Transnational Journal of Mathematical Analysis and Applications
Vol. 11, Issue 1, 2023, Pages 1-18
ISSN 2347-9086
Published Online on March 4, 2023

http://jyotiacademicpress.org

AN OPTIMAL ENERGY CONTROL FOR A SERIALLY CONNECTED EULER-BERNOULLI BEAM

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Abstract

A system of a serially connected Euler-Bernoulli beam and its optimal energy control are studied by means of semigroup of linear operators in the present paper. The system is formulated by partial differential equations with the boundary conditions. The spectral analysis and semigroup generation of the system are discussed in the appropriate Hilbert spaces. Finally, an optimal energy control is proposed, and existence and uniqueness of the optimal control are demonstrated. Eventually, an approximation theorem is proved in terms of semigroup approach and geometric method.

1. Introduction

The vibration and control of serially connected strings and Euler-Bernoulli beams with linear feedback controls at joins have been studied extensively in the last two decades (see, e.g., [2-4, 7, 10, 12, 13, 15]). In addition to the analysis of the distribution of eigenvalues, one also needs

2020 Mathematics Subject Classification: 35B35, 93C20.

Keywords and phrases: serially connected Euler-Bernoulli beam, optimal energy control, semigroup of linear operators.

Communicated by Francisco Bulnes.

Received February 06, 2023

to establish the so-called spectrum determined growth condition in order to conclude exponential stability for these infinite-dimensional systems from spectral analysis. In the case of serially connected strings, the first results on exponential stability were obtained in [2] for a 2-connected strings with linear feedbacks at the middle of the span. The stability of N-connected strings under joints feedback was studied in [3].

In this paper, we consider the following serially connected beams with linear feedback control:

$$y_{tt}(x, t) + y_{xxxx}(x, t) = 0, \quad L_{j-1} < x < L_j, \quad j = 1, 2, \cdots, n. \tag{1.1}$$

The boundary condition are

$$\begin{cases} y(0) = y_{xx}(0) = 0, \\ y_x(L_n) = y_{xxx}(L_n) = 0. \end{cases} \tag{1.2}$$

The linear feedback control at the joint points L_j, $j = 1, \cdots, n-1$, takes the form

$$\begin{cases} y(L_j^-, t) = y(L_j^+, t), \\ y_{xx}(L_j^-, t) = y_{xx}(L_j^+, t), \\ y_{tx}(L_j^+, t) = y_{tx}(L_j^-, t) = (-1)^j r_j y_t(L_j^-, t) + p_j^2 y_{xx}(L_j^-, t), \\ y_{xxx}(L_j^+, t) - y_{xxx}(L_j^-, t) = -q_j^2 y_t(L_j^-, t) + (-1)^j s_j y_{xx}(L_j^-, t), \end{cases} \tag{1.3}$$

where $0 = L_0 < L_1 < \cdots < L_n$ and

$$p_j^2 \geq 0, \quad q_j^2 \geq 0, \quad p_j^2 + q_j^2 > 0, \quad r_j, s_j \in \mathbb{R},$$

$$p_j^2\alpha^2 + q_j^2\beta^2 + (r_j - s_j)\alpha\beta \geq 0, \quad \forall \alpha, \beta \in \mathbb{R}. \tag{1.4}$$

Let us defines the energy of system (1.1)-(1.4) as

$$E(t) = \frac{1}{2}\sum_{j=1}^{n}\int_{L_{j-1}}^{L_j}[y_t^2(x, t) + y_{xx}^2(x, t)]dx.$$

Then a simple computation shows that $\dot{E}(t) \leq 0$ and hence the system is dissipative.

Without loss of generality, we may assume that n is odd. For $j = 1, 2, \cdots, n,$ we set

$$\begin{cases} u_j(x, t) = \frac{1}{2}[y_t(L_j + \frac{(-1)^j - 1}{2} l_j + (-1)^{j+1} l_j x, t) \\ \qquad + \frac{(-1)^{j+1}}{l_j^2} y_{xx}(L_j + \frac{(-1)^j - 1}{2} l_j + (-1)^{j+1} l_j x, t)], \\ v_j(x, t) = \frac{1}{2}[y_t(L_j + \frac{(-1)^j - 1}{2} l_j + (-1)^{j+1} l_j x, t) \\ \qquad - \frac{(-1)^{j+1}}{l_j^2} y_{xx}(L_j + \frac{(-1)^j - 1}{2} l_j + (-1)^{j+1} l_j x, t)], \end{cases} \tag{1.5}$$

