_{x \to 0^+} \frac{a + b\ln(x) + cx\ln(x)}{x} = b \lim_{x \to 0^+} \frac{\ln(x)}{x} = \infty,$$

and

$$\lim_{x \to 0^+} \left(1 - \frac{1}{x^d}\right) \ln\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right] = -\infty,$$

we have

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} \left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{1-\frac{1}{x^d}}$$

$$= \lim_{x \to 0^+} e^{\left(1-\frac{1}{x^d}\right)\ln\left[1+\frac{a+b\ln(x)+cx\ln(x)}{x}\right]} = \lim_{y \to -\infty} e^y = 0,$$

and

$$\lim_{x \to 1^-} F(x) = \lim_{x \to 1^-} \left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{1-\frac{1}{x^d}} = (1 + a)^0 = 1.$$

Also, after differentiation and diverse developments, for any $x \in (0, 1)$, we have

$$F'(x) = x^{-d-2}\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{-\frac{1}{x^d}}$$

$$\times \left\{ d[a + b\ln(x) + cx\ln(x) + x]\ln\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]\right.$$

$$\left. + (1 - x^d)[a + b\ln(x) - b - cx]\right\}.$$

Since $b \leq 0$ and $c \leq 0$, it is clear that $-b \geq 0$, $b\ln(x) \geq 0$, $-cx \geq 0$ and $cx\ln(x) \geq 0$. By taking into account that $a \geq 0$ and $d > 0$, all the main terms are non-negative. As a result, $F(x)$ is non-decreasing for $x \in (0, 1)$. This ends the proof; $F(x)$ is a valid CDF. □

As for the VPP CDF of the fifth kind, let us notice that, since $a \geq 0, b \leq 0, c \leq 0$, the condition $a - b - c \neq 0$ means that we can not have $a = b = c = 0$ simultaneously.

Based on our current knowledge, the VPP CDF of the sixth kind appears to be a novel discovery within the realm of probability and statistics.

Figure 6 illustrates Proposition 7.1 by displaying $F(x)$ for several values of a, b and c satisfying $a \geq 0, b \leq 0, c \leq 0$ such that $a - b - c \neq 0$, and by fixing $d = 1$ to simplify the situation.

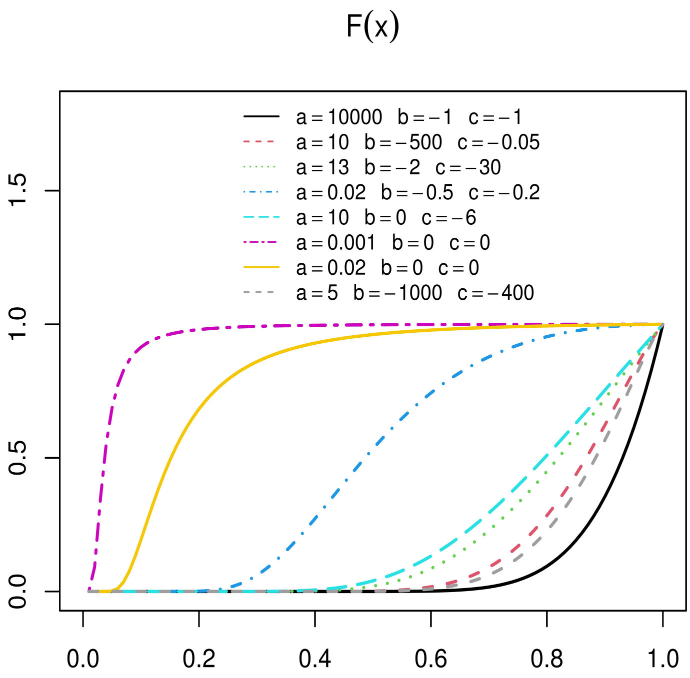

Figure 6. Plots of $F(x)$ in Equation (14) for several values of a, b and c satisfying $a \geq 0$, $b \leq 0$, $c \leq 0$ such that $a - b - c \neq 0$, and $d = 1$.

This figure demonstrates the validity of the VPP CDF of the sixth kind for the parameters under consideration. Notably, the figure reveals an array of versatile convex and concave shapes, affirming the robustness and adaptability of the model within diverse scenarios.

Below, we present a few examples of new one-parameter VPP CDFs that have been derived from this original VPP CDF.

• The function defined by

$$F(x) = \left[1 + a\frac{1 - \ln(x)}{x}\right]^{1-\frac{1}{x}}, \quad x \in (0, 1),$$

and completed by $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $a > 0$. It is derived from Equation (14) by selecting $b = -a$, $c = 0$, and $d = 1$.

