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ON THE RATIO OF THE ARITHMETIC AND GEOMETRIC MEANS OF THE FIRST *n* TERMS OF SOME GENERAL SEQUENCES

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Abstract

The aim of this note is to study the limit behaviour of the ratio of the arithmetic mean and the geometric mean of the first n terms of some general sequences. In fact, this note generalizes some of the previously known results and extends the results to several well-known sequences in number theory.

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1. Introduction

Suppose that a_n is a sequence of positive real numbers and let

$$A_n = A(a_1, a_2, \ldots, a_n),$$

and

$$G_n = G(a_1, a_2, \ldots, a_n)$$

denote the arithmetic and geometric means of the numbers $a_1, a_2, \ldots, a_n$, respectively.

The limiting behaviour of the ratio of A_n and G_n have attracted the attention of many mathematicians in recent years, and interesting results have been obtained for some special sequences of positive numbers. For example, consider the sequence of the first n positive integers. The well-known Stirling's approximation $n! \sim \frac{\sqrt{2\pi} n^n \sqrt{n}}{e^n}$ implies that

$$\lim_{n\to\infty} \frac{A(1, 2, \ldots, n)}{G(1, 2, \ldots, n)} = \frac{e}{2}.$$

As a generalization of this limit, Kubelka [11] proved that for any $\alpha > 0$,

$$\lim_{n\to\infty} \frac{A(1^\alpha, 2^\alpha, \ldots, n^\alpha)}{G(1^\alpha, 2^\alpha, \ldots, n^\alpha)} = \frac{e^\alpha}{\alpha + 1}.$$

Recently, similar results have been considered for the sequence p_n of prime numbers. In his paper [3], Hassani established some inequalities and asymptotic formulas for the ratio $\frac{A(p_1, p_2, \ldots, p_n)}{G(p_1, p_2, \ldots, p_n)}$ that led to

$$\lim_{n\to\infty} \frac{A(p_1, p_2, \ldots, p_n)}{G(p_1, p_2, \ldots, p_n)} = \frac{e}{2}.$$

For more inequalities and asymptotic formulas for the ratio $\frac{A(p_1, p_2, \ldots, p_n)}{G(p_1, p_2, \ldots, p_n)}$, see also [1].

Our interest in this paper is to show that the ratio $\frac{e^{\alpha}}{\alpha + 1}$ (with $\alpha > 0$) appears surprisingly in studying the limit behaviour of the ratio of the arithmetic and geometric means of the first n terms of some general sequences. Furthermore, we show that the general results of this paper apply to some well-known sequences in number theory, for example to the sequence of k-free numbers $(k \geq 2)$, the sequence of k-full numbers $(k \geq 2)$, the sequence of prime numbers, the sequence of numbers with $k \geq 2$ prime factors in their prime factorization and the sequence of perfect powers.

2. Main Results

Theorem 2.1. *Let* A_n *be a strictly increasing sequence of positive integers such that*

$$A_n \sim cn^s, \tag{1}$$

where $c > 0$ *and* $s \geq 1$ *are fixed real numbers. The following limits hold for all* $\alpha > 0$

$$\lim_{n \to \infty} \frac{\frac{\sum_{i=1}^{n} A_i^{\alpha}}{n}}{\left(\prod_{i=1}^{n} A_i^{\alpha}\right)^{\frac{1}{n}}} = \frac{e^{\alpha s}}{\alpha s + 1}, \tag{2}$$

$$\lim_{x \to \infty} \frac{\frac{\sum_{A_i \leq x} A_i^{\alpha}}{\psi(x)}}{\left(\prod_{A_i \leq x} A_i^{\alpha}\right)^{\frac{1}{\psi(x)}}} = \frac{e^{\alpha s}}{\alpha s + 1}, \tag{3}$$

where $\psi(x)$ *is the counting function of the sequence. Namely,* $\psi(x) = \sum_{A_i \leq x} 1.$

Proof. We have the following general proposition [13, page 332]. Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be two series of positive terms such that $\frac{a_i}{b_i} \to 1$. If $\sum_{i=1}^{\infty} b_i$ diverges, then we have $\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \to 1$.

