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STRICT OSCILLATION CRITERIA FOR FIRST-ORDER TWO DIMENSIONAL LINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

The Riccati equation method is used to establish new strict oscillatory criteria for systems of two linear First order ordinary differential equations. These criteria can be used for detection of oscillating linear matrix Hamiltonian systems.

1. Introduction

Let $a_{jk}(t)$ $(j, k = 1, 2)$ be real-valued continuous functions on $[t_0, +\infty)$. Consider the linear system

$$\begin{cases} \phi' = a_{11}(t)\phi + a_{12}(t)\psi, \\ \psi' = a_{21}(t)\phi + a_{22}(t)\psi, \quad t \geq t_0, \end{cases} \tag{1.1}$$

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and associated with it the Riccati equation

$$z' + a_{12}(t)z^2 + B(t)z - a_{21}(t) = 0, \quad t \geq t_0, \tag{1.2}$$

where $B(t) = a_{11}(t) - a_{22}(t)$, $t \geq t_0$. The solutions $z(t)$ of this equation, existing on an interval $[t_1, t_2)$, $(t_0 \leq t_1 < t_2 \leq +\infty)$ are connected with solutions $(\phi(t), \psi(t))$ of the system (1.1) by the relations (see [1])

$$\phi(t) = \phi(t_1)\exp\left\{\int_{t_1}^{t} [a_{12}(\tau)z(\tau) + B(\tau)]d\tau\right\}, \; \phi(t_1) \neq 0, \; \psi(t) = z(t)\phi(t), \tag{1.3}$$

$t \in [t_1, t_2)$. Let $z_0(t)$ be a solution of Equation (1.2) with $z_0(t_0) = i$. It was shown in [2] that $z_0(t)$ exists on $[t_0, +\infty)$. Set: $x_0(t) \equiv \mathfrak{Re}\, z_0(t)$, $y_0(t) \equiv \operatorname{Im} z_0(t)$, $t \geq t_0$. Then for every real-valued solution $(\phi(t), \psi(t))$ of the system (1.1) we can write the equalities (see [2])

$$\phi(t) = \mu \frac{J_{S/2}(t)}{\sqrt{y_0(t)}} \sin\left(\int_{t_0}^{t} a_{12}(\tau)y_0(\tau)d\tau + \nu\right), \tag{1.4}$$

$$\psi(t) = \mu\sqrt{x_0^2(t) + y_0^2(t)}\,\frac{J_{S/2}(t)}{\sqrt{y_0(t)}} \cos\left(\int_{t_0}^{t} a_{12}(\tau)y_0(\tau)d\tau + \nu - \alpha_0(t)\right), \tag{1.5}$$

where μ, ν are some real constants, $J_{S/2}(t) \equiv \exp\left\{\int_{t_0}^{t} \frac{a_{11}(\tau) + a_{22}(\tau)}{2} d\tau\right\}$,

$$\alpha_0(t) \equiv \arcsin \frac{x_0(t)}{\sqrt{x_0^2 9t0 + y_0^2(t)}} = \arctan \frac{x_0(t)}{y_0(t)}, \; t \geq t_0.$$

Definition 1.1. A connected component of the set of zeroes of $\phi(t)$ $(\psi(t))$ of a real-valued solution $(\phi(t), \psi(t))$ of the system (1.1) is called a null-element of $\phi(t)$ $(\psi(t))$ and is denoted by $N(\phi)$ $(N(\psi))$.

Definition 1.2. Two null-elements $N_1(\phi)$ and $N_2(\phi)$ ($N_1(\psi)$ and $N_2(\psi)$) of $\phi(t)$ ($\psi(t)$) of a solution $(\phi(t), \psi(t))$ of the system (1.1) are called congenerous if for every $t_j \in N_j(\phi)$ ($\in N_j(\psi)$), $j = 1, 2$ the inequality

$$\left| \int_{t_1}^{t} a_{12}(\tau) y_0(\tau) d\tau \right| < \pi \left(\left| \int_{t_1}^{t} a_{21}(\tau) y_1(\tau) d\tau \right| < \pi \right), \; r \in [t_1, t_2],$$

is satisfied, where $y_1(t) \equiv \operatorname{Im} z_1(t)$, $z_1(t)$ is a solution of the Riccati equation

$$z' + a_{21}(t) z^2 - B(t) z - a_{12}(t) = 0, \quad t \geq t_0,$$

with $z_1(t_0) = i$.

The congeniality relation is an equivalence (see [2]).

Definition 1.3. An equivalence class of congenerous null-elements of $\phi(t)$ ($\psi(t)$) of a solution $(\phi(t), \psi(t))$ of the system (1.1) is called a null-class of $\phi(t)$ ($\psi(t)$) and is denoted by $n(\phi)$ ($n(\psi)$).

Definition 1.4. The system (1.1) is called oscillatory if for its every real-valued non trivial solution $(\phi(t), \psi(t))$ the functions $\phi(t)$ and $\psi(t)$ have arbitrary large zeroes.