• The function specified by

$$F(x) = \left[1 + \frac{1}{x} - \ln(x)\right]^{1-\frac{1}{x^d}}, \quad x \in (0, 1),$$

with the complement of $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $d > 0$. It is obtained using Equation (14) by choosing $a = 1$, $b = 0$, and $c = -1$.

The two CDFs showcased here feature unique VPP expressions, each governed by a single parameter. They offer versatile applicability for a wide range of modelling and data analysis purposes.

A complementary study is provided below.

7.2. Complements

Table 7 contains extended or modified versions of the CDF of the sixth kind based on the schemes presented in Table 1.

Table 7. Extended or modified versions of the CDF in Equation (14) based on the schemes presented in Table 1

Scheme	Parameter(s)	CDF for $x \in (0, 1)$
Power	$\alpha > 0,\ a \geq 0,\ b \leq 0,\ c \leq 0,$ $a - b - c \neq 0,\ d > 0$	$\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{\alpha\left(1 - \frac{1}{x^d}\right)}$
Type II	$a \geq 0,\ b \leq 0,\ c \leq 0,\ a - b - c \neq 0,\ d > 0$	$1 - \left[1 + \frac{a + b\ln(1-x) + c(1-x)\ln(1-x)}{1-x}\right]^{1 - \frac{1}{(1-x)^d}}$
Transmuted	$\lambda \in [-1, 1],\ a \geq 0,\ b \leq 0,\ c \leq 0,$ $a - b - c \neq 0,\ d > 0$	$(1+\lambda)\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{1 - \frac{1}{x^d}}$ $- \lambda\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{2\left(1 - \frac{1}{x^d}\right)}$
Marshall-Olkin	$\theta > 0,\ a \geq 0,\ b \leq 0,\ c \leq 0,$ $a - b - c \neq 0,\ d > 0$	$\frac{\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{1 - \frac{1}{x^d}}}{\theta + (1-\theta)\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{1 - \frac{1}{x^d}}}$
Topp-Leone	$\alpha > 0,\ a \geq 0,\ b \leq 0,\ c \leq 0,$ $a - b - c \neq 0,\ d > 0$	$\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{\alpha\left(1 - \frac{1}{x^d}\right)}$ $\times \left\{2 - \left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{1 - \frac{1}{x^d}}\right\}^{\alpha}$
Weibull	$\alpha > 0,\ \beta > 0,\ a \geq 0,\ b \leq 0,$ $c \leq 0,\ a - b - c \neq 0,\ d > 0$	$1 - e^{-\left[-\frac{1}{\beta}\ln\left\{1 - \left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{1 - \frac{1}{x^d}}\right\}\right]^{\alpha}}$
Odd Fréchet	$\alpha > 0,\ a \geq 0,\ b \leq 0,\ c \leq 0,$ $a - b - c \neq 0,\ d > 0$	$e^{-\left\{\frac{1 - \left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{1 - \frac{1}{x^d}}}{\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{1 - \frac{1}{x^d}}}\right\}^{\alpha}}$
Sin	$a \geq 0,\ b \leq 0,\ c \leq 0,\ a - b - c \neq 0,\ d > 0$	$\sin\left\{\frac{\pi}{2}\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{1 - \frac{1}{x^d}}\right\}$
Cos	$a \geq 0,\ b \leq 0,\ c \leq 0,\ a - b - c \neq 0,\ d > 0$	$1 - \cos\left\{\frac{\pi}{2}\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{1 - \frac{1}{x^d}}\right\}$

Table 7. (Continued)

Scheme	Parameter(s)	CDF for $x \in (0, 1)$
Tan	$a \geq 0,\ b \leq 0,\ c \leq 0,\ a - b - c \neq 0,\ d > 0$	$\tan\left\{\frac{\pi}{4}\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{1-\frac{1}{x^d}}\right\}$
Sec	$a \geq 0,\ b \leq 0,\ c \leq 0,\ a - b - c \neq 0,\ d > 0$	$\sec\left\{\frac{\pi}{3}\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{1-\frac{1}{x^d}}\right\} - 1$

This table broadens the scope of the VPP CDF of the sixth kind by integrating diverse variants, some of which have additional tuning parameters.

We can utilize this CDF to generate a diverse spectrum of continuous distributions spanning various ranges. In essence, for any CDF that characterizes a continuous distribution (irrespective of its support) denoted as $G(x)$ and from $F(x)$ as described in Equation (14), we can create a new array of distributions by defining the following CDF:

$$J(x) = F[G(x)] = \left[1 + \frac{a + b\ln[G(x)] + cx\ln[G(x)]}{G(x)}\right]^{1-\frac{1}{[G(x)]^d}}, \quad x \in \mathbb{R}. \qquad (15)$$

Some examples are given below.

