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GENERATION OF INFINITE SEQUENCES OF PAIRWISE RELATIVELY PRIME INTEGERS

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Abstract

In this short, note we give a method to obtain infinite sequences of pairwise relatively prime integers. This subject is of interest to many mathematicians. Finally, we apply our results to certain Diophantine equations.

1. Introduction

We have the following well-known theorem.

Theorem 1.1. *There exist infinitely many prime numbers.*

There are several proof of this theorem (see [2], Chapter 1).

The following idea is of interest to many mathematicians.

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If we generate an infinite sequence of pairwise relatively prime integers greater than 1 without using for this purpose the fact that there exist infinitely many primes then we clearly obtain a proof of Theorem 1.1.

Apparently this idea is of Hurwitz (1891) (see [2], Chapter 1).

However in a letter to Euler (July, 1730), Goldbach proved that the Fermat numbers $F_n = 2^{2^n} + 1 (n \geq 0)$ are pairwise relatively prime and consequently he proved Theorem 1.1 (see [2], Chapter 1).

The generation of infinite sequences of pairwise relatively prime integers is of great interest to mathematicians.

In 1947, Bellman proposed a method to generate infinite sequences of pairwise relatively prime integers (see [2], Chapter 1).

Many particular cases of Bellman's method were discovered independently by mathematicians, for example, Edwards (1964) and Mohanti (1978). Some properties of particular Bellman's sequences were studied by Guy and Nowakowsky (1975) and Odoni (1985) (see [2], Chapter 1).

In this note, we obtain infinite sequences of pairwise relatively prime integers in function of two arbitrary positive integers A and B relatively prime. We also apply our results to certain Diophantine equations (in the last section).

2. Main Results

Theorem 2.1. *Let s_1 and s_2 be positive integers not necessarily distinct. Let A and B be relatively prime positive integers such that $A^{s_1} > B^{s_2}$. Suppose that either A is even and B is odd or A is odd and B is even. Let us consider the strictly increasing sequence of odd positive integers*

$$a_0 = A^{s_1} - B^{s_2}, \tag{1}$$

$$a_k = A^{2^{k-1}s_1} + B^{2^{k-1}s_2} \quad (k \geq 1). \tag{2}$$

That is, the sequence

$$A^{s_1} - B^{s_2},\ A^{s_1} + B^{s_2},\ A^{2s_1} + B^{2s_2},\ A^{4s_1} + B^{4s_2},\ A^{8s_1} + B^{8s_2},\ \ldots. \tag{3}$$

Then $gcd(a_i, a_j) = 1$ *if* $(i \neq j)$. *That is, the* a_k $(k \geq 0)$ *are pairwise relatively prime.*

Proof. It is sufficient to prove the theorem for the sequence with $s_1 = s_2 = 1$. That is, the sequence

$$a_0 = A - B, \tag{4}$$

$$a_k = A^{2^{k-1}} + B^{2^{k-1}} \quad (k \geq 1). \tag{5}$$

That is, the sequence

$$A - B,\ A + B,\ A^2 + B^2,\ A^4 + B^4,\ A^8 + B^8,\ \ldots. \tag{6}$$

Since substituting A by A^{s_1} and B by B^{s_2} into (4) and (5) we obtain the sequence (1) and (2).

We proceed by mathematical induction. $A - B$ and $A + B$ are relatively prime since

$$(A + B) + (A - B) = 2A,$$

$$(A + B) - (A - B) = 2B.$$

If $A + B$ and $A - B$ are not relatively prime (since they are odd), then A and B also are not relatively prime, that is absurd by hypothesis. Therefore $A - B$ and $A + B$ are relatively prime.

Suppose that

$$A-B,\ A+B,\ A^2+B^2,\ A^4+B^4,\dots,\ A^{2^{k-1}}+B^{2^{k-1}} \qquad (k \geq 1), \tag{7}$$

are pairwise relatively prime.

Note that (use mathematical induction and the formula $a^2-b^2=(a-b)(a+b)$) if $k \geq 1$

$$A^{2^k}-B^{2^k}=(A-B)(A+B)(A^2+B^2)(A^4+B^4)\cdots(A^{2^{k-1}}+B^{2^{k-1}}). \tag{8}$$

Now, note also that the numbers $A^{2^k}-B^{2^k}$ and $A^{2^k}+B^{2^k}$ are relatively prime since

$$(A^{2^k}+B^{2^k})+(A^{2^k}-B^{2^k})=2A^{2^k},$$

$$(A^{2^k}+B^{2^k})-(A^{2^k}-B^{2^k})=2B^{2^k}.$$

If $A^{2^k}+B^{2^k}$ and $A^{2^k}-B^{2^k}$ are not relatively prime (since they are odd), then A and B also are not relatively prime, that is absurd by hypothesis. Therefore $A^{2^k}-B^{2^k}$ and $A^{2^k}+B^{2^k}$ are relatively prime. This fact and Equations (7) and (8) imply that

$$A-B,\ A+B,\ A^2+B^2,\ A^4+B^4,\dots,\ A^{2^{k-1}}+B^{2^{k-1}},\ A^{2^k}+B^{2^k}$$

are pairwise relatively prime and the theorem is proved.