where $l_j = L_j - L_{j-1}$, $j = 1, 2, \cdots, n,$ $0 \leq x \leq 1.$ Then system (1.1)-(1.4) can be transformed into the form of

$$\begin{cases} \frac{\partial}{\partial t}\begin{bmatrix} u(x, t) \\ v(x, t) \end{bmatrix} = K\frac{\partial^2}{\partial x^2}\begin{bmatrix} u(x, t) \\ v(x, t) \end{bmatrix} \\ [A, B][u_x(0), v_x(0), u(0), v(0)]^T = 0, \\ [E, F][u_x(1), v_x(1), u(1), v(1)]^T = 0, \end{cases} \tag{1.6}$$

where

$$u(x, t) = [u_1(x, t), u_2(x, t), \dots, u_n(x, t)]^T,$$

$$v(x, t) = [v_1(x, t), v_2(x, t), \dots, v_n(x, t)]^T,$$

and

$(2n \times 2n)$- matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & P_{21} & 0 & \dots & 0 & 0 & P_{22} & 0 & \dots & 0 \\ 0 & 0 & P_{41} & \dots & 0 & 0 & 0 & P_{42} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & P_{(n-1)1} & 0 & 0 & 0 & \dots & P_{(n-1)2} \end{bmatrix}, \tag{1.6a}$$

$$B = \begin{bmatrix} P_{n1} & 0 & 0 & \dots & 0 & P_{n2} & 0 & 0 & \dots & 0 \\ 0 & \widetilde{P}_{21} & 0 & \dots & 0 & 0 & \widetilde{P}_{22} & 0 & \dots & 0 \\ 0 & 0 & \widetilde{P}_{41} & \dots & 0 & 0 & 0 & \widetilde{P}_{42} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \widetilde{P}_{(n-1)1} & 0 & 0 & 0 & \dots & \widetilde{P}_{(n-1)2} \end{bmatrix}, \tag{1.6b}$$

$$E = \begin{bmatrix} P_{11} & 0 & \dots & 0 & 0 & P_{12} & 0 & \dots & 0 & 0 \\ 0 & P_{31} & \dots & 0 & 0 & 0 & P_{32} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_{(n-2)1} & 0 & 0 & 0 & \dots & P_{(n-2)2} & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \tag{1.6c}$$

$$F = \begin{bmatrix} \widetilde{P}_{11} & 0 & \dots & 0 & 0 & \widetilde{P}_{12} & 0 & \dots & 0 & 0 \\ 0 & \widetilde{P}_{31} & \dots & 0 & 0 & 0 & \widetilde{P}_{32} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \widetilde{P}_{(n-2)1} & 0 & 0 & 0 & \dots & \widetilde{P}_{(n-2)2} & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \tag{1.6d}$$

where for $j = 1, 2, \cdots, n,$

$$p_{n1} = \begin{bmatrix} 1 \\ \\ 1 \end{bmatrix}, \quad P_{n2} = \begin{bmatrix} 1 \\ \\ -1 \end{bmatrix},$$

$$P_j 1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{l_j} & \frac{1}{l_j+1} \\ \frac{-1}{l_j} & \frac{1}{l_j+1} \end{bmatrix}, \quad P_j 2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{l_j} & \frac{1}{l_j+1} \\ \frac{1}{l_j} & \frac{1}{-l_j+1} \end{bmatrix},$$

$$\widetilde{P}_j 1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ p_j^2 - r_j & 0 \\ q_j^2 + s_j & 0 \end{bmatrix}, \quad \widetilde{P}_j 2 = \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ -p_j^2 - r_j & 0 \\ q_j^2 - s_j & 0 \end{bmatrix}.$$

Now, we confine ourselves to system (1.1)-(1.4) with A, B, E, F specified by (1.6). Divide by $\rho\omega_1$ both sides of those equations which contain nonzero factors ρ in the system $\tilde{M}C = 0$, then we have becomes

$$\tilde{M}C = 0, \tag{1.7}$$

where

$$\tilde{M} = [M_1 \;\; M_2 \;\; M_3 \;\; M_4], \tag{1.8}$$

and for $1 \le k \le 4$.