Definition 1.5. The system (1.1) is called strict oscillatory if for its every real-valued non trivial solution $(\phi(t), \psi(t))$ the functions $\phi(t)$ and $\psi(t)$ have infinitely many null-classes.

Notice that from the strict oscillation of the system (1.1) it follows its oscillation, but from the oscillation of the system (1.1) does not follow its strict oscillation (see [2]).

In this paper, we use the Riccati equation method for establishing some new strict oscillatory criteria for the system (1.1). They can be used for, e.g., detection of oscillating linear matrix Hamiltonian systems (see [3]).

2. Auxiliary Propositions

Hereafter we will assume that the coefficient functions $a_{12}(t)$ and $a_{21}(t)$ have unbounded supports (the case when one of them has a bounded support is trivial).

Definition 2.1. A real-valued solution of Equation (1.2) is called t_1-regular if it exists on $[t_1, +\infty)$.

Definition 2.2. A real-valued solution $x(t)$ of Equation (1.2) is called t_1-normal, if there exists a neighbourhood $U_\delta(x(t_1)) \equiv (x(t_1) - \delta, x(t_1) + \delta)$ of $x(t_1)$ such that every solution $\tilde{x}(t)$ of Equation (1.2) with $\tilde{x}(t_1) \in U_\delta(x(t_1))$ is t_1-regular. Otherwise it is called t_1-extremal.

Denote by $reg(t_1)$ the set of all $x_{(0)} \in \mathbb{R}$, for which the solutions $x(t)$ of Equation (1.2) with $x(t_1) = x_{(0)}$ are t_1-regular.

Lemma 2.1. *If* $a_{12}(t) \geq 0,\ t \geq t_0$ *and Equation* (1.2) *has* a t_1-*regular solution for some* $t_1 \geq t_0$, *then it has the unique* t_1-*extremal solution* $x_*(t)$ *and* $reg(t_1) = [y_*(t_1), +\infty)$.

See the proof in [4].

For any continuous function $u(t)$ on $[t_0, +\infty)$ set

$$\nu_u(t) \equiv \int_t^{+\infty} a_{12}(\tau) \exp\left\{-\int_t^{\tau} [2a_{12}(s)u(s) + B(s)]ds\right\} d\tau, \quad t \geq t_0.$$

Theorem 2.1. *Let* $a_{12}(t) \geq 0,\ t \geq t_0$, *and let Equation* (1.2) *have a* t_1-*regular solution* $x(t)$. *In order that* $x(t)$ *is* t_1-*normal it is necessary and sufficient that* $\nu_x(t_1) < +\infty$.

See the proof in [4].

Furthermore, we will assume that the set $([t_1, +\infty)\setminus supp a_{12}(t)) \cap ([t_1, +\infty)\setminus supp B(t)(t))$ has a null measure. Denote by Ω the set of all positive and continuously-differentiable on $[t_0, +\infty)$ functions $f(t)$ on $[t_0, +\infty)$ for which the set $([t_1, +\infty)\setminus supp a_{12}(t)) \cap ([t_1, +\infty)\setminus supp f'(t)(t))$ has a null measure. Finally for arbitrary functions $u(t)$ and $v(t)$ on $[t_0, +\infty)$ set

$$\left(\frac{u(t)}{v(t)}\right)_0 \equiv \begin{cases} \dfrac{u(t)}{v(t)}, & \text{if } v(t) \neq 0, \\ 0, & \text{if } v(t) = 0. \end{cases}$$

Lemma 2.2. *Let* $x(t)$ *be a* t_1*-regular solution of Equation* (1.2) *and let* $a_{12}(t) \geq 0$, $t \geq t_0$. *Then for every* $f \in \Omega$, *the following inequality is valid.*

$$x(t) \leq \frac{c(t_1, x)}{f(t)}$$

$$+ \frac{1}{f(t)} \int_{t_0}^{t} \left[f(\tau)a_{21}(\tau) + \frac{f(\tau)a_{12}(\tau)}{4} \left(\frac{f(\tau)B(\tau) - f'(\tau)}{f(\tau)a_{12}(\tau)} \right)_0^2 \right] d\tau, \quad t \geq t_1, \quad (2.1)$$

where

$$c(t_1, x) \equiv f(t_1)x(t_1) - \int_{t_0}^{t_1} \left[f(\tau)a_{21}(\tau) + \frac{f(\tau)a_{12}(\tau)}{4} \left(\frac{f(\tau)B(\tau) - f'(\tau)}{f(\tau)a_{12}(\tau)} \right)_0^2 \right] d\tau.$$

Proof. By (1.2), we have

$$x'(t) + a_{12}(t)x^2(t) + B(t)x(t) - a_{21}(t) = 0, \quad t \geq t_1.$$

Multiply both sides of this equality by $f(t) (\in \Omega)$ and integrate from t_1 to t. We obtain $f(t)x(t) - f(t_1)x(t_1) + \int_{t_1}^{t} [f(\tau)a_{12}(\tau)x^2(\tau) + (f(\tau)B(\tau) - f'(\tau))$