Therefore, we have (see (1))

$$\sum_{i=1}^{n} A_i^{\alpha} \sim c^{\alpha} \sum_{i=1}^{n} i^{\alpha s} = c^{\alpha} \int_0^n x^{\alpha s} dx + O(n^{\alpha s}) = \frac{c^{\alpha} n^{\alpha s+1}}{\alpha s+1} + O(n^{\alpha s})$$

$$= \frac{c^{\alpha} n^{\alpha s+1}}{\alpha s+1} + O(n^{\alpha s}) \sim \frac{c^{\alpha} n^{\alpha s+1}}{\alpha s+1} = \frac{(cn^{s})^{\alpha} n}{\alpha s+1} \sim \frac{A_n^{\alpha} n}{\alpha s+1}. \tag{4}$$

Equation (1) gives

$$\ln A_n = \ln c + s \ln n + o(1), \tag{5}$$

and the Stirling's formula $n! \sim \frac{\sqrt{2\pi} n^n \sqrt{n}}{e^n}$ gives

$$\sum_{i=1}^{n} \ln i = n \ln n - n + o(n). \tag{6}$$

Equations (5) and (6) give

$$\sum_{i=1}^{n} \ln A_i = n \ln c + s(n \ln n - n + o(n)) + o(n)$$

$$= n \ln c + sn \ln n - sn + o(n).$$

Hence

$$\ln \left(\prod_{i=1}^{n} A_i \right)^{\frac{1}{n}} = \ln c + s \ln n - s + o(1),$$

and consequently

$$\left(\prod_{i=1}^{n} A_i\right)^{\frac{1}{n}} \sim cn^s e^{-s} \sim \frac{A_n}{e^s},$$

and

$$\left(\prod_{i=1}^{n} A_i^{\alpha}\right)^{\frac{1}{n}} \sim c^{\alpha} n^{\alpha s} e^{-\alpha s} \sim \frac{A_n^{\alpha}}{e^{\alpha s}}. \tag{7}$$

Equations (4) and (7) give Equation (2).

Since $\psi(A_n) = n$, Equation (4) can be written in the form

$$\sum_{A_i \le A_n} A_i^{\alpha} \sim \frac{\psi(A_n) A_n^{\alpha}}{\alpha s + 1}. \tag{8}$$

If $x \in [A_n, A_{n+1})$, then $\sum_{A_i \le A_n} A_i^{\alpha} = \sum_{A_i \le x} A_i^{\alpha}$ and consequently we have (see (8))

$$1 \leftarrow \frac{\sum_{A_i \le A_n} A_i^{\alpha}}{\frac{\psi(A_n) A_{n+1}^{\alpha}}{\alpha s + 1}} \le \frac{\sum_{A_i \le x} A_i^{\alpha}}{\frac{\psi(x) x^{\alpha}}{\alpha s + 1}} \le \frac{\sum_{A_i \le A_n} A_i^{\alpha}}{\frac{\psi(A_n) A_n^{\alpha}}{\alpha s + 1}} \rightarrow 1$$

since, by Equation (1), $A_{n+1} \sim A_n$. Therefore by the compression theorem we have

$$\sum_{A_i \le x} A_i^{\alpha} \sim \frac{\psi(x) x^{\alpha}}{\alpha s + 1}. \tag{9}$$

Analogously if $x \in [A_n, A_{n+1})$ Equation (7) gives

$$\frac{1}{e^{\alpha s}} \leftarrow \frac{\left(\prod_{A_i \leq A_n} A_i^{\alpha}\right)^{\frac{1}{\psi(A_n)}}}{A_{n+1}^{\alpha}} \leq \frac{\left(\prod_{A_i \leq x} A_i^{\alpha}\right)^{\frac{1}{\psi(x)}}}{x^{\alpha}}$$

$$\leq \frac{\left(\prod_{A_i \leq A_n} A_i^{\alpha}\right)^{\frac{1}{\psi(A_n)}}}{A_n^{\alpha}} \to \frac{1}{e^{\alpha s}},$$

and therefore we have

$$\lim_{x \to \infty} \frac{\left(\prod_{A_i \leq x} A_i^{\alpha}\right)^{\frac{1}{\psi(x)}}}{x^{\alpha}} = \frac{1}{e^{\alpha s}}. \tag{10}$$

Equations (9) and (10) give Equation (3). The theorem is proved.