• The function defined by

$$J(x) = \left[1 + \frac{a + b\theta\ln(x) + c\theta x^{\theta}\ln(x)}{x^{\theta}}\right]^{1-\frac{1}{x^{\theta d}}}, \quad x \in (0, 1),$$

and completed by $J(x) = 0$ for $x \leq 0$ and $J(x) = 1$ for $x \geq 1$, is a valid CDF for $a \geq 0,\ b \leq 0,\ c \leq 0$ such that $a - b - c \neq 0,\ d > 0$, and $\theta > 0$. It is derived from Equation (15) by selecting the CDF of the power distribution with parameter θ for $G(x)$.

• The function specified by

$$J(x) = \left[1 + \frac{a + b\ln(1 - e^{-\lambda x}) + c(1 - e^{-\lambda x})\ln(1 - e^{-\lambda x})}{1 - e^{-\lambda x}}\right]^{1 - \frac{1}{(1 - e^{-\lambda x})^d}}, \quad x > 0,$$

with the addition of $J(x) = 0$ for $x \leq 0$ is a valid CDF for $a \geq 0$, $b \leq 0$, $c \leq 0$ such that $a - b - c \neq 0$, $d > 0$, and $\lambda > 0$. It is obtained using Equation (15) by choosing the CDF of the exponential distribution with parameter λ for $G(x)$.

• The function defined by

$$J(x) = \{1 + (1 + e^{-\beta x})[a - b\ln(1 + e^{-\beta x}) - c(1 + e^{-\beta x})^{-1}\ln(1 + e^{-\beta x})]\}^{1 - (1 + e^{-\beta x})^d}, \quad x \in \mathbb{R},$$

is a valid CDF for $a \geq 0$, $b \leq 0$, $c \leq 0$ such that $a - b - c \neq 0$, $d > 0$, and $\beta > 0$. It is derived from Equation (15) by selecting the CDF of the logistic distribution with parameter β for $G(x)$.

The three showcased CDFs exhibit unique VPP expressions, each contingent on five parameters. These functions offer remarkable versatility, catering to a broad spectrum of modelling and data analysis needs.

Another VPP CDF of the proposed collection is presented in the next section.

8. VPP CDF of the Seventh Kind

8.1. Main result

The subsequent proposition presents a novel VPP CDF referred to as "of the seventh kind." It is formulated through the functional expression $x^{V(x)}$, where $V(x)$ is itself of the form $V(x) = x^{W(x)}$, with $W(x)$ remaining to be specified.

Proposition 8.1. *The function specified by*

$$F(x) = x^{x^{-a(1-x^b)}}, \quad x \in (0, 1), \tag{16}$$

and completed by $F(x) = 0$ *for* $x \leq 0$ *and* $F(x) = 1$ *for* $x \geq 1$, *is a valid CDF for* $a \geq 0$ *and* $b \geq 0$.

Proof. For any $x \in (0, 1)$, $a \geq 0$ and $b \geq 0$, we have $F(x) = x^{x^{-a(1-x^b)}} \geq 0$. For any $x \leq 0$ or $x \geq 1$, it is immediate that $F(x) \geq 0$. Furthermore, we have

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} x^{x^{-a(1-x^b)}} = \lim_{x \to 0^+} e^{\frac{\ln(x)}{x^{a(1-x^b)}}} = \lim_{y \to -\infty} e^y = 0,$$

and

$$\lim_{x \to 1^-} F(x) = \lim_{x \to 1^-} x^{x^{-a(1-x^b)}} = 1^{\frac{1}{1}} = 1.$$

Also, after some differentiation developments, we obtain

$$F'(x) = x^{-a(1-x^b)+x^{-a(1-x^b)}-1}\{abx^b[\ln(x)]^2 - a(1-x^b)\ln(x) + 1\}.$$

Since $a \geq 0$ and $b \geq 0$, we have $x^{-a(1-x^b)+x^{-a(1-x^b)}-1} \geq 0$, $abx^b[\ln(x)]^2 \geq 0$, $-a\ln(x) \geq 0$ and $1 - x^b \geq 0$, and we immediately conclude that $F'(x) \geq 0$. Therefore, $F(x)$ is non-decreasing for $x \in (0, 1)$. The proof is completed; $F(x)$ is a valid CDF. □

Based on our current understanding, the VPP CDF of the seventh kind appears to be a novel contribution to the field of probability and statistics. For any $x \in (0, 1)$, we may eventually express it as follows:

$$F(x) = e^{\ln(x)x^{-a(1-x^b)}}.$$

We can also mention that the CDF of the $(0, 1)$-supported uniform distribution is obtained by selecting $a = 0$ or $b = 0$.

Figure 7 illustrates Proposition 8.1 by showing $F(x)$ for several values of a and b satisfying $a \geq 0$ and $b \geq 0$.