$x(\tau) - f(\tau)a_{21}(\tau)]d\tau = 0, \quad t \geq t_1$. Allocating a full square under the integral of the obtained equality and dividing both sides of it by $f(t)$ we obtain

$$x(t) - \frac{f(t_1)}{f(t)} + \frac{1}{f(t)} \int_{t_1}^{t} f(\tau)a_{12}(\tau)\left[x(t) + \left(\frac{f(\tau)B(\tau) - f'(\tau)}{2f(\tau)a_{12}(\tau)}\right)_0\right]^2 d\tau$$

$$- \frac{1}{f(t)} \int_{t_0}^{t} \left[f(\tau)a_{21}(\tau) + \frac{f(\tau)a_{12}(\tau)}{4}\left(\frac{f(\tau)B(\tau) - f'(\tau)}{f(\tau)a_{12}(\tau)}\right)_0^2\right] d\tau = 0, \; t \geq t_1.$$

From here it follows (2.1). The lemma is proved.

Set

$$Q(t) \equiv \left(\frac{B(t)}{2a_{12}(t)}\right)_0 + \int_{t_0}^{t}\left[a_{21}(\tau) + \left(\frac{B^2(\tau)}{4a_{12}(\tau)}\right)_0\right] d\tau, \quad t \geq t_0.$$

Lemma 2.3. *Let* $a_{12}(t) \geq 0, \; t \geq t_0$. *Then if the system* (1.1) *is not strict oscillatory, then Equation* (1.2) *has a* t_1*-regular solution for some* $t_1 \geq t_0$.

Proof. From the nonnegativity condition on $a_{12}(t)$ it follows that the integral $\int_{t_0}^{+\infty} a_{12}(\tau)y_0(\tau)d\tau$ converges (recall that $y_0(t) > 0, \; t \geq t_0$). Indeed if $\int_{t_0}^{+\infty} a_{12}(\tau)y_0(\tau)d\tau = +\infty$, then for some solution $(\phi(t), \psi(t))$ of the system (1.1) the function $\phi(t)$ has infinite many null-classes. Then by virtue of Lemma 4.2 from [2] the system (1.1) is strict oscillatory which contradicts one of the conditions of the lemma. So the integral $\int_{t_0}^{+\infty} a_{12}(\tau)y_0(\tau)d\tau$ is convergent. Therefore from (1.4), it follows that for some real-valued solution $(\phi_0(t),\ \psi_0(t))$ of the system (1.1) $\phi_0(t) \neq 0,\ t \geq t_1$ for some $t_1 \geq t_0$. By (1.3) from here it follows that $x_0(t) \equiv \dfrac{\psi_0(t)}{\phi_0(t)},\ t \geq t_1$ is a t_1-regular solution of Equation (1.2). The lemma is proved.

In Equation (1.2) substitute

$$z = y + \lambda + \int_{t_0}^{t} \left[\left(\frac{B^2(\tau)}{4a_{12}(\tau)} \right)_0 + a_{21}(\tau) \right] d\tau, \quad t \geq t_0. \tag{2.2}$$

We obtain

$$y' + a_{12}(t)\,(y + \lambda + Q(t))^2 = 0, \quad t \geq t_0, \tag{2.3}$$

or

$$y' + a_{12}(t)y^2 + 2a_{12}(t)\{\lambda + Q(t)\} + \{\lambda + Q(t)\}^2 = 0, \quad t \geq t_0, \tag{2.4}$$

For arbitrary $\lambda \in \mathbb{R}$ set

$$A_\lambda^{\pm} \equiv \{t \geq t_0 : \pm(\lambda + Q(t)) \geq 0\}.$$

Lemma 2.4. *Let the following conditions be satisfied*:

(1) $a_{12}(t) \geq 0, \quad t \geq t_0$;

(2) $\int a_{12}(t)(\lambda + Q(t))^2 dt = +\infty$;

(3) $\int_{A_\lambda^-}^{A_\lambda^+} a_{12}(t)dt = +\infty$;

(4) *Equation* (2.3) *has a* t_1*-regular solution for some* $t_1 \geq t_0$. *Then for the unique* t_1*-extremal solution* $y_*(t)$ *of Equation* (2.3) *the equality*

$$\lim_{t \to =\infty} y_*(t) = -\infty \tag{2.5}$$

is satisfied.

Proof. By virtue of Lemma 2.1 from (4) it follows that Equation (2.1) has the unique t_1-extremal solution which we denote by $y_*(t)$. By (2.3) we have

$$y_*(t) = y_*(t_1) - \int_{t_1}^{t} a_{12}(\tau)(y_*(\tau) + \lambda + Q(\tau))^2 d\tau, \; t \geq t_1. \tag{2.6}$$

From here and from the nonnegativity of $a_{12}(t)$ it follows that $y_*(t)$ is non increasing on $[t_1, +\infty)$. Suppose (2.5) is false. Then there exists a finite limit $y_*(+\infty) = \lim_{t \to +\infty} y_*(t)$. Two cases are possible

(a) $y_*(+\infty) \geq 0$,

(b) $y_*(+\infty) < 0$.