Theorem 2.2. *The equation*

$$A_n \sim cn^s \tag{11}$$

is equivalent to

$$\psi(x) \sim \frac{x^{\frac{1}{s}}}{c^{\frac{1}{s}}}. \tag{12}$$

Proof. Clearly (12) implies (11) if we substituting $x = A_n$ into (12). On the other hand, Equation (11) can be written in the form $A_n \sim c(\psi(A_n))^s$. That is, $\frac{\psi(A_n)}{\frac{A_n^{\frac{1}{s}}}{c^{\frac{1}{s}}}} \to 1$, and consequently if $x \in [A_n, A_{n+1})$ we have

$$1 \leftarrow \frac{\psi(A_n)}{\frac{A_{n+1}^{\frac{1}{s}}}{c^{\frac{1}{s}}}} \leq \frac{\psi(x)}{\frac{x^{\frac{1}{s}}}{c^{\frac{1}{s}}}} \leq \frac{\psi(A_n)}{\frac{A_n^{\frac{1}{s}}}{c^{\frac{1}{s}}}} \to 1,$$

since $A_n \sim A_{n+1}$. Therefore, by the compression theorem we obtain (12). That is, (11) implies (12). The theorem is proved.

Example 2.3. The sequence $A_n = n$ satisfies Theorem 2.1, since $A_n = n \sim n$. In this case we obtain the result by Kubelka (see the introduction)

$$\lim_{n\to\infty} \frac{\frac{1^\alpha + 2^\alpha + \cdots + n^\alpha}{n}}{(1^\alpha 2^\alpha \cdots n^\alpha)^{\frac{1}{n}}} = \frac{e^\alpha}{\alpha + 1}$$

since in this case $s = 1$.

In general, all sequence A_n with positive density $\rho > 0$, that is, $\psi(x) \sim \rho x$, satisfies Theorem 2.1. For example, the sequence of square-free numbers and in general the sequence of k-free numbers $(k \geq 2)$, since (as it is well-known) they have positive density $\frac{1}{\zeta(k)}$, where $\zeta(k)$ denotes the zeta function (see, for example, [8]). The sequence of square-full numbers and in general the sequence of k-full numbers $(k \geq 2)$ also satisfies Theorem 2.1, since if $\psi(x)$ is the number of k-full numbers not exceeding x then (as it is well-known) $\psi(x) \sim cx^{\frac{1}{k}}$, where the constant c depends of k (see, for example, [9]).

The sequence $A_n = P_n$ of perfect powers also satisfies Theorem 2.1, since (as it is well-known) $P_n \sim n^2$ (see, for example, [10]) etc.

The following definition was established in [4].

Definition 2.4. Let $f(x)$ be a function defined on the interval $[a, \infty)$ such that $f(x) > 0, \lim_{x\to\infty} f(x) = \infty$ and with continuous derivative $f'(x) > 0.$ The function $f(x)$ is of slow increase if and only if the following condition holds:

$$\lim_{x\to\infty} \frac{xf'(x)}{f(x)} = 0. \tag{13}$$

Typical functions of slow increase are $\ln x, \ln^2 x, \ln \ln x, \frac{\ln x}{\ln \ln x}$, etc.

The functions $f(x)$ of slow increase have the following property (see [4]): for all $\alpha > 0$ the following limit holds:

$$\lim_{x\to\infty} \frac{f(x)}{x^{\alpha}} = 0.$$

Besides, if $f(x)$ is of slow increase then $c f(x)$ and $f(x)^{\alpha}$ are also functions of slow increase, where $c > 0$ and $\alpha > 0$ are real numbers (see [4]).