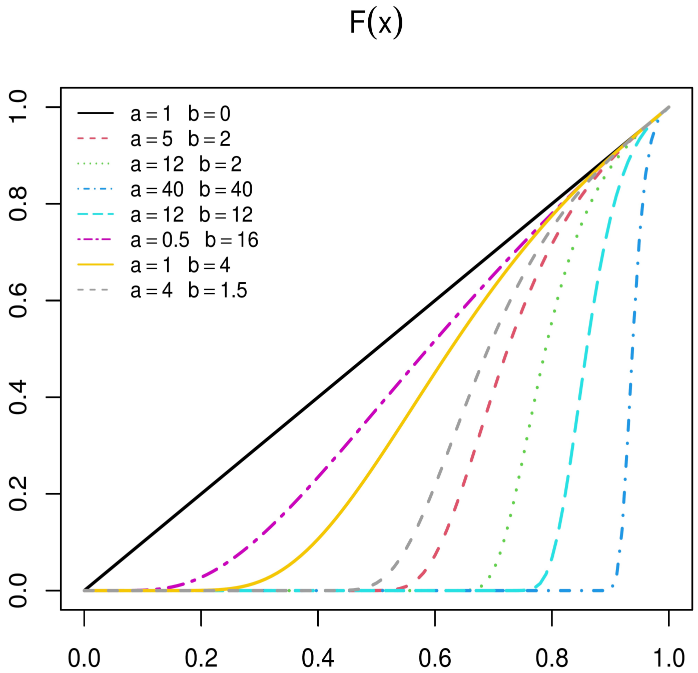

Figure 7. Plots of $F(x)$ in Equation (16) for several values of a and b satisfying $a \geq 0$ and $b \geq 0$.

This figure showcases the validity of the VPP CDF of the seventh kind for the parameters being considered. Particularly noteworthy, the figure illustrates a diverse range of convex or "convex then concave" shapes more or less angular under the diagonal line of equation $y = x$, affirming the robust and adaptable nature of the model across various scenarios.

Subsequently, we provide several examples of newly derived one-parameter VPP CDFs based on this CDF.

- The function defined by

$$F(x) = x^{x^{-(1-x^b)}}, \quad x \in (0, 1),$$

with the complement of $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $b \geq 0$. It is obtained using Equation (16) by choosing $a = 1$.

- The function specified by

$$F(x) = x^{x^{-a(1-x)}}, \quad x \in (0, 1),$$

and completed by $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $a \geq 0$. It is derived from Equation (16) by selecting $b = 1$.

The two presented CDFs exhibit distinctive VPP expressions, each characterized by a single parameter. They provide versatile applicability across a broad spectrum of modeling and data analysis needs.

8.2. Complements

Table 8 presents extended or modified versions of the CDF in Equation (16) based on the schemes presented in Table 1.

Table 8. Extended or modified versions of the CDF in Equation (16) based on the schemes presented in Table 1

Scheme	Parameter(s)	CDF for $x \in (0, 1)$
Power	$\alpha > 0,\ a \geq 0,\ b \geq 0$	$x^{\alpha x^{-a(1-x^b)}}$
Type II	$a \geq 0,\ b \geq 0$	$1 - (1-x)^{(1-x)^{-a[1-(1-x)^b]}}$
Transmuted	$\lambda \in [-1, 1],\ a \geq 0,\ b \geq 0$	$(1+\lambda)x^{x^{-a(1-x^b)}} - \lambda x^{2x^{-a(1-x^b)}}$
Marshall-Olkin	$\theta > 0,\ a \geq 0,\ b \geq 0$	$\dfrac{x^{x^{-a(1-x^b)}}}{\theta + (1-\theta)x^{x^{-a(1-x^b)}}}$
Topp-Leone	$\alpha > 0,\ a \geq 0,\ b \geq 0$	$x^{\alpha x^{-a(1-x^b)}}\left(2 - x^{x^{-a(1-x^b)}}\right)^{\alpha}$
Weibull	$\alpha > 0,\ \beta > 0,\ a \geq 0,\ b \geq 0$	$1 - e^{-\left[-\frac{1}{\beta}\ln\left(1-x^{x^{-a(1-x^b)}}\right)\right]^{\alpha}}$
Odd Fréchet	$\alpha > 0,\ a \geq 0,\ b \geq 0$	$e^{-\left(\frac{1-x^{x^{-a(1-x^b)}}}{x^{x^{-a(1-x^b)}}}\right)^{\alpha}}$
Sin	$a \geq 0,\ b \geq 0$	$\sin\left(\frac{\pi}{2}x^{x^{-a(1-x^b)}}\right)$
Cos	$a \geq 0,\ b \geq 0$	$1 - \cos\left(\frac{\pi}{2}x^{x^{-a(1-x^b)}}\right)$
Tan	$a \geq 0,\ b \geq 0$	$\tan\left(\frac{\pi}{4}x^{x^{-a(1-x^b)}}\right)$
Sec	$a \geq 0,\ b \geq 0$	$\sec\left(\frac{\pi}{3}x^{x^{-a(1-x^b)}}\right) - 1$

This table expands the horizons of the VPP CDF of the seventh kind by incorporating a range of variants. This enriches the available array of VPP CDFs.