$$M_k = \begin{bmatrix} Q_0 k & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & Q_2 k & R_2 k & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & Q_{n-k} & R(n-1)k \\ Q_{1k}e^{\omega_k \rho l_1} & R_{1k}e^{\omega_k \rho l_1} & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & Q_{(n-2)k}e^{\omega_k \rho l_{n-2}} & Q_{(n-2)k}e^{\omega_k \rho l_{n-1}} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & Q_{n1}e^{\omega_k \rho l_n} \end{bmatrix}, \tag{1.9}$$

with

$$Q_{01} = [1-i \quad 1+i]^T, \quad Q_{02} = [1+i \quad 1-i]^T,$$

$$Q_{03} = Q_{01}, \quad Q_{04} = Q_{02}, \quad Q_{n1} = Q_{01},$$

$$Q_{n2} = Q_{02} \cdot i, \quad Q_{n3} = -Q_{n1}, \quad Q_{n4} = -Q_{n2}, \tag{1.10}$$

where ρ, ω_k, $1 \le k \le 4$ and l_n were defined in the Section 3 of [4].

For $j = 1, 3, \cdots, n-2$, $l = 2, 4, \cdots, n-1$,

$$Q_{j1} = \left[1-i+\frac{(1+i)p_j^2-(1-i)r_j}{\rho\omega_1}, -(1+i)+\frac{(1-i)q_j^2+(1+i)s_j}{\rho\omega_1}, 1-i, 1+i\right]^T, \tag{1.11a}$$

$$Q_{j2} = \left[-(1-i)+\frac{(1-i)p_j^2-(1+i)r_j}{\rho\omega_1}, -(1+i)+\frac{(1+i)q_j^2+(1-i)s_j}{\rho\omega_1}, 1+i, 1-i\right]^T, \tag{1.11b}$$

$$Q_{j3} = \left[-(1-i)+\frac{(1+i)p_j^2-(1-i)r_j}{\rho\omega_1}, 1+i+\frac{(1-i)q_j^2+(1+i)s_j}{\rho\omega_1}, 1-i, 1+i\right]^T, \tag{1.11c}$$

$$Q_{j4} = \left[1-i+\frac{(1-i)p_j^2-(1+i)r_j}{\rho\omega_1}, 1+i+\frac{(1+i)q_j^2+(1-i)s_j}{\rho\omega_1}, 1+i, 1-i\right]^T, \tag{1.11d}$$

$$Q_{l1} = \left[1+i+\frac{(1-i)p_l^2-(1+i)r_l}{\rho\omega_1}, -(1-i)+\frac{(1+i)q_l^2+(1-i)s_l}{\rho\omega_1}, 1+i, 1-i\right]^T, \tag{1.12a}$$

$$Q_{l2} = \left[1 + i + \frac{(1+i)p_l^2 - (1-i)r_l}{\rho\omega_1}, 1 - i + \frac{(1-i)q_l^2 + (1+i)s_l}{\rho\omega_1}, 1 - i, 1 + i\right]^T,$$

(1.12b)

$$Q_{l3} = \left[-(1+i) + \frac{(1-i)p_l^2 - (1+i)r_l}{\rho\omega_1}, 1 - i + \frac{(1+i)q_l^2 + (1-i)s_l}{\rho\omega_1}, 1 + i, 1 - i\right]^T,$$

(1.12d)

$$Q_{l4} = \left[-(1+i) + \frac{(1+i)p_l^2 - (1-i)r_l}{\rho\omega_1}, -(1-i) + \frac{(1-i)q_l^2 + (1+i)s_l}{\rho\omega_1}, 1 - i, 1 + i\right]^T,$$

(1.12d)