Theorem 2.5. *Let A_n be a strictly increasing sequence of positive integers such that*

$$A_n \sim n^s f(n), \tag{14}$$

where $s \geq 1$ is a real number and $f(x)$ is a function of slow increase. The following limits hold for all $\alpha > 0$

$$\lim_{n\to\infty} \frac{\frac{\sum_{i=1}^{n} A_i^{\alpha}}{n}}{\left(\prod_{i=1}^{n} A_i^{\alpha}\right)^{\frac{1}{n}}} = \frac{e^{\alpha s}}{\alpha s + 1}, \tag{15}$$

$$\lim_{x\to\infty} \frac{\frac{\sum_{A_i \le x} A_i^{\alpha}}{\psi(x)}}{\left(\prod_{A_i \le x} A_i^{\alpha}\right)^{\frac{1}{\psi(x)}}} = \frac{e^{\alpha s}}{\alpha s + 1}, \tag{16}$$

where $\psi(x)$ *is the counting function of the sequence. Namely,* $\psi(x) = \sum_{A_i \le x} 1.$

Proof. In [4, Theorem 22] is proved the following formula:

$$\sum_{i=1}^{n} A_i^{\alpha} \sim \frac{nA_n^{\alpha}}{\alpha s + 1}, \tag{17}$$

and in [4, Theorem 24] is proved the following formula:

$$\lim_{n\to\infty} \frac{\left(\prod_{i=1}^{n} A_i^{\alpha}\right)^{\frac{1}{n}}}{A_n^{\alpha}} = \frac{1}{e^{\alpha s}}. \tag{18}$$

Equations (17) and (18) give Equation (15).

In [4, Theorem 22] is also proved the following formula:

$$\sum_{A_i \le x} A_i^{\alpha} \sim \frac{\psi(x) x^{\alpha}}{\alpha s + 1}, \tag{19}$$

and the formula

$$\lim_{x\to\infty} \frac{\left(\prod_{A_i \le x} A_i^{\alpha}\right)^{\frac{1}{\psi(x)}}}{x^{\alpha}} = \frac{1}{e^{\alpha s}} \tag{20}$$

can be proved from Equation (18) in the same way as in Theorem 1.1. Equations (19) and (20) give Equation (16). The theorem is proved.

Example 2.6. Let p_n be the n-th prime number. The prime number theorem is $p_n \sim n \ln n$. Therefore the sequence of primes $A_n = p_n$ satisfies Theorem 2.5. In this case $s = 1$ and $f(x) = \ln x$. Consequently, we have the limits (where $\pi(x)$ denotes the prime counting function)

$$\lim_{n\to\infty} \frac{\frac{\sum_{i=1}^{n} p_i^{\alpha}}{n}}{\left(\prod_{i=1}^{n} p_i^{\alpha}\right)^{\frac{1}{n}}} = \frac{e^{\alpha}}{\alpha+1}, \tag{21}$$

$$\lim_{x\to\infty} \frac{\frac{\sum_{p_i \le x} p_i^{\alpha}}{\pi(x)}}{\left(\prod_{p_i \le x} p_i^{\alpha}\right)^{\frac{1}{\pi(x)}}} = \frac{e^{\alpha}}{\alpha+1}, \tag{22}$$

Equation (21) with $\alpha = 1$ was proved by Hassani (see the Introduction).

The sequence A_n of numbers with k prime factors in their prime factorization, where k is an arbitrary but fixed positive integer, also satisfies Theorem 2.5 (see [4, Example 11]) etc.

In the following theorem we obtain asymptotic expansions for (21) and (22).

Theorem 2.7. *Let* $\alpha > 0$ *be and let* m *be an arbitrary but fixed positive integer.*

The following asymptotic expansion holds:

$$\frac{\frac{\sum_{p_i \le x} p_i^{\alpha}}{\pi(x)}}{\left(\prod_{p_i \le x} p_i^{\alpha}\right)^{\frac{1}{\pi(x)}}} = \frac{e^{\alpha}}{\alpha+1} + \sum_{h=1}^{m-1} \frac{b_{\alpha,h}}{\ln^{h} x} + o\left(\frac{1}{\ln^{m-1} x}\right), \tag{23}$$

where a method for to determinate the coefficients $b_{\alpha,h}$, *depending of* α, *is given below* (*in the proof*).