This CDF can be leveraged to create a wide variety of continuous distributions that cover diverse supports. Essentially, for any continuous distribution characterized by a CDF (regardless of its support) denoted as $G(x)$ and from $F(x)$ as described in Equation (16), we can generate a new family of distributions by defining the following CDF:

$$J(x) = F[G(x)] = [G(x)]^{[G(x)]^{-a\{1-[G(x)]^{b}\}}}, \quad x \in \mathbb{R}. \tag{17}$$

Some precise CDFs of this type are given below.

- The function defined by

$$J(x) = x^{\theta x^{-\theta a(1-x^{\theta b})}}, \quad x \in (0, 1),$$

with the addition of $J(x) = 0$ for $x \leq 0$ and $J(x) = 1$ for $x \geq 1$, is a valid CDF for $a \geq 0, b \geq 0$ and $\theta > 0$. It is obtained using Equation (17) by choosing the CDF of the power distribution with parameter θ for $G(x)$.

- The function specified by

$$J(x) = (1 - e^{-\lambda x})^{(1-e^{-\lambda x})^{-a[1-(1-e^{-\lambda x})^{b}]}}, \quad x > 0,$$

and completed by $J(x) = 0$ for $x \leq 0$, is a valid CDF for $a \geq 0, b \geq 0$ and $\lambda > 0$. It is derived from Equation (17) by selecting the CDF of the exponential distribution with parameter λ for $G(x)$.

- The function defined by

$$J(x) = (1 + e^{-\beta x})^{-(1+e^{-\beta x})^{a\{1-(1+e^{-\beta x})^{-b}\}}}, \quad x \in \mathbb{R},$$

is a valid CDF for $a \geq 0, b \geq 0$ and $\beta > 0$. It is obtained using Equation (17) by choosing the CDF of the logistic distribution with parameter β for $G(x)$.

These three CDFs have unique VPP expressions that depend on three parameters, and they are defined on various supports. These features are extremely adaptable and can be used for a variety of modeling and data analysis purposes.

9. VPP CDF of the Eighth Kind

9.1. Main Result

The subsequent proposition presents a novel VPP CDF referred to as "of the eighth kind." It is formulated through the functional expression $(1-x)^{V(x)}$, where $V(x)$ is defined as a ratio of distinct power and logarithmic functions.

Proposition 9.1. *The function specified by*

$$F(x) = 1 - (1-x)^{\frac{x^a}{[-\ln(x)]^b}}, \quad x \in (0, 1), \tag{18}$$

with the addition of $F(x) = 0$ *for* $x \leq 0$ *and* $F(x) = 1$ *for* $x \geq 1$, *is a valid CDF for* $a \geq -1$ *and* $b \geq 0$ *such that* $1 + a + b \neq 0$.

Proof. For any $x \in (0, 1)$, $a > -1$ and $b \geq 0$, it is clear that $F(x) = 1 - (1-x)^{\frac{x^a}{[-\ln(x)]^b}} \geq 0$. For any $x \leq 0$ or $x \geq 1$, it is immediate that $F(x) \geq 0$. Furthermore, if $1 + a + b \neq 0$, we have

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} 1 - (1-x)^{\frac{x^a}{[-\ln(x)]^b}} = 1 - \lim_{x \to 0^+} e^{\ln(1-x)\frac{x^a}{[-\ln(x)]^b}}$$

$$= 1 - \lim_{x \to 0^+} e^{-\frac{x^{a+1}}{[-\ln(x)]^b}} = 1 - e^0 = 0,$$

and

$$\lim_{x\to 1^-} F(x) = \lim_{x\to 1^-} 1 - (1-x)^{\frac{x^a}{[-\ln(x)]^b}} = 1 - \lim_{x\to 1^-} e^{\ln(1-x)\frac{x^a}{[-\ln(x)]^b}}$$

$$= 1 - \lim_{x\to 1^-} e^{\frac{\ln(1-x)}{(1-x)^b}} = 1 - \lim_{x\to -\infty} e^y = 1 - 0 = 1.$$