It follows from (2.6) that

$$I(t_1) \equiv \int_{t_1}^{+\infty} a_{12}(\tau)(y_*(\tau) + \lambda + Q(\tau))^2 d\tau < +\infty. \tag{2.7}$$

Assume the case (a) takes place. Then from (2) it follows that $I(t_1) = +\infty$, which contradicts (2.7). If the case (b) takes place then from (3) it follows that again $I(t_1) = +\infty$, which contradicts (2.7). So (2.5) is valid. The lemma is proved.

Lemma 2.5. *Let the conditions* (1) *and* (4) *of Lemma* 2.4 *and for some* $\lambda \in \mathbb{R}$ *and* $\alpha \geq 1$ *the following conditions be satisfied*:

(5) $$\int_{t_0}^{+\infty} a_{12}(t) \exp\left\{ -4\lambda \int_{t_0}^{t} a_{12}(\tau) d\tau - 2\int_{t_0}^{t} B(\tau) d\tau - 4\int_{t_0}^{t} a_{12}(\tau) d\tau \int_{t_0}^{\tau} a_{21}(s) ds \right.$$
$$\left. - \int_{t_0}^{t} a_{12}(\tau) d\tau \int_{t_0}^{\tau} \left(\frac{B^2(s)}{a_{12}(s)} \right)_0 ds \right\} dt < +\infty;$$

(6) $\liminf_{t\to+\infty} \frac{1}{t^\alpha} \int_{t_0}^{t} a_{12}(\tau)(t-\tau)^{\alpha-1}[\lambda + Q(\tau)]d\tau < +\infty;$

(7) $\limsup_{t\to+\infty} \frac{1}{t^\alpha} \int_{t_0}^{t} a_{12}(\tau)(t-\tau)^{\alpha} d\tau > 0.$

Then (2.5) *is valid.*

Proof. Assume (2.5) is false. Then there exists a finite limit $y_*(+\infty) \equiv \lim_{t\to+\infty} y_*(t)$ from here and from (2.6) it follows, that

$$0 \le I \equiv \lim_{t\to+\infty} \frac{1}{t^\alpha} \int_{t_1}^{t} a_{12}(\tau)(y_*(\tau) + \lambda + Q(\tau))^2 d\tau$$

$$\le \lim_{t\to+\infty} \frac{1}{t} \int_{t_1}^{+\infty} a_{12}(\tau)\left(\frac{t-\tau}{t}\right)^{\alpha-1} (y_*(\tau) + \lambda + Q(\tau))^2 d\tau = 0.$$

Set $\rho(t) \equiv y_*((t) - y_*(+\infty)$, $t \ge t_1$. Obviously

$$\rho(t) \to 0, \text{ for } t \to +\infty. \tag{2.9}$$

Then we have

$$I = \limsup_{t\to+\infty} \frac{1}{t^\alpha} \int_{t_1}^{t} a_{12}(\tau)(t-\tau)^{\alpha-1}(y_*(+\infty) + \lambda + Q(\tau) + \rho(\tau))^2 d\tau$$

$$= \limsup_{t\to+\infty} \left[\frac{y_*(+\infty)}{t^\alpha} \int_{t_1}^{t} a_{12}(\tau)(t-\tau)^{\alpha-1}\left[1 + \frac{2\rho(\tau)}{y_*(+\infty)}\right] d\tau \right.$$

$$+ \frac{2y_*(+\infty)}{t^\alpha} \int_{t_1}^{t} a_{12}(\tau)(t-\tau)^{\alpha-1}[\lambda + Q(\tau)]d\tau$$

$$+ \frac{1}{t^\alpha} \int_{t_1}^{t} a_{12}(\tau)[\lambda + Q(\tau) + \rho(\tau)]^2 d\tau$$

$$\geq \limsup_{t\to+\infty}\left[\frac{y_*^2(+\infty)}{t^\alpha}\int\limits_{t_1}^{t} a_{12}(\tau)(t-\tau)^{\alpha-1}\left[1+\frac{2\rho(\tau)}{y_*(+\infty)}\right]d\tau\right.$$

$$\left. + \frac{2y_*(+\infty)}{t^\alpha}\int\limits_{t_1}^{t} a_{12}(\tau)(t-\tau)^{\alpha-1}[\lambda + Q(\tau)]d\tau. \right] \qquad (2.10)$$

Due to (2.9) chose $t_2 > t_1$ so large that $\left|\frac{2\rho(t)}{y_*(+\infty)}\right| < \frac{1}{2}$, $t \geq t_2$. Then taking into account the conditions (5) and (6) from (2.10) we obtain (from 5) it follows that, $y_*(+\infty) < 0$)

$$I \geq \limsup_{t\to+\infty}\left[\frac{y_*^2(+\infty)}{t^\alpha}\int\limits_{t_1}^{t_2} a_{12}(\tau)(t-\tau)^{\alpha-1}\left[1+\frac{2\rho(\tau)}{y_*(+\infty)}\right]d\tau\right.$$

$$\left. + \frac{y_*^2(+\infty)}{2t^\alpha}\int\limits_{t_1}^{t} a_{12}(\tau)(t-\tau)^{\alpha-1}d\tau\right] = \limsup_{t\to+\infty}\frac{y_*^2(+\infty)}{2t^\alpha}\int\limits_{t_1}^{t} a_{12}(\tau)(t-\tau)^{\alpha-1}d\tau.$$

From here and from (7) it follows that $I > 0$, which contradicts (2.8). The obtained contradiction proves (2.5). The lemma is proved.