The following asymptotic expansion holds.

$$\frac{\frac{\sum_{i=1}^{n} p_i^{\alpha}}{n}}{\left(\prod_{i=1}^{n} p_i^{\alpha}\right)^{\frac{1}{n}}} = \frac{e^{\alpha}}{\alpha+1} + \sum_{h=1}^{m-1} \frac{f_h(\ln\ln n)}{\ln^h n} + o\left(\frac{1}{\ln^{m-1} n}\right) \tag{24}$$

where the $f_h(x)$ *are polynomials. A method for to determinate the polynomials* $f_h(x)$ *is given below* (*in the proof*).

Proof. We have the following Taylor's polynomial

$$e^x = 1 + \sum_{k=1}^{n} \frac{x^k}{k!} + o(x^n) \quad (x \to 0). \tag{25}$$

On the other hand, we have the following asymptotic expansion (see either [6] or [7]):

$$\sum_{p \le x} p^{\alpha} = \sum_{k=1}^{m} \frac{(k-1)!\, x^{\alpha+1}}{(\alpha+1)^k \ln^k x} + o\left(\frac{x^{\alpha+1}}{\ln^m x}\right) \tag{26}$$

We also have the following formula well-known [14], where a is a positive constant.

$$\vartheta(x) = \sum_{p \le x} \log p = x + O\left(\frac{x}{e^{a\sqrt{\ln x}}}\right) = x + o\left(\frac{x}{\ln^{m+1} x}\right), \tag{27}$$

and the following Panaitopol's asymptotic expansion [12]:

$$\frac{1}{\pi(x)} = \frac{\ln x}{x} - \frac{1}{x} + \sum_{k=1}^{m} \frac{a_k}{x \ln^k x} + o\left(\frac{1}{x \ln^m x}\right), \tag{28}$$

where the coefficients a_k can be obtained recursively (see [12]).

Now, we have

$$M(x) = \frac{\dfrac{\sum_{p_i \le x} p_i^{\alpha}}{\pi(x)}}{\left(\prod_{p_i \le x} p_i^{\alpha}\right)^{\frac{1}{\pi(x)}}} = \frac{1}{\pi(x)}\left(\sum_{p \le x} p^{\alpha}\right)\exp\left(\frac{1}{\pi(x)}(-\alpha)\vartheta(x)\right). \tag{29}$$

Substituting (25), (26), (27) and (28) into (29) we obtain

$$M(x) = \left(\sum_{k=1}^{m} \frac{(k-1)!\, x^{\alpha}}{(\alpha+1)^k \ln^k x} + o\left(\frac{x^{\alpha}}{\ln^m x}\right)\right)\left(\ln x - 1 + \sum_{k=1}^{m} \frac{a_k}{\ln^k x} + o\left(\frac{1}{\ln^m x}\right)\right)$$

$$\exp\left((-\alpha)\left(\ln x - 1 + \sum_{k=1}^{m} \frac{a_k}{\ln^k x} + o\left(\frac{1}{\ln^m x}\right)\right)\right)$$

$$= \left(\sum_{k=1}^{m} \frac{(k-1)!}{(\alpha+1)^k \ln^k x} + o\left(\frac{1}{\ln^m x}\right)\right)\left(\ln x - 1 + \sum_{k=1}^{m} \frac{a_k}{\ln^k x} + o\left(\frac{1}{\ln^m x}\right)\right)$$

$$\times e^{\alpha} \exp\left((-\alpha)\sum_{k=1}^{m} \frac{a_k}{\ln^k x}\right)\left(1 + o\left(\frac{1}{\ln^m x}\right)\right)$$

$$= e^{\alpha}\left(\frac{1}{\alpha+1} + \sum_{k=2}^{m} \frac{(k-1)!}{(\alpha+1)^k \ln^{k-1} x} + \left(\sum_{k=1}^{m-1} \frac{(k-1)!}{(\alpha+1)^k \ln^k x}\right)\right.$$

$$\left.\times\left(-1 + \sum_{k=1}^{m-1} \frac{a_k}{\ln^k x}\right) + o\left(\frac{1}{\ln^{m-1} x}\right)\right)\left(1 + \sum_{h=1}^{m-1} \frac{1}{h!}\left((-\alpha)\sum_{k=1}^{m-1} \frac{a_k}{\ln^k x}\right)^h + o\left(\frac{1}{\ln^{m-1} x}\right)\right)$$