Also, after some differentiation techniques and appropriate developments, we obtain

$$F'(x) = x^{a-1}[-\ln(x)]^{-b-1}(1-x)^{x^a[-\ln(x)]^{-b}-1}$$

$$\{-(1-x)\ln(1-x)[b - a\ln(x)] - x\ln(x)\}.$$

It is immediate that $x^{a-1} \geq 0, [-\ln(x)]^{-b-1} \geq 0$ and $(1-x)^{x^a[-\ln(x)]^{-b}-1} \geq 0.$ Let us now focus on the last main term in brackets, say

$$T(x) = -(1-x)\ln(1-x)[b - a\ln(x)] - x\ln(x).$$

We can write

$$T(x) = -b(1-x)\ln(1-x) + a(1-x)\ln(1-x)\ln(x) - x\ln(x).$$

Since $b \geq 0,$ we have $-b(1-x)\ln(1-x) \geq 0$ and in the case $a \geq 0,$ it is clear that $a(1-x)\ln(1-x)\ln(x) \geq 0,$ and always $-\ln(x) \geq 0,$ implying that that $T(x) \geq 0.$

For the case $a \in [-1, 0),$ by using the following logarithmic inequality: $\ln(1+u) \geq \frac{u}{1+u}$ for $u > -1,$ we get

$$T(x) \geq -b(1-x)\ln(1-x) + a(1-x)\frac{-x}{1-x}\ln(x) - x\ln(x)$$

$$= -b(1-x)\ln(1-x) - (1+a)x\ln(x) \geq 0.$$

Hence, for any $b \geq 0$ and $a \geq -1$, we have $T(x) \geq 0$, which implies that $F'(x) \geq 0$, so $F(x)$ is non-decreasing for $x \in (0, 1)$. The proof is completed; $F(x)$ is a valid CDF. □

Since $a \geq -1$ and $b \geq 0$, the condition $1 + a + b \neq 0$ means that we can not have $a = -1$ and $b = 0$ simultaneously.

Thus, the VPP CDF of the eighth kind uses the same mathematical ingredient as the VPP CDF of the first kind but is rearranged in a different manner. When $a = 1$ and $b = 0$ it corresponds to the CDF at the heart of the study in Tahir et al. [36]. We can also mention that the CDF of the $(0, 1)$-supported uniform distribution is obtained by selecting $a = 0$ and $b = 0$. On the basis of what we currently know for the other values of the parameters, it seems to be a novel addition to statistics and probability.

For any $x \in (0, 1)$, we may eventually express it as follows:

$$F(x) = 1 - e^{\ln(1-x)\frac{x^a}{[-\ln(x)]^b}}.$$

Figure 8 provides graphical support to Proposition 8.1 by plotting $F(x)$ for several values of a and b satisfying $a \geq -1$ and $b \geq 0$ such that $1 + a + b \neq 0$.

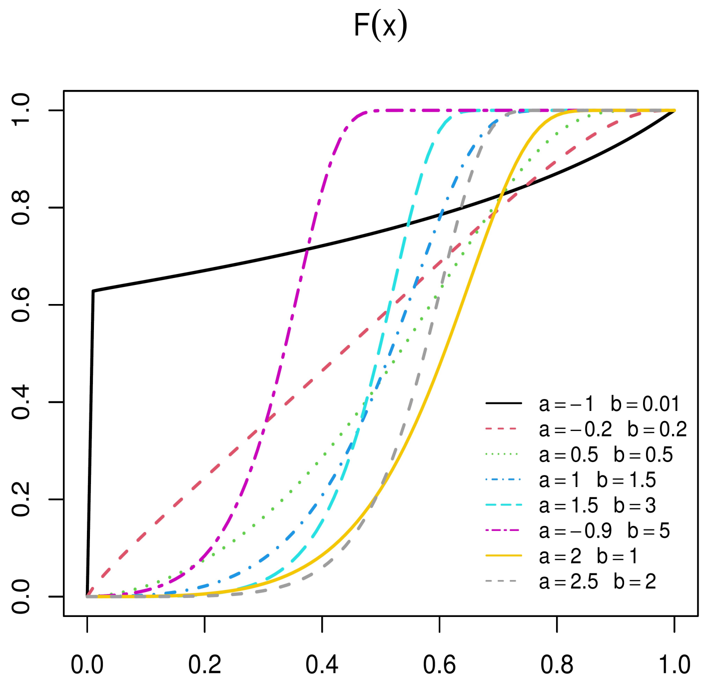

Figure 8. Plots of $F(x)$ in Equation (18) for several values of a and b satisfying $a \geq -1$ and $b \geq 0$ such that $1 + a + b \neq 0$.

The validity of the VPP CDF of the eighth kind for the parameters under consideration is demonstrated in this figure. Notably, the figure displays a wide variety of shapes, some of them that are roughly angular beneath the axis of equation $x = 0$.

