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ASYMPTOTIC EXPANSIONS

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Abstract

In this note we prove asymptotic expansions for the sums $\sum_{p>x} \frac{1}{p^k}$, where $k > 1$, $\sum_{k=2}^{n} \pi(k)$, $k\,\pi(x) - \pi(k\,x)\,(k = 2, 3, 4, \ldots)$ and $\pi\left(\frac{x}{\log^m x}\right)$.

1. Introduction

In Lemma 1.2 of [1], Alladi and Erdős proved the formula (p denotes a positive prime)

$$\sum_{p>x} \frac{1}{p^k} = \frac{1}{(k-1)x^{k-1}\log x} + O\left(\frac{1}{x^{k-1}\log^2 x}\right)$$

where $k > 1$. In this note we obtain a more precise result for the sum $\sum_{p>x} \frac{1}{p^k}$. That is, we prove the following theorem.

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Theorem 1.1. *Let* $k > 1$ *be and* h *be an arbitrary but fixed positive integer. The following formula holds*:

$$\sum_{p>x} \frac{1}{p^k} = \left(\sum_{i=1}^{h} \frac{(-1)^{i+1}(i-1)!}{(k-1)^i} \frac{1}{x^{k-1} \log^i x} \right) + o\left(\frac{1}{x^{k-1} \log^h x} \right). \qquad (1)$$

Proof. We have (L'Hospital's rule)

$$\lim_{x \to \infty} \frac{\displaystyle\int_x^{\infty} \frac{1}{t^k \log^i t} dt}{\displaystyle\frac{1}{(k-1)x^{k-1} \log^i x}} = 1,$$

where i is a positive integer.

Integration by parts give us

$$\int_x^{\infty} \frac{1}{t^k \log^i t} dt = \frac{1}{k-1} \frac{1}{x^{k-1} \log^i x} - \frac{i}{k-1} \int_x^{\infty} \frac{1}{t^k \log^{i+1} t} dt. \qquad (2)$$

Let $\pi(x)$ be (as usual) the prime counting function. The following formula is well-known (see [3], Chapter 7), where h is an arbitrary but fixed positive integer.

$$\pi(x) = \left(\sum_{i=1}^{h} \frac{(i-1)!\, x}{\log^i x} \right) + o\left(\frac{x}{\log^h x} \right).$$

Abel summation (see [2], Chapter XXII) give us

$$\sum_{p \le x} \frac{1}{p^k} = \frac{\pi(x)}{x^k} + k \int_2^x \frac{\pi(t)}{t^{k+1}} dt = \frac{\pi(x)}{x^k} + k \int_2^{\infty} \frac{\pi(t)}{t^{k+1}} dt - k \int_x^{\infty} \frac{\pi(t)}{t^{k+1}} dt.$$

Therefore

$$\sum_{p>x} \frac{1}{p^k} = k \int_x^{\infty} \frac{\pi(t)}{t^{k+1}} dt - \frac{\pi(x)}{x^k} = k \sum_{i=1}^{h} \int_x^{\infty} \frac{(i-1)!}{t^k \log^i t} dt$$

$$- \sum_{i=1}^{h} \frac{(i-1)!}{x^{k-1} \log^i x} + o\left(\frac{1}{x^{k-1} \log^h x} \right). \qquad (3)$$

If we eliminate the integrals in Equation (3) (see below) we obtain the formula

$$\sum_{p>x} \frac{1}{p^k} = \left(\sum_{i=1}^{h} \frac{a_i}{x^{k-1} \log^i x} \right) + o\left(\frac{1}{x^{k-1} \log^h x} \right). \tag{4}$$

Now, we shall determinate the coefficients a_i. If $i = 1$, then we have (see (3) and (2))

$$k \int_x^{\infty} \frac{1}{t^k \log t}\, dt = \frac{1}{x^{k-1} \log x}$$

$$= k\left(\frac{1}{k-1} \frac{1}{x^{k-1} \log x} - \frac{1}{k-1} \int_x^{\infty} \frac{1}{t^k \log^2 t}\, dt \right) - \frac{1}{x^{k-1} \log x}$$

$$= \frac{1}{k-1} \frac{1}{x^{k-1} \log x} + \left(-\frac{1}{k-1} - 1 \right) \int_x^{\infty} \frac{1}{t^k \log^2 t}\, dt. \tag{5}$$

Therefore (see (5)) $a_1 = \frac{1}{k-1}$ and we put $A_2 = -\frac{1}{k-1} - 1$.