3. Strict Oscillation Criteria

For any $f \in \Omega$ set:

$$Q_f(t_1;t) \equiv \int\limits_{t_1}^{t}\frac{a_{12}(\tau)}{f(\tau)}d\tau\int\limits_{t_1}^{\tau}\left[2f(s)a_{21}(s)+\frac{1}{2}\left(\frac{f(\tau)B(\tau)-f'(\tau)}{f(\tau)a_{12}(\tau)}\right)_0^2\right]ds + \int\limits_{t_1}^{t}B(\tau)d\tau;$$

$$I_f \equiv \int\limits_{t}^{+\infty} a_{12}(\tau)\exp\{-Q_f(t;\tau)\}d\tau, \quad t \geq t_0.$$

Theorem 3.1. *Let* $a_{12}(t) \geq 0,\ t \geq t_0,$ *and let for some* $f \in \Omega$ *the following relations be satisfied*:

(A$_1$) $\displaystyle\int_{t_0}^{+\infty} \frac{a_{12}(\tau)}{f(\tau)}\, d\tau < +\infty;$

(B$_1$) $I_f(t_0) = +\infty.$

Then the system (1.1) *is strict oscillatory.*

Proof. Suppose, that the system (1.1) is not strict oscillatory. Then by Lemma 2.3 from the nonnegativity of $a_{12}(t)$ it follows that Equation (1.2) has a t_1-regular solution for some $t_1 \geq t_0$. By Lemma 2.1 from here and from the nonnegativity of $a_{12}(t)$ it follows that Equation (1.2) has a t_1-normal solution $x_0(t)$. Then by virtue of Theorem 2.1, we have

$$\nu_{x_0}(t_1) < +\infty. \tag{3.1}$$

By Lemma 2.2, we get

$$\int_{t_1}^{t} [2a_{12}(\tau)x_0(\tau) + B(\tau)]\, d\tau \leq 2c(t_1; x_0)\int_{t_1}^{t} \frac{a_{12}(\tau)}{f(\tau)}\, d\tau$$

$$+ \int_{t_1}^{t} \frac{a_{12}(\tau)}{f(\tau)}\, d\tau \int_{t_0}^{\tau} \left[2f(s)a_{21}(s) + \frac{1}{2}\left(\frac{f(\tau)B(\tau) - f'(\tau)}{f(\tau)a_{12}(\tau)}\right)_0^2\right] ds$$

$$+ \int_{t_1}^{t} B(\tau)\, d\tau, \quad t \geq t_1.$$

From here and from the conditions (A$_1$) and (B$_1$) of the theorem it follows that $\nu_{x_0}(t_0) \geq MI_f(t_0) = +\infty,$ where

$$M \equiv \exp\left\{2c(t_1;\, x_0)\int_{t_1}^{+\infty}\frac{a_{12}(\tau)}{f(\tau)}dtau - \int_{t_0}^{t_1}B(\tau)d\tau\right.$$

$$\left. - \int_{t_0}^{t_1}\frac{a_{12}(\tau)}{f(\tau)}d\tau\int_{t_0}^{\tau}\left[2f(s)a_{21}(s)+\frac{1}{2}\left(\frac{f(\tau)B(\tau)-f'(\tau)}{f(\tau)a_{12}(\tau)}\right)_0^2\right]ds\right\} > 0,$$

which contradicts (3.1). The obtained contradiction completes the proof of the theorem.

Theorem 3.2. *Let* $a_{12}(t) \ge 0,\ t \ge t_0$ *and let for some* $f \in \Omega$ *the conditions*

(B_1) $I_f(t_0) = +\infty$;

(A_2) $$\liminf_{t\to+\infty}\left\{\int_{t_0}^{t}\left[4a_{21}(\tau)+\left(\frac{B(\tau)}{a_{12}(\tau)}\right)_0^2\right]d\tau\right.$$

$$\left. - \frac{1}{f(t)}\int_{t_0}^{t}\left[4f(\tau)a_{21}(\tau)+f(\tau)a_{12}(\tau)\left(\frac{f(\tau)B(\tau)-f'(\tau)}{f(\tau)a_{12}(\tau)}\right)_0^2\right]d\tau\right\} < +\infty$$

as well as for some $\lambda \in \mathbb{R}$ *the conditions*

(B_2) $\displaystyle\int_{A_\lambda^-} a_{12}(\tau)d\tau = +\infty$;