We now present some precise examples of newly derived one-parameter VPP CDFs based on $F(x)$.

- The function defined by

$$F(x) = 1 - (1 - x)^{\frac{1}{x[-\ln(x)]^b}}, \quad x \in (0, 1),$$

and completed by $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $b > 0$. It is derived from Equation (18) by choosing $a = -1$.

- The function specified by

$$F(x) = 1 - (1 - x)^{\left[\frac{x}{-\ln(x)}\right]^{a}}, \quad x \in (0, 1),$$

with the complement of $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, is a valid CDF for $a \geq 0$. It is obtained using Equation (18) by selecting $b = a$.

These two CDFs exhibit distinct VPP characteristics and have functionalities modulated by a single parameter. They provide versatile applicability across a wide range of modelling purposes.

9.2. Complements

Table 9 shows extended or modified versions of the CDF in Equation (18) based on the schemes presented in Table 1.

Table 9. Extended or modified versions of the CDF in Equation (18) based on the schemes presented in Table 1

Scheme	Parameter(s)	CDF for $x \in (0, 1)$
Power	$\alpha > 0,\ a \geq -1,\ b \geq 0,\ 1 + a + b \neq 0$	$\left[1 - (1 - x)^{\frac{x^a}{[-\ln(x)]^b}}\right]^{\alpha}$
Type II	$a \geq -1,\ b \geq 0,\ 1 + a + b \neq 0$	$x^{\frac{(1-x)^a}{[-\ln(1-x)]^b}}$
Transmuted	$\lambda \in [-1, 1],\ a \geq -1,\ b \geq 0,$ $1 + a + b \neq 0$	$(1 + \lambda)\left[1 - (1 - x)^{\frac{x^a}{[-\ln(x)]^b}}\right] - \lambda\left[1 - (1 - x)^{\frac{x^a}{[-\ln(x)]^b}}\right]^2$
Marshall-Olkin	$\theta > 0,\ a \geq -1,\ b \geq 0,\ 1 + a + b \neq 0$	$\dfrac{1 - (1 - x)^{\frac{x^a}{[-\ln(x)]^b}}}{1 - (1 - \theta)(1 - x)^{\frac{x^a}{[-\ln(x)]^b}}}$
Topp-Leone	$\alpha > 0,\ a \geq -1,\ b \geq 0,\ 1 + a + b \neq 0$	$\left[1 - (1 - x)^{\frac{x^a}{[-\ln(x)]^b}}\right]^{\alpha}\left[1 + (1 - x)^{\frac{x^a}{[-\ln(x)]^b}}\right]^{\alpha}$
Weibull	$\alpha > 0,\ \beta > 0,\ a \geq -1,\ b \geq 0,$ $1 + a + b \neq 0$	$1 - e^{-\left[-\frac{1}{\beta}\frac{x^a}{[-\ln(x)]^b}\ln(1-x)\right]^{\alpha}}$
Odd Fréchet	$\alpha > 0,\ a \geq -1,\ b \geq 0,\ 1 + a + b \neq 0$	$e^{-\left(\frac{(1-x)^{\frac{x^a}{[-\ln(x)]^b}}}{1-(1-x)^{\frac{x^a}{[-\ln(x)]^b}}}\right)^{\alpha}}$
Sin	$a \geq -1,\ b \geq 0,\ 1 + a + b \neq 0$	$\cos\left[\frac{\pi}{2}(1 - x)^{\frac{x^a}{[-\ln(x)]^b}}\right]$
Cos	$a \geq -1,\ b \geq 0,\ 1 + a + b \neq 0$	$1 - \sin\left[\frac{\pi}{2}(1 - x)^{\frac{x^a}{[-\ln(x)]^b}}\right]$

Table 9. (Continued)

Scheme	Parameter(s)	CDF for $x \in (0, 1)$
Tan	$a \geq -1, b \geq 0, 1 + a + b \neq 0$	$\tan\left\{\frac{\pi}{4}\left[1-(1-x)^{\frac{x^a}{[-\ln(x)]^b}}\right]\right\}$
Sec	$a \geq -1, b \geq 0, 1 + a + b \neq 0$	$\sec\left\{\frac{\pi}{3}\left[1-(1-x)^{\frac{x^a}{[-\ln(x)]^b}}\right]\right\}-1$

The type II CDF modification can be viewed as the complementary VPP CDF of the first kind. Overall, this table expands the horizons of the VPP CDF of the eighth kind by integrating a series of variants. This enriches the available range of CDF VPPs.