$$R_{j1} = [1 + i, \ \ 1 - i, \ \ -(1+i), \ \ 1 - i]^T,$$

$$R_{j2} = [1 + i, \ \ -(1-i), \ \ -(1-i), \ \ 1 + i]^T,$$

$$R_{j3} = [-(1+i), \ \ -(1-i), \ \ -(1+i), \ \ 1 - i]^T,$$

$$R_{j4} = [-(1+i), \ \ 1 - i, \ \ -(1-i), \ \ 1 + i]^T, \tag{1.13}$$

$$R_{l1} = [1 - i, \ \ 1 + i, \ \ -(1-i), \ \ 1 + i]^T,$$

$$R_{l2} = [-(1-i), \ \ 1 + i, \ \ -(1+i), \ \ 1 - i]^T,$$

$$R_{l3} = [-(1-i), \ \ -(1+i), \ \ -(1-i), \ \ 1 + i]^T,$$

$$R_{l4} = [1 - i, \ \ -(1+i), \ \ -(1+i), \ \ 1 - i]^T. \tag{1.14}$$

2. Spectral Analysis and Semigroup Generation

In this section, we derive the characteristic equation satisfied by eigenvalues of system (1.1)-(1.4). To begin with, we put system (1.1)-(1.4) into the framework of evolutionary equations in an underlying Hilbert space $\mathcal{H}$. Take $\mathcal{H} = (L^2(0, 1))^{2n}$ and define $\mathcal{A} : D(A)(\subset \mathcal{H}) \to \mathcal{H}$ by

$$\mathcal{A}\begin{bmatrix} u \\ v \end{bmatrix} = K\frac{\partial^2}{\partial x^2}\begin{bmatrix} u \\ v \end{bmatrix}, \tag{2.1}$$

where

$$D(\mathcal{A}) = \left\{ [u, v]^T \in (H^2(0, 1))^{2n} \middle| \begin{array}{l} [A, B][u_x(0), v_x(0), u(0), v(0)]^T = 0, \\ [E, F][u_x(1), v_x(1), u(1), v(1)]^T = 0 \end{array} \right\},$$

and $H^2(0, 1)$ denotes the usual Sobolev space. With this setting, system (1.1)-(1.4) can be considered as an abstract equation in $\mathcal{H}$:

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \mathcal{A}\begin{bmatrix} u \\ v \end{bmatrix}. \tag{2.2}$$

Obviously, $\mathcal{A}$ is densely defined in $\mathcal{H}$. Next, we consider the eigenvalue problem for $\mathcal{A}$. For any given $\Phi = [f, g]^T \in \mathcal{H}$, solve the following equation:

$$(\lambda - \mathcal{A})\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{2.3}$$

i.e.,

$$\begin{cases} \dfrac{\partial^2}{\partial x^2}\begin{bmatrix} u \\ v \end{bmatrix} = \lambda K^{-1}\begin{bmatrix} u \\ v \end{bmatrix} - K^{-1}\Phi, \\ [A, B][u_x(0), v_x(0), u(0), v(0)]^T = 0, \\ [E, F][u_x(1), v_x(1), u(1), v(1)]^T = 0, \end{cases} \tag{2.4}$$

which can be further written as a first-order ordinary differential equation of the following form:

$$\begin{cases} \dfrac{\partial}{\partial x}\begin{bmatrix} u_x \\ v_x \\ u \\ v \end{bmatrix} = \begin{bmatrix} 0_{2n} & \lambda K^{-1} \\ I_{2n} & 0_{2n} \end{bmatrix}\begin{bmatrix} u_x \\ v_x \\ u \\ v \end{bmatrix} - \begin{bmatrix} K^{-1}\Phi \\ 0 \end{bmatrix}, \\ [A,\ B][u_x(0),\ v_x(0),\ u(0),\ v(0)]^T = 0, \\ [E,\ F][u_x(1),\ v_x(1),\ u(1),\ v(1)]^T = 0, \end{cases} \tag{2.5}$$

where I_{2n} denotes the $2n \times 2n$ identity matrix. Set

$$K_\lambda = \begin{bmatrix} 0_{2n} & \lambda K^{-1} \\ I_{2n} & 0_{2n} \end{bmatrix}. \tag{2.6}$$

Then the solution to the governing equation of 2.5 is

$$\begin{bmatrix} u_x(x) \\ v_x(x) \\ u(x) \\ v(x) \end{bmatrix} = e^{K_\lambda x}\begin{bmatrix} u_x(0) \\ v_x(0) \\ u(0) \\ v(0) \end{bmatrix} - \int_0^x e^{K_\lambda (x-s)}\begin{bmatrix} K^{-1}\Phi \\ 0 \end{bmatrix} ds. \tag{2.7}$$

In order for (2.7) to satisfy (2.5), the last two boundary conditions should be fulfilled, i.e.,

$$\begin{cases} [A,\ B][u_x(0),\ v_x(0),\ u(0),\ v(0)]^T = 0, \\ [E,\ F]e^{K_\lambda}[u_x(0),\ v_x(0),\ u(0),\ v(0)]^T = \int_0^1 [E,\ F]e^{K_\lambda (1-s)}[K^{-1}\Phi,\ 0]^T\, ds, \end{cases} \tag{2.8}$$