(C_2) $\displaystyle\int_{A_\lambda^+} a_{12}(\tau)[\lambda + Q(\tau)]d\tau = +\infty$

be satisfied. Then the system (1.1) *is strict oscillatory.*

Proof. Suppose the system (1.1) is not strict oscillatory. Then by Lemmas 2.1 and 2.3 from the nonnegativity of $a_{12}(t)$ it follows that Equation (1.2) has a t_1-extremal solution for some $t_1 \ge t_0$. Then by (2.2) Equation (2.3) has a t_1-extremal solution $y_*(t)$, and

$$y_*(t) = y_*(t_1) - \int_{t_1}^{t}a_{12}(\tau)\big(y_*(\tau)+\lambda+Q(\tau)\big)^2 d\tau,\quad t \ge t_1.$$

By Lemma 2.4 from the conditions (B_2) and (C_2) it follows that

$$\lim_{t\to+\infty} y_*(t) = -\infty. \tag{3.2}$$

From the condition (A_2) it follows that there exists an infinitely large sequence $\{\theta_n\}_{n=1}^{+\infty}$ such that

$$S \equiv \sup_{n\geq 1}\left\{\int_{t_0}^{\theta_n}\left[4a_{21}(\tau) + \left(\frac{B(\tau)}{a_{12}(\tau)}\right)_0^2\right]d\tau\right.$$

$$\left. - \frac{1}{f(\theta_n)}\int_{t_0}^{\theta_n}\left[4f(\tau)a_{21}(\tau) + f(\tau)a_{12}(\tau)\left(\frac{f(\tau)B(\tau) - f'(\tau)}{a_{12}(\tau)f(\tau)}\right)_0^2\right]d\tau\right\} < +\infty.$$

Then due to (3.2) chose $t_2 = \theta_n$ so large that

$$y_*(t_2) < -\lambda - S/4. \tag{3.3}$$

Let $x_0(t)$ be a solution of Equation (1.2) with

$$x_0(t_0) = \frac{1}{f(t_2)}\int_{t_0}^{t_2}\left[f(\tau)a_{21}(\tau) + \frac{f(\tau)a_{12}(\tau)}{4}\left(\frac{f(\tau)B(\tau) - f'(\tau)}{a_{12}(\tau)f(\tau)}\right)_0^2\right]d\tau. \tag{3.4}$$

Let $x_*(t)$ be the t_1-extremal solution of Equation (1.2). In virtue of (2.2) we have

$$x_*(t_2) = y_*(t_2) + \lambda + \int_{t_0}^{t_2}\left[\left(\frac{B^2(\tau)}{4a_{12}(\tau)}\right)_0 + a_{21}(\tau)\right]d\tau.$$

From here and from (3.3), we obtain

$$x_*(t_2) < \lambda + \int_{t_0}^{t_2}\left[\left(\frac{B^2(\tau)}{4a_{12}(\tau)}\right)_0 + a_{21}(\tau)\right]d\tau - \lambda - S/4$$

$$\leq \frac{1}{f(t_2)}\int_{t_0}^{t_2}\left[f(\tau)a_{21}(\tau) + \frac{f(\tau)a_{12}(\tau)}{4}\left(\frac{f(\tau)B(\tau) - f'(\tau)}{a_{12}(\tau)f(\tau)}\right)_0^2\right]d\tau = x_0(t_2).$$

By Lemma 2.1 from here it follows that $x_0(t)$ is t_2-normal. Then by virtue of Theorem 2.1

$$\nu_{x_0}(t_2) < +\infty. \tag{3.5}$$

By Lemma 2.2 taking into account (3.4), we have

$$x_0(t) \le \frac{1}{f(t)} \int_{t_0}^{t} \left[f(\tau) a_{21}(\tau) + \frac{f(\tau) a_{12}(\tau)}{4} \left(\frac{f(\tau) B(\tau) - f'(\tau)}{a_{12}(\tau) f(\tau)} \right)_0^2 \right] d\tau, \quad t \ge t_0.$$

From here we get

$$- 2a_{12}(t) x_0(t) - B(t) \ge - \frac{2a_{12}(t)}{f(t)} \int_{t_0}^{t} \left[f(\tau) a_{21}(\tau) \right.$$

$$\left. + \frac{f(\tau) a_{12}(\tau)}{4} \left(\frac{f(\tau) B(\tau) - f'(\tau)}{a_{12}(\tau) f(\tau)} \right)_0^2 \right] d\tau - B(t), \quad t \ge t_0.$$

Hence

$$\nu_{x_0}(t_2) = M_1 \int_{t_1}^{+\infty} a_{12}(t) \exp\left\{ - \int_{t_0}^{t} \frac{a_{12}(\tau)}{f(\tau)} d\tau \int_{t_0}^{\tau} \left[2f(s) a_{21}(s) \right. \right.$$

$$\left. \left. + \frac{1}{2} \left(\frac{f(s) B(s) - f'(s)}{f(s) a_{12}(s)} \right)_0^2 \right] ds - \int_{t_0}^{t} B(\tau) d\tau \right\} dt,$$

where $M_1 = \exp\left\{ \int_{t_0}^{t_2} [2a_{12}(s) x_0(s) + b(s)] ds \right\} > 0$. From here and from (B_1) it follows that $\nu_{x_0}(t_2) = +\infty$, which contradicts (3.5). The obtained contradiction completes the proof of the theorem.