This CDF can be leveraged to create a wide range of continuous distributions covering various statistical scenarios. Essentially, for any continuous distribution characterized by a CDF (whatever its support) denoted $G(x)$ and $F(x)$ as described in Equation (18), we can generate a new family of distributions by defining the following CDF:

$$J(x) = F[G(x)] = 1 - [1 - G(x)]^{\frac{[G(x)]^a}{\{-\ln[G(x)]\}^b}}, \quad x \in \mathbb{R}. \tag{19}$$

In the particular case where $a = 1$ and $b = 0$, it corresponds to the CDF at the family studied in Tahir et al. [36]. Some precise CDFs of this type are given below.

- The function defined by

$$J(x) = 1 - (1 - x^{\theta})^{\frac{x^{a\theta}}{[-\theta\ln(x)]^b}}, \quad x \in (0, 1),$$

and completed by $J(x) = 0$ for $x \leq 0$ and $J(x) = 1$ for $x \geq 1$, is a valid CDF for $a \geq -1, b \geq 0$ such that $1 + a + b \neq 0$ and $\theta > 0$. It is derived from Equation (19) by choosing the CDF of the power distribution with parameter θ for $G(x)$.

• The function specified by

$$J(x) = 1 - e^{-\lambda x \frac{(1-e^{-\lambda x})^{a}}{[-\ln(1-e^{-\lambda x})]^{b}}}, \quad x > 0,$$

with the addition of $J(x) = 0$ for $x \leq 0$, is a valid CDF for $a \geq -1$, $b \geq 0$ such that $1 + a + b \neq 0$ and $\lambda > 0$. It is obtained using Equation (19) by selecting the CDF of the exponential distribution with parameter λ for $G(x)$.

• The function defined by

$$J(x) = 1 - [1 - (1 + e^{-\beta x})^{-1}]^{\frac{(1+e^{-\beta x})^{-a}}{[\ln(1+e^{-\beta x})]^{b}}}, \quad x \in \mathbb{R},$$

is a valid CDF for $a \geq -1$, $b \geq 0$ such that $1 + a + b \neq 0$ and $\beta > 0$. It is derived from Equation (19) by choosing the CDF of the logistic distribution with parameter β for $G(x)$.

Thus, original VPP CDFs with diverse supports can be derived from the findings. The collection of VPP CDFs is now finalized. It is important to note that products or mixtures of CDFs remain as CDFs. Therefore, leveraging the existing collection, a wide array of VPP CDFs can be derived as needed.

10. Conclusion

In conclusion, the pursuit of accurate and exible CDFs of distributions supported on the unit interval $(0, 1)$ stands as a significant challenge in the domain of probability and statistics. The necessity for such distributions is evident in their applicability to proportional and rate-type data, prevalent across diverse fields such as biology, finance, environmental science, and reliability analysis. This article introduces a novel collection of CDFs utilizing VPP functions to address this persistent challenge, allowing for dependence on one or more tuning parameters. Table 10 summarizes the main eight VPP CDFs developed in this article.

Table 10. Summary of the main VPP CDFs developed in this article

Kind	Parameter(s)	CDF for $x \in (0, 1)$
First	$\alpha > 0,\ a \geq 0,\ b > -1$	$x^{\frac{[-\ln(x)]^b}{x^a}}$
Second	$b \leq 0,\ a \geq 1_{\{c \neq 0\}} - b,\ a > 0,\ c \in [0, 1]$	$x^{a+bx+cx\ln(x)}$
Third	$a \in [0, 1],\ b \geq 1$	$(1-x)^{a(1-x)^b} - (1-x)$
Fourth	$b \geq 0,\ a \geq 1 - b$	$\frac{1}{a}\{[1 + ax + bx\ln(x)]^x - x^x\}$
Fifth	$a \geq 0,\ b \leq 0,\ c \leq 0,\ a - b - c \neq 0,\ d \geq 0$	$(1+a)\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{-\frac{1}{x^d}}$
Sixth	$a \geq 0,\ b \leq 0,\ c \leq 0,\ a - b - c \neq 0,\ d > 0$	$\left[1 + \frac{a + b\ln(x) + cx\ln(x)}{x}\right]^{1-\frac{1}{x^d}}$
Seventh	$a \geq 0,\ b \geq 0$	$x^{x^{-a(1-x^b)}}$
Eight	$a \geq -1,\ b \geq 0,\ 1 + a + b \neq 0$	$1 - (1-x)^{\frac{x^a}{[-\ln(x)]^b}}$

Furthermore, this collection is enriched by extended or modified versions through standard transformation schemes, including modern trigonometric schemes. The development of these CDFs represents an original contribution to the field, providing a versatile and robust toolkit for modelling and analyzing $(0, 1)$-supported data. Thus, the logical perspective of this work is the in-depth analysis of proportional or rate-like data in modern statistical challenges.

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