Define

$$H(\lambda) = \begin{bmatrix} [A,\ B] \\ [E,\ F]e^{K_\lambda} \end{bmatrix}. \tag{2.9}$$

Then for

$$h(\lambda) = \det H(\lambda) \neq 0, \tag{2.10}$$

it has

$$R(\lambda,\ \mathcal{A})\Phi = [0_{2n},\ I_{2n}]e^{K_\lambda x}\begin{bmatrix} u_x(0) \\ v_x(0) \\ u(0) \\ v(0) \end{bmatrix} - \int_0^x [0_{2n},\ I_{2n}]e^{K_\lambda(x-s)}\begin{bmatrix} K^{-1}\Phi \\ 0 \end{bmatrix} ds, \tag{2.11}$$

where

$$\begin{bmatrix} u_x(0) \\ v_x(0) \\ u(0) \\ v(0) \end{bmatrix} = H^{-1}(\lambda)\begin{bmatrix} 0 \\ \int_0^1 [E,\ F]e^{K_\lambda(1-s)}\begin{bmatrix} K^{-1}\Phi \\ 0 \end{bmatrix} ds \end{bmatrix}. \tag{2.12}$$

Therefore, in this case, $\lambda \in \rho(\mathcal{A})$ and $R(\lambda,\ \mathcal{A})$ is compact.

On the order hand, if $h(\lambda) = 0$, for any $4n \times 1$ nonzero column vector $Z = (u_x(0),\ v_x(0),\ u(0),\ v(0))^T$ satisfying $H(\lambda)Z = 0$, by setting $\Phi = 0$ in (2.7), we have

$$\begin{bmatrix} u_x(0) \\ v_x(0) \\ u(0) \\ v(0) \end{bmatrix} = e^{K_\lambda x} Z \neq 0,$$

and hence $(u_x(0), v_x(0), u(0), v(0))^T \neq 0$. Therefore,

$$\begin{bmatrix} u \\ v \end{bmatrix} = [0_{2n}, I_{2n}] \begin{bmatrix} u_x \\ v_x \\ u \\ v \end{bmatrix} = [0_{2n}, I_{2n}] e^{K_\lambda x} Z \neq 0, \tag{2.13}$$

and satisfies

$$\mathcal{A} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix}.$$

In other words, $\lambda \in \sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$.

To sum up, we have obtained the following Theorem:

Theorem 1. *Let* $h(\lambda) = \det H(\lambda)$ *be defined by* (2.10). *Then* $h(\lambda)$ *is an entire function of* λ, *and the following statements hold*:

(1) $\lambda \in \sigma(\mathcal{A})$ *if and only if* $h(\lambda) = 0$, *i.e.*,

$$\sigma(\mathcal{A}) = \{\lambda | h(\lambda) = 0\}. \tag{2.14}$$

The eigenvalues are symmetric with respect to the real axis.

(2) *For each* $\lambda \in \sigma(\mathcal{A})$, *the corresponding eigenfunction* $[u, v]^T$ *is given by* (2.13), *where* Z *is any nonzero solution of the algebraic equation* $H(\lambda)Z = 0$.

(3) $\mathcal{A}$ *is a densely defined discrete operator in* $\mathcal{H}$, *i.e.*, $\mathcal{A}$ *is densely defined in* $\mathcal{H}$ *and* $R(\lambda, \mathcal{A}) = (\lambda - \mathcal{A})^{-1}$ *is compact for any* $\lambda \in \rho(\mathcal{A})$.

(4) $\mathcal{A}$ *is an infinitesimal generator of a* C_0*-semigroup in* $\mathcal{H}$.

3. An Optimal Energy Control

In this section, let us discuss an optimal control problem of the following system:

$$\frac{dy}{dt} = Ay(t) + \mathcal{B}u((y(t),\, t),$$

$$y(0) = y_0, \tag{3.1}$$

where both state space $\mathcal{H}$ and control space $\mathcal{Y}$ are Hilbert spaces, the state function $y(t)$ on $[0, T]$ is valued in $H, \mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $S(t),\, t \geq 0.\ \mathcal{B}$ is a bounded linear operator from $L^2([0, T] : \mathcal{Y})$ to $L^2([0, T] : \mathcal{H}),\, u(y(t),\, t)$ is a control of the system.