$$= e^{\alpha}\left(\frac{1}{\alpha+1}+\sum_{k=2}^{m}\frac{(k-1)!}{(\alpha+1)^k\ln^{k-1}x}+\left(\sum_{k=1}^{m-1}\frac{(k-1)!}{(\alpha+1)^k\ln^k x}\right)\right.$$

$$\left.\times\left(-1+\sum_{k=1}^{m-1}\frac{a_k}{\ln^k x}\right)\right)\left(1+\sum_{h=1}^{m-1}\frac{1}{h!}\left((-\alpha)\sum_{k=1}^{m-1}\frac{a_k}{\ln^k x}\right)^h\right)+o\left(\frac{1}{\ln^{m-1}x}\right)$$

$$= \frac{e^{\alpha}}{\alpha+1}+\sum_{h=1}^{m-1}\frac{b_{\alpha,h}}{\ln^h x}+o\left(\frac{1}{\ln^{m-1}x}\right). \tag{30}$$

That is, Equation (23).

We have the following Taylor's formula:

$$\frac{1}{1+x} = 1+\sum_{k=1}^{n}(-1)^k x^k + o(x^n) \quad (x \to 0). \tag{31}$$

Cipolla [2] proved the following asymptotic expansion for $\ln p_n$, where p_n denotes the n-th prime number,

$$\ln p_n = \ln n + \ln\ln n + \sum_{i=1}^{r}\frac{g_i(\ln\ln n)}{\ln^i n}+o\left(\frac{1}{\ln^r n}\right), \tag{32}$$

where the $g_i(x)$ are polynomials of degree i and rational coefficients. Cipolla [2] gives a recursive method to obtain the polynomials $g_i(x)$.

Next, we obtain an asymptotic expansion for $\frac{1}{\ln p_n}$.

If we put

$$x = \frac{\ln\ln n}{\ln n}+\sum_{i=1}^{r}\frac{g_i(\ln\ln n)}{\ln^{i+1}n}+o\left(\frac{1}{\ln^{r+1}n}\right), \tag{33}$$

and use Equation (31) then we obtain

$$\frac{1}{\ln p_n} = \frac{1}{\ln n}\frac{1}{1+x}$$

$$= \frac{1}{\ln n}\left(1 - x + x^2 - \cdots + (-1)^{r+1}x^{r+1} + (-1)^{r+2}x^{r+2}(1+o(1))\right)$$

$$= \frac{1}{\ln n}\left(1 - x + x^2 - \cdots + (-1)^{r+1}x^{r+1} + o\left(\frac{1}{\ln^{r+1} n}\right)\right)$$

$$= \frac{1}{\ln n}\left(1 + \sum_{i=1}^{r+1}(-1)^i\left(\frac{\ln\ln n}{\ln n} + \sum_{j=1}^{r}\frac{g_j(\ln\ln n)}{\ln^{j+1} n}\right)^i\right) + o\left(\frac{1}{\ln^{r+2} n}\right)$$

$$= \frac{1}{\ln n} + \sum_{i=1}^{r+1}\frac{h_i(\ln\ln n)}{\ln^{i+1} n} + o\left(\frac{1}{\ln^{r+2} n}\right), \tag{34}$$

where the $h_i(x)$ are polynomials of rational coefficients. This is the asymptotic expansion for $\frac{1}{\ln p_n}$. That is,

$$\frac{1}{\ln p_n} = \frac{1}{\ln n} + \sum_{i=1}^{r+1}\frac{h_i(\ln\ln n)}{\ln^{i+1} n} + o\left(\frac{1}{\ln^{r+2} n}\right). \tag{35}$$