If $i \geq 2$, then (see (3) and (2))

$$A_i \int_x^{\infty} \frac{1}{t^k \log^i t}\, dt + k((i-1)!) \int_x^{\infty} \frac{1}{t^k \log^i t}\, dt - \frac{(i-1)!}{x^{k-1} \log^i x}$$

$$= (A_i + k((i-1)!)) \left(\frac{1}{k-1} \frac{1}{x^{k-1} \log^i x} - \frac{i}{k-1} \int_x^{\infty} \frac{1}{t^k \log^{i+1} t}\, dt \right)$$

$$- \frac{(i-1)!}{x^{k-1} \log^i x} = \frac{1}{k-1} (A_i + (i-1)!) \frac{1}{t^{k-1} \log^i x}$$

$$+ (-ia_i - i!) \int_x^{\infty} \frac{1}{t^k \log^{i+1} t}\, dt. \tag{6}$$

Therefore (see (6))

$$a_i = \frac{1}{k-1} (A_i + (i-1)!), \tag{7}$$

and we put (see (6))

$$A_{i+1} = -ia_i - i!. \tag{8}$$

We shall prove that for $i \geq 1$

$$a_i = \frac{(-1)^{i+1}(i-1)!}{(k-1)^i}. \tag{9}$$

Clearly (9) is true for $i = 1$ and $i = 2$ (see (5) and (6)).

Suppose that Equation (9) is true. Equations (7), (8) and (9) give

$$a_{i+1} = \frac{1}{k-1}(A_{i+1} + i!) = \frac{-i}{k-1}a_i = (-1)^{i+2}\frac{i!}{(k-1)^{i+1}}.$$

Therefore Equation (9) is proved. Equations (4) and (9) give (1). The theorem is proved.

Theorem 1.2. *The following asymptotic formula holds*:

$$\sum_{j=2}^{n} \pi(j) = \sum_{j=1}^{m} \frac{c_j}{2^j}\frac{n^2}{\log^j n} + o\left(\frac{n^2}{\log^m n}\right), \tag{10}$$

where the positive integers c_j *satisfy the recurrence relation*

$$c_1 = 1, \tag{11}$$

$$c_{j+1} = jc_j + 2^j j! \quad (j \geq 1). \tag{12}$$

Therefore $c_2 = 3,\ c_3 = 14,\ c_4 = 90,$ *etc.*

Proof. We have the formula (see the proof of Theorem 1.1)

$$\pi(i) = \left(\sum_{j=1}^{m} \frac{(j-1)!\, i}{\log^j i}\right) + f(i)\left(\frac{i}{\log^m i}\right), \tag{13}$$

where $f(i) \to 0$. Hence

$$\sum_{i=k}^{n} \pi(i) = \left(\sum_{j=1}^{m} \sum_{i=k}^{n} \frac{(j-1)!\, i}{\log^{j} i} \right) + \sum_{i=k}^{n} f(i) \left(\frac{i}{\log^{m} i} \right). \tag{14}$$

Note that (L'Hospital's rule)

$$\lim_{x \to \infty} \frac{\displaystyle\int_{a}^{x} \frac{t}{\log^{h} t}\, dt}{\dfrac{x^2}{2 \log^{h} x}} = 1. \tag{15}$$

Therefore

$$\sum_{i=k}^{n} f(i) \left(\frac{i}{\log^{m} i} \right) = o\left(\frac{n^2}{\log^{m} n} \right). \tag{16}$$

The function $\dfrac{x}{\log^{j} x}$ is positive and strictly increasing. Consequently,

$$\sum_{i=k}^{n} \left(\frac{i}{\log^{j} i} \right) = \int_{k}^{n} \frac{x}{\log^{j} x}\, dx + O\left(\frac{n}{\log^{j} n} \right) = \int_{k}^{n} \frac{x}{\log^{j} x}\, dx + o\left(\frac{n^2}{\log^{m} n} \right). \tag{17}$$

Substituting (16) and (17) into (14) we obtain

$$\sum_{i=k}^{n} \pi(i) = \left(\sum_{j=1}^{m} \int_{k}^{n} \frac{(j-1)!\, x}{\log^{j} x}\, dx \right) + o\left(\frac{n^2}{\log^{m} n} \right). \tag{18}$$

Now, we have (use integration by parts in the first integral)

$$K_j \int_{k}^{n} \frac{x}{\log^{j} x}\, dx + \int_{k}^{n} \frac{j!\, x}{\log^{j+1} x}\, dx = K_j \frac{n^2}{2 \log^{j} n} + \left(\frac{jK_j}{2} + j! \right) \int_{k}^{n} \frac{x}{\log^{j+1} x}\, dx + O(1).$$

In this form we eliminate successively all integrals into (18) and obtain (10). Where

$$K_1 = 1,$$

$$K_{j+1} = K_j \frac{j}{2} + j! \quad (j \geq 1).$$

If we put $K_j = \frac{c_j}{2^{j-1}}$ then we obtain (11) and (12). The theorem is proved.

We have the following general theorem.