In this section, we shall discuss a specific optimal control, that is, the minimum energy control of the system (3.1). We know that the minimum energy control in an abstract space is, in general, the minimum norm control. So, from mathematics point of view, the existence and uniqueness of the optimal control are essential. If these are true, then how to obtain the optimal control is a significant problem. The main content of this paper is to solve these essential and significant issue.

From the theory of operator semigroup, we see that for every control element $u(y(\cdot),\, \cdot) \in L^2([0, T] : \mathcal{Y})$, the system (3.1) has an unique mild solution

$$y(t) = S(t)y_0 + \int_0^t S(t-s)\mathcal{B}(u(y(s),\, s))ds. \tag{3.2}$$

Let $\varphi(\cdot)$ be an arbitrary element in $C([0, T],\, \mathcal{H})$, and

$$\rho = \inf_{u \in L^2([0, T],\, \mathcal{Y})} \|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)\mathcal{B}u(y(s),\, s)ds\|,$$

define the admissible control set of the system (3.1) as follows:

$$U_{ad} = \{u \in L^2([0, T], \mathcal{Y}) : \|\varphi(t) - S(t)y_0$$

$$- \int_0^t S(t-s)\mathcal{B}u(y(s), s)\| \leq \rho + \epsilon\}, \tag{3.3}$$

where ϵ is any positive number.

It can be seen from (2.2) that U_{ad} is not empty and contains infinitely many elements related to φ and ϵ. The minimum energy control problem is actually to find the element u, satisfying

$$\|u_0\| = \min\{\|u\| : u \in U_{ad}\}, \tag{3.4}$$

where u_0 is said to be a minimum energy control element.

Lemma 3.1. *The admissible control set* U_{ad} *defined by* (2.2) *is a closed convex set in Hilbert space* $L^2([0, T] : \mathcal{Y})$.

Proof. Convexity: For any $u_1, u_2 \in U_{ad}$ and a real number λ, $0 < \lambda < 1$, it is easy to see from (2.2) that

$$\|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)\mathcal{B}u_i(y(s), s)\| \leq \rho + \epsilon, \quad i = 1, 2, \tag{3.5}$$

and hence

$$\|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)B(\lambda u_1(y(s), s) + (1-\lambda)u_2(y(s), s))ds\|$$

$$\leq \lambda\|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)\mathcal{B}u_1(y(s), s)ds\|$$

$$+ (1-\lambda)\|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)\mathcal{B}u_2(y(s), s)ds\|. \tag{3.6}$$

Since $\lambda u_1 + (1-\lambda)u_2 \in L^2([0, T]; \mathcal{Y})$, it follows that $\lambda u_1 + (1-\lambda)u_2 \in U_{ad}$, this implies that U_{ad} is a convex subset of $L^2([0, T]; \mathcal{Y})$.

Closedness: Suppose $\{u_n\} \subset U_{ad}$, and $\lim_{n\to\infty}\|u_n - u^*\| = 0$. It can be shown that $u^* \in U_{ad}$. In fact, from the definition of U_{ad} we see that

$$\|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)\mathcal{B}u_n(y(s), s)ds\| \leq \rho + \epsilon, \quad n = 1, 2, \cdots.$$

Since $S(t), t \geq 0$ is a C_0-semigroup in Hilbert space $\mathcal{H}$, there is a constant $M > 0$ such that $\sup_{0\leq t\leq T}\|S(t)\| \leq M$. On the other hand, since $y(s)$ is differentiable on $[0, T]$, it is continuous on $[0, T]$, and hence $\{y(s) : s \in [0, T]\}$ is a bounded set in $L^2([0, T] : \mathcal{Y})$. Thus there is a constant $N > 0$ such that $\|Bu(y(s), s)\| \leq N (0 \leq s \leq T)$ and

$$\begin{aligned}
&\|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)\mathcal{B}u^*(y(s), s)ds\| \\
&\leq \|\varphi(t) - S(t)y_0 - \int_0^t u_n(y(s), s)\mathcal{B}u(y(s), s)ds\| \\
&\quad + \|\int_0^t S(t-s)\mathcal{B}[u_n(y(s), s) - u^*(y(s), s)]\| \\
&\leq \rho + \epsilon + M\|u_n - u^*\| \cdot NT.
\end{aligned} \tag{3.7}$$

Letting $n \to \infty$ leads to

$$\|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)\mathcal{B}u^*(y(s), s)ds\| \leq \rho + \epsilon.$$

Thus, $u^* \in U_{ad}$, and U_{ad} is a closed set. The proof is complete.