Theorem 3.3. *Let* $a_{12}(t) \geq 0,\ t \geq t_0$ *and let for some* $f \in \Omega,\ \lambda \in \mathbb{R}$ *and* $\alpha \geq 1$ *the following conditions be satisfied*:

(B_1) $I_f(t_0) = +\infty;$

$$(\mathrm{A}_2)\ \liminf_{t\to+\infty}\left\{\int_{t_0}^{t}\left[4a_{21}(\tau)+\left(\frac{B(\tau)}{a_{12}(\tau)}\right)_0^2\right]d\tau\right.$$

$$\left.-\frac{1}{f(t)}\int_{t_0}^{t}\left[4f(\tau)a_{21}(\tau)+f(\tau)a_{12}(\tau)\left(\frac{f(\tau)B(\tau)-f'(\tau)}{f(\tau)a_{12}(\tau)}\right)_0^2\right]d\tau\right\} < +\infty;$$

$$(\mathrm{A}_3)\ \int_{t_0}^{+\infty} a_{12}(t)\exp\left\{-4\lambda\int_{t_0}^{t}a_{12}(\tau)d\tau-2\int_{t_0}^{t}B(\tau)d\tau-4\int_{t_0}^{t}a_{12}(\tau)d\tau\int_{t_0}^{\tau}a_{21}(s)ds\right.$$

$$\left.-\int_{t_0}^{t}a_{12}(\tau)d\tau\int_{t_0}^{\tau}\left(\frac{B^2(s)}{a_{12}(s)}\right)_0 ds\right\}dt < +\infty;$$

$$(\mathrm{B}_3)\ \liminf_{t\to+\infty}\frac{1}{t^\alpha}\int_{t_0}^{t}a_{12}(\tau)(t-\tau)^{\alpha-1}[\lambda+Q(\tau)]d\tau < +\infty;$$

$$(\mathrm{C}_3)\ \limsup_{t\to+\infty}\frac{1}{t^\alpha}\int_{t_0}^{t}a_{12}(\tau)(t-\tau)^{\alpha}d\tau > 0.$$

Then the system (1.1) *is strict oscillatory.*

Proof. Suppose the system (1.1) is not strict oscillatory. Then by Lemmas 2.1 and 2.3, Equation (1.2) has a t_1-extremal solution for some $t_1 \geq t_0$. Then by (2.2), Equation (2.3) has a t_1-extremal solution $y_*(t)$, and

$$y_*(t) = y_*(t_1) - \int_{t_1}^{t} a_{12}(\tau)(t_*(\tau)+\lambda+Q(t))^2 d\tau, \quad t \geq t_1.$$

By Lemma 2.5 from the conditions (A_3)-(C_3) it follows that $\lim\limits_{t\to+\infty} y_*(t) = -\infty$. Further as in the proof of Theorem 3.2. The theorem is proved.

Theorem 3.4. *Let the following conditions be satisfied*:

$$(C_3)\ \limsup_{t\to+\infty} \frac{1}{t^\alpha} \int_{t_0}^{t} a_{12}(\tau)(t-\tau)^\alpha d\tau > 0, \quad \alpha \ge 1;$$

$$(A_4)\ \limsup_{t\to+\infty} \frac{1}{t^\alpha} \int_{t_0}^{t} a_{12}(\tau)(t-\tau)^{\alpha-1}(\lambda - \varepsilon + Q(\tau))d\tau \ge 0,\ \varepsilon > 0,\ \lambda \in \mathbb{R};$$

$$(B_4)\ \int_{t_0}^{+\infty} a_{12}(\tau)\exp\left\{4\lambda\int_{t_0}^{\tau} B(s)ds + \int_{t_0}^{\tau} B(s)ds\int_{t_0}^{s}\left[a_{21}(\xi) + \left(\frac{B^2(\xi)}{4a_{12}(\xi)}\right)_0\right]d\xi\right\} d\tau = +\infty.$$

Then the system (1.1) *is strict oscillatory*.