Theorem 1.3. *Let* α *be a positive real number and* h *be a positive integer. The following asymptotic formula holds*:

$$\pi(\alpha x) = \sum_{s=1}^{h} \frac{\alpha P_s(\log \alpha) x}{\log^s x} + o\left(\frac{x}{\log^h x}\right) \tag{19}$$

where $P_s(y)\,(s = 1, 2, 3, \ldots)$ *is the polynomial*

$$P_s(y) = \sum_{i=1}^{s} (i-1)! \binom{-i}{s-i} y^{s-i}. \tag{20}$$

Therefore $P_s(y)$ *has degree* $s-1$, *integer coefficients alternating in sign and leading coefficient* $(-1)^{s+1}$. *We have*

$$P_1(y) = 1, \quad P_2(y) = -y + 1, \quad P_3(y) = y^2 - 2y + 2, \tag{21}$$

$$P_4(y) = -y^3 + 3y^2 - 6y + 6, \ldots. \tag{22}$$

Proof. The Taylor's formula for the binomial formula is

$$\frac{1}{(1+z)^m} = 1 + \binom{-m}{1} z + \binom{-m}{2} z^2 + \cdots + \binom{-m}{h-m} z^{h-m} + o(z^{h-m}), \tag{23}$$

where $m = 1, 2, \ldots, h$ and $\binom{-m}{n}$ denotes the generalized binomial number

$$\binom{-m}{n} = \frac{-m(-m-1)\cdots(-m-(n-1))}{n!}$$

$$= (-1)^n \frac{m(m+1)\cdots(m+(n-1))}{n!}. \tag{24}$$

We have the formula (see the proof of Theorem 1.1)

$$\pi(\alpha x) = \sum_{m=1}^{h} \frac{\alpha(m-1)!\, x}{\log^m(\alpha x)} + o\left(\frac{x}{\log^h x}\right)$$

$$= \sum_{m=1}^{h} \frac{\alpha(m-1)!\, x}{\log^m(x)} \frac{1}{\left(1 + \frac{\log \alpha}{\log x}\right)^m} + o\left(\frac{x}{\log^h x}\right). \tag{25}$$

Substituting $z = \frac{\log \alpha}{\log x}$ into (23) and then substituting (23) into (25) we obtain (19) and (20). The theorem is proved.

Theorem 1.4. *Let* $k \geq 2$ *be a fixed positive integer. The following asymptotic formula holds*:

$$k\,\pi(x) - \pi(k\,x) = \sum_{s=2}^{h} \frac{(k(s-1)! - kP_s(\log k))x}{\log^s x} + o\left(\frac{x}{\log^h x}\right), \tag{26}$$

where $h = 2, 3, 4, \ldots$. *Therefore*

$$(k\,\pi(x) - \pi(k\,x)) \sim (k \log k)\frac{x}{\log^2 x}. \tag{27}$$

In particular, if $k = 2$,

$$(2\pi(x) - \pi(2x)) \sim (2 \log 2) \frac{x}{\log^2 x}. \tag{28}$$

Proof. It is an immediate consequence of Theorem 1.3. The theorem is proved.

We also can obtain (as a direct consequence of Theorem 1.3) an asymptotic expansion for $\pi(x + \epsilon x) - \pi(x)$, where $\epsilon > 0$ is small. That is, for the number of primes in the small interval $[x, x + \epsilon x]$. If we take the first terms of this expansion we obtain the formula

$$\pi(x + \epsilon x) - \pi(x) = \epsilon \frac{x}{\log x} + (\epsilon - (1 + \epsilon) \log(1 + \epsilon)) \frac{x}{\log^2 x} + o\left(\frac{x}{\log^2 x}\right), \tag{29}$$

If $\epsilon \to 0$ we have $(\epsilon - (1 + \epsilon) \log(1 + \epsilon)) \sim -\frac{1}{2} \epsilon^2$. Therefore if ϵ is small the constant $(\epsilon - (1 + \epsilon) \log(1 + \epsilon))$ is negative. We have used $\log(1 + x) = x - \frac{1}{2} x^2 + o(x^2)$.

Theorem 1.5. *Let* m *be and let* h *be arbitrary but fixed positive integers. The following asymptotic formula holds*:

$$\pi\left(\frac{x}{\log^m x}\right) = \sum_{k=1}^{h} \frac{x P_{m+k}(\log \log x)}{\log^{m+k} x} + o\left(\frac{x}{\log^{m+h} x}\right), \tag{30}$$

where $P_{m+s}(y)\,(s = 1, 2, 3, \ldots)$ *is the polynomial*

$$P_{m+s}(y) = \sum_{i=1}^{s} (i-1)! \binom{-i}{s-i} (-1)^{s-i} m^{s-i} y^{s-i}. \tag{31}$$

Therefore $P_{m+s}(y)$ *has degree* $s-1$, *positive integer coefficients and leading coefficient* m^{s-1}. *We have*

$$P_{m+1}(y) = 1, \quad P_{m+2}(y) = my + 1, \quad P_{m+3}(y) = m^2y^2 + 2my + 2. \tag{32}$$

Proof. The proof is the same as the proof of Theorem 1.3. The theorem is proved.

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