Substituting $x = p_n$ into (23) we obtain

$$\frac{\frac{\sum_{i=1}^{n} p_i^{\alpha}}{n}}{\left(\prod_{i=1}^{n} p_i^{\alpha}\right)} = \frac{e^{\alpha}}{\alpha+1} + \sum_{h=1}^{m-1}\frac{b_{\alpha,h}}{\ln^h p_n} + o\left(\frac{1}{\ln^{m-1} n}\right), \tag{36}$$

since $\ln p_n \sim \ln n$. Finally, substituting (35) (with $r = m - 3$) into (36) we find that

$$\frac{\frac{\sum_{i=1}^{n} p_i^{\alpha}}{n}}{\left(\prod_{i=1}^{n} p_i^{\alpha}\right)^{\frac{1}{n}}} = \frac{e^{\alpha}}{\alpha+1} + \sum_{h=1}^{m-1} b_{\alpha,h}\left(\frac{1}{\ln n} + \sum_{i=1}^{m-2} \frac{h_i(\ln\ln n)}{\ln^{i+1} n} + o\left(\frac{1}{\ln^{m-1} n}\right)\right)^h$$

$$+ o\left(\frac{1}{\ln^{m-1} n}\right) = \frac{e^{\alpha}}{\alpha+1} + \sum_{h=1}^{m-1} b_{\alpha,h}\left(\frac{1}{\ln n} + \sum_{i=1}^{m-2} \frac{h_i(\ln\ln n)}{\ln^{i+1} n}\right)^h$$

$$+ o\left(\frac{1}{\ln^{m-1} n}\right) = \frac{e^{\alpha}}{\alpha+1} + \sum_{h=1}^{m-1} \frac{f_h(\ln\ln n)}{\ln^h n} + o\left(\frac{1}{\ln^{m-1} n}\right), \tag{37}$$

where the $f_h(x)$ are polynomials. This is the asymptotic expansion (24) that we desired. The theorem is proved.

Example 2.8. We choose $m = 3$. Equation (30) becomes

$$\frac{\frac{\sum_{p_i \le x} p_i^{\alpha}}{\pi(x)}}{\left(\prod_{p_i \le x} p_i^{\alpha}\right)^{\frac{1}{\pi(x)}}} = e^{\alpha}\left(\frac{1}{\alpha+1} + \sum_{k=2}^{3} \frac{(k-1)!}{(\alpha+1)^k \ln^{k-1} x}\right.$$

$$+ \left(\sum_{k=1}^{2} \frac{(k-1)!}{(\alpha+1)^k \ln^k x}\right)\left(-1 + \sum_{k=1}^{2} \frac{a_k}{\ln^k x}\right)\Bigg)$$

$$\left(1 + \sum_{h=1}^{2} \frac{1}{h!}\left((-\alpha)\sum_{k=1}^{2} \frac{a_k}{\ln^k x}\right)^h\right) + o\left(\frac{1}{\ln^2 x}\right)$$

$$= \frac{e^{\alpha}}{\alpha+1} + \frac{e^{\alpha}\alpha^2}{(\alpha+1)^2}\frac{1}{\ln x} + \frac{e^{\alpha}\alpha^2(\alpha+3)^2}{2(\alpha+1)^3}\frac{1}{\ln^2 x} + o\left(\frac{1}{\ln^2 x}\right),$$

since $a_1 = -1$ and $a_2 = -3$. Now, $r = m - 3 = 0$ and Equation (34) becomes

$$\frac{1}{\ln p_n} = \frac{1}{\ln n}\left(1 + \sum_{i=1}^{1}(-1)^i\left(\frac{\ln\ln n}{\ln n}\right)^i\right) + o\left(\frac{1}{\ln^2 n}\right)$$

$$= \frac{1}{\ln n} - \frac{\ln\ln n}{\ln^2 n} + o\left(\frac{1}{\ln^2 n}\right).$$