Proof. Suppose the system (1.1) is not strict oscillatory. Then by Lemmas 2.1 and 2.3, it follows from the nonnegativity of $a_{12}(t)$ that Equation (1.2) has a t_1-extremal solution for some $t_1 \ge t_0$. Then by (2.2), Equation (2.3) has also a t_1-extremal solution $y_*(t)$ and

$$y_*(t) = y_*(t_1) - \int_{t_1}^{t} a_{12}(\tau)(y_*(\tau) + \lambda + Q(\tau))^2 d\tau, \quad t \ge t_1. \tag{3.6}$$

From the condition (B_4) it follows that $y_*(t) \ge 0$, $t \ge t_2$ for some $t_2 \ge t_1$. Without loss of generality, we can take that $t_2 = t_1$. Then from (3.6), it follows that $\int_{t_1}^{+\infty} a_{12}(\tau)(y_*(\tau) + \lambda + Q(\tau))^2 d\tau < +\infty$. From here we have

$$0 \le I_1 \equiv \limsup_{t \to +\infty} \frac{1}{t^{\alpha}} \int_{t_1}^{t} a_{12}(\tau)(t - \tau)^{\alpha - 1}(y_*(\tau) + \lambda + Q(\tau))^2 d\tau$$

$$\le \lim_{t \to +\infty} \frac{1}{t} \int_{t_1}^{+\infty} a_{12}\left(\frac{t - \tau}{t}\right)^{\alpha - 1}[y_*(\tau) + \lambda + Q(\tau)]^2 d\tau = 0.$$

But on the other hand from the nonnegativity of $y_*(t)$, from (2.1), (A_4) and (B_4) it follows that $I_1 > 0$. We come to the contradiction. The theorem is proved.

Theorem 3.5. *Let* $a_{12}(t) \ge 0,\ t \ge t_0$ *and let the following conditions be satisfied*:

(B_1) $I_f(t_0) = +\infty$;

$$(\mathrm{A}_5)\quad \liminf_{t \to +\infty}\left\{ \int_{t_0}^{t}\left[4a_{12}(\tau) + \left(\frac{B(\tau)}{a_{12}(\tau)}\right)_0^2\right] d\tau \right.$$

$$\left. - \frac{1}{f(t)} \int_{t_0}^{t}\left[4f(\tau)a_{21}(\tau) + f(\tau)a_{12}(\tau)\left(\frac{f(\tau)B(\tau) - f'(\tau)}{f(\tau)a_{12}(\tau)}\right)_0^2\right] d\tau \right\} = -\infty.$$

Then the system (1.1) *is strict oscillatory*.

Proof. Suppose the system (1.1) is not strict oscillatory. Then by virtue of Lemmas 2.1 and 2.3 from the nonnegativity of $a_{12}(t)$, it follows that Equation (1.2) has a t_1-extremal solution for some $t_1 \ge t_0$. Therefore by (2.2), Equation (2.3) has also a t_1-extremal solution $y_*(t)$ and

$$y_*(t) = y_*(t_1) - \int_{t_1}^{t} a_{12}(\tau)(y_*(\tau) + \lambda + Q(\tau))^2 d\tau, \quad t \ge t_1.$$

Consequently $y_*(t) \le y_*(t_1)$, $t \ge t_1$. From here and from the condition (A_5), it follows that for some $t_2 \ge t_1$ the following inequality holds:

$$y_*(t_2) < -\lambda - S_{t_2}/2, \tag{3.7}$$

where

$$S_{t_2} \equiv \int_{t_0}^{t_2}\left[4a_{12}(\tau) + \left(\frac{B(\tau)}{a_{12}(\tau)}\right)_0^2\right]d\tau$$

$$-\frac{1}{f(t_2)}\int_{t_0}^{t_2}\left[4f(\tau)a_{21}(\tau) + f(\tau)a_{12}(\tau)\left(\frac{f(\tau)B(\tau) - f'(\tau)}{f(\tau)a_{12}(\tau)}\right)_0^2\right]d\tau.$$

Let $x_0(t)$ be a solution of Equation (1.2) with

$$x_0(t_2) = \frac{1}{f(t_2)}\int_{t_0}^{t_2}\left[4f(\tau)a_{21}(\tau) + f(\tau)a_{12}(\tau)\left(\frac{f(\tau)B(\tau) - f'(\tau)}{f(\tau)a_{12}(\tau)}\right)_0^2\right]d\tau.$$

Then by (2.2) we have $x_*(t_2) = y_*(t_2) + \int_{t_0}^{t_2}\left[\left(\frac{B^2(\tau)}{4a_{12}(\tau)}\right) + a_{21}(\tau)\right]d\tau$, where $x_*(t)$ is the t_2-extremal solution of Equation (1.2). From here and from (3.5) it follows

$$x_*(t_2) < \lambda + \int_{t_0}^{t_2}\left[a_{21}(\tau) + \left(\frac{B^2(\tau)}{4a_{12}(\tau)}\right)_0\right]d\tau - \lambda - S_{t_2}/4$$

$$= \frac{1}{f(t_2)}\int_{t_0}^{t_2}\left[4f(\tau)a_{21}(\tau) + f(\tau)a_{12}(\tau)\left(\frac{f(\tau)B(\tau) - f'(\tau)}{f(\tau)a_{12}(\tau)}\right)_0^2\right]d\tau = x_0(t_2).$$

Hence by Lemma 2.1 $x_0(t)$ is t_2-normal. Further as in the proof of Theorem 3.2. The theorem is proved.

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