Theorem 2. *There exists an unique minimum energy control element in the admissible control set* U_{ad} *of the system* (1.1).

Proof. Since $L^2([0, T] : \mathcal{Y})$ is a Hilbert space, it is naturally a strict convex Banach Space. From the preceding Lemma, we have seen that U_{ad} is a closed convex set in $L^2([0, T] : \mathcal{Y})$, it follows from [2] that there is an unique element $u_0 \in U_{ad}$ such that

$$\|u_0\| = \min\{\|u\| : u \in U_{ad}\}.$$

According to the definition (2.3), u_0 is just the desired minimum energy control element of the system (1.1). The proof is complete.

Finally, we shall show that the minimum energy control element can be approached.

Theorem 3. *Suppose that* u_0 *is the minimum energy control element of the system* (1.1), *then there exists a sequence* $\{u_n\} \subset U_{ad}$ *such that* $\{u_n\}$ *converges strongly to* u_0 *in* $L^2([0, T] : \mathcal{Y})$, *namely*,

$$\lim_{n \to \infty} \|u_n - u_0\| = 0.$$

Proof. Let $\{u_n\}$ be a minimizing sequence in the admissible control set U_{ad}, then it follows that

$$\|u_{n+1}\| \leq \|u_n\|, \quad n = 1, 2, \cdots, \tag{3.8}$$

and

$$\lim_{n \to \infty} \|u_n\| = \inf\{\|u\| : u \in U_{ad}\}. \tag{3.9}$$

It is obvious that $\{u_n\}$ is a bounded sequence in $L^2([0, T]; \mathcal{Y})$, and so there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ weakly converges to an element $\tilde{u}$ in $L^2([0, T]; \mathcal{Y})$ (see [3]).

Since U_{ad} is a closed convex set in $L^2([0, T]; \mathcal{Y})$ (see Lemma 2.1), we see from Mazur's Theorem that U_{ad} is a weakly closed set in $L^2([0, T] : \mathcal{Y})$, thus $\tilde{u} \in U_{ad}$. Combining (3.2) and employing the properties of limits of weakly convergent sequence on norm yield

$$\inf\{\|u\| : u \in U_{ad}\} \leq \|\tilde{u}\| \leq \underline{\lim}_{k\to\infty} \|u_{n_k}\|$$

$$= \lim_{n_k\to\infty} \|u_{n_k}\| = \lim_{n\to\infty} \|u_n\| = \inf\{\|u\|;\ u \in U_{ad}\}. \quad (3.10)$$

Thus, we have

$$\lim_{n\to\infty} \|u_n\| = \|\tilde{u}\|, \quad (3.11)$$

and

$$\|\tilde{u}\| = \inf\{\|u\|;\ u \in U_{ad}\}. \quad (3.12)$$

Since $\{u_{n_k}\}$ is weakly convergent to $\tilde{u}$, it follows from (3.3) that $\{u_{n_k}\}$ converges to $\tilde{u}$. Therefore, we see in terms of Theorem 2 and (3.4) that $\tilde{u} = u_0$, namely, $\tilde{u}$ is the minimum energy control element. Thus, $\{u_{n_k}\}$ strongly converges to the minimum energy control element in $L^2([0, T] : \mathcal{Y})$. Without loss of generality, we can rewrite $\{u_{n_k}\}$ by $\{u_n\}$, then the conclusion of theorem is now obtained.

The Theorem 3 points out that the minimum energy control element can be approached by a weakly convergent sequence in the control space, which provides the theoretical basis of approximate computation for finding the minimum energy control element.

4. Conclusion

In this paper, we have investigated a kind of optimal energy control for a serially connected Euler-Bernoulli Beam formulated by partial differential equations with initial and boundary conditions. After a discussion of minimum energy problem for the beam system, we have proposed and proved the existence and uniqueness Theorem 3 of the optimal energy control in terms of semigroup approach of linear operators. Finally, we gave an approximation result Theorem 3.2 that points out that the minimum energy control element can be approached by a weakly convergent sequence in the control space, and provides the theoretical basis of approximate computation for finding the optimal energy control element.

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