Finally, Equation (37) becomes

$$\frac{\frac{\sum_{i=1}^{n} p_i^{\alpha}}{n}}{\left(\prod_{i=1}^{n} p_i^{\alpha}\right)^{\frac{1}{n}}} = \frac{e^{\alpha}}{\alpha+1} + \sum_{h=1}^{2} b_{\alpha,h}\left(\frac{1}{\ln n} + \sum_{i=1}^{1}\frac{h_i(\ln\ln n)}{\ln^{i+1} n}\right)^h + o\left(\frac{1}{\ln^2 n}\right)$$

$$= \frac{e^{\alpha}}{\alpha+1} + \sum_{h=1}^{2} b_{\alpha,h}\left(\frac{1}{\ln n} - \frac{\ln\ln n}{\ln^2 n}\right)^h + o\left(\frac{1}{\ln^2 n}\right)$$

$$= \frac{e^{\alpha}}{\alpha+1} + \frac{e^{\alpha}\alpha^2}{(\alpha+1)^2}\frac{1}{\ln n}$$

$$+ \frac{-\frac{e^{\alpha}\alpha^2}{(\alpha+1)^2}\ln\ln n + \frac{e^{\alpha}\alpha^2(\alpha+3)^2}{2(\alpha+1)^3}}{\ln^2 n} + o\left(\frac{1}{\ln^2 n}\right).$$

Theorem 2.9. *Let A_n be a sequence of positive real numbers (in particular integers) such that*

$$\frac{A_n}{A_{n-1}} \sim Cn^{\alpha} f(n)^{\beta}, \tag{38}$$

where $C > 0$, $\alpha > 0$ and β are fixed real numbers and $f(x)$ is a function of slow increase.

The following limit holds.

$$\lim_{n\to\infty} \frac{\dfrac{\dfrac{A_1}{A_0} + \dfrac{A_2}{A_1} + \cdots + \dfrac{A_n}{A_{n-1}}}{n}}{\left(\dfrac{A_1}{A_0}\dfrac{A_2}{A_1}\cdots\dfrac{A_n}{A_{n-1}}\right)^{\frac{1}{n}}} = \frac{e^{\alpha}}{\alpha+1}. \tag{39}$$

Proof. In [5, Theorem 5] was proved the formula

$$\frac{\left(\dfrac{A_1}{A_0}\dfrac{A_2}{A_1}\cdots\dfrac{A_n}{A_{n-1}}\right)^{\frac{1}{n}}}{\dfrac{A_n}{A_{n-1}}} \to \frac{1}{e^{\alpha}}. \tag{40}$$

On the other hand, we have (L'Hospital's rule and (13))

$$\lim_{x\to\infty} \frac{\int_a^x t^{\alpha} f(t)^{\beta} dt}{\dfrac{x^{\alpha+1}}{\alpha+1} f(x)^{\beta}} = 1.$$

Note that the function $Cx^{\alpha} f(x)^{\beta}$ is strictly increasing from a certain positive integer a and $\lim_{x\to\infty} Cx^{\alpha} f(x)^{\beta} = \infty$. Therefore we have

$$\begin{aligned}
\frac{A_1}{A_0} + \frac{A_2}{A_1} + \cdots + \frac{A_n}{A_{n-1}} &\sim \sum_{i=1}^{n} Ci^{\alpha} f(i)^{\beta} = \sum_{i=1}^{a-1} Ci^{\alpha} f(i)^{\beta} + \sum_{i=a}^{n} Ci^{\alpha} f(i)^{\beta} \\
&= \sum_{i=1}^{a-1} Ci^{\alpha} f(i)^{\beta} + \int_a^n Cx^{\alpha} f(x)^{\beta} dx + O\left(n^{\alpha} f(n)^{\beta}\right) \\
&\sim C \frac{n^{\alpha+1}}{\alpha+1} f(n)^{\beta} \sim \frac{A_n}{A_{n-1}} \frac{n}{\alpha+1}. \qquad (41)
\end{aligned}$$

Equations (40) and (41) give Equation (39). The theorem is proved.

Example 2.10. Let us consider the sequence B_n of Bell numbers. It is well-known (see, for instance, [5]) that $\frac{B_n}{B_{n-1}} \sim \frac{n}{\ln n}$. Therefore Theorem 2.9 is applicable and we have

$$\lim_{n\to\infty} \frac{\dfrac{\frac{B_1}{B_0} + \frac{B_2}{B_1} + \cdots + \frac{B_n}{B_{n-1}}}{n}}{\left(\frac{B_1}{B_0}\frac{B_2}{B_1}\cdots\frac{B_n}{B_{n-1}}\right)^{\frac{1}{n}}} = \frac{e}{2}.$$

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