Corollary 2.2. *The number*

$$A^{2^k s_1}-B^{2^k s_2}$$

have at least k *distinct prime factors.*

Proof. It is an immediate consequence of Equations (7) and (8) if we substitute in these equations A by A^{s_1} and B by B^{s_2}. The corollary is proved.

Corollary 2.3 (Goldbach)**.** *The Fermat numbers are pairwise relatively prime.*

Proof. It is an immediate consequence of Theorem 2.1 if we put $s_1 = s_2 = 1$, $A = 2$ and $B = 1$. The corollary is proved.

Corollary 2.4. *If* $A^{2^k s_1} - B^{2^k s_2}$ *is a r-th perfect power* (*that is,* $A^{2^k s_1} - B^{2^k s_2} = C^r$), *then the* $k+1$ *numbers*

$$(A^{s_1} - B^{s_2}), (A^{s_1} - B^{s_2}), (A^{2s_1} + B^{2s_2}), (A^{4s_1} + B^{4s_2}), \ldots, (A^{2^{k-1} s_1} + B^{2^{k-1} s_2}) \tag{9}$$

are also all simultaneously r-th perfect powers.

Proof. It is an immediate consequence of Equations (7) and (8). The corollary is proved.

Theorem 2.5. *Let* s_1 *and* s_2 *be positive integers not necessarily distinct. Let* A *and* B *be relatively prime positive integers such that* $A^{s_1} > B^{s_2}$. *Suppose that* A *and* B *are odd. Let us consider the strictly increasing sequence of positive integers*

$$b_0 = \frac{A^{s_1} - B^{s_2}}{2}, \tag{10}$$

$$b_k = \frac{A^{2^{k-1} s_1} + B^{2^{k-1} s_2}}{2} \quad (k \geq 1). \tag{11}$$

That is, the sequence

$$\frac{A^{s_1} - B^{s_2}}{2}, \frac{A^{s_1} + B^{s_2}}{2}, \frac{A^{2s_1} + B^{2s_2}}{2}, \frac{A^{4s_1} + B^{4s_2}}{2}, \frac{A^{8s_1} + B^{8s_2}}{2}, \ldots. \tag{12}$$

Then $gcd(b_i, b_j) = 1$ *if* $(i \neq j)$. *That is, the* b_k $(k \geq 0)$ *are pairwise relatively prime.*

Proof. As in Theorem 2.1 it is sufficient to prove the theorem for the sequence with $s_1 = s_2 = 1$. That is, the sequence

$$a_0 = \frac{A - B}{2}, \tag{13}$$

$$a_k = \frac{A^{2^{k-1}} + B^{2^{k-1}}}{2} \quad (k \geq 1). \tag{14}$$

That is, the sequence

$$\frac{A - B}{2}, \frac{A + B}{2}, \frac{A^2 + B^2}{2}, \frac{A^4 + B^4}{2}, \frac{A^8 + B^8}{2}, \ldots. \tag{15}$$

We proceed by mathematical induction. $\frac{A-B}{2}$ and $\frac{A+B}{2}$ are relatively prime since

$$\left(\frac{A + B}{2}\right) + \left(\frac{A - B}{2}\right) = A,$$

$$\left(\frac{A + B}{2}\right) - \left(\frac{A - B}{2}\right) = B.$$

If $\frac{A+B}{2}$ and $\frac{A-B}{2}$ are not relatively prime, then A and B also are not relatively prime, that is absurd by hypothesis. Therefore $\frac{A-B}{2}$ and $\frac{A+B}{2}$ are relatively prime.

Suppose that

$$\frac{A-B}{2},\ \frac{A+B}{2},\ \frac{A^2+B^2}{2},\ \frac{A^4+B^4}{2},\ \ldots,\ \frac{A^{2^{k-1}}+B^{2^{k-1}}}{2} \quad (k \geq 1) \qquad (16)$$

are pairwise relatively prime.

Equation (8) becomes in this case $(k \geq 1)$

$$\frac{A^{2^k}-B^{2^k}}{2^{k+1}} = \left(\frac{A-B}{2}\right)\left(\frac{A+B}{2}\right)\left(\frac{A^2+B^2}{2}\right)\cdots\left(\frac{A^{2^{k-1}}+B^{2^{k-1}}}{2}\right). \qquad (17)$$

Note that the sum of the squares of two odd numbers has only one 2 in its prime factorization, that is,

$$(2n_1+1)^2+(2n_2+1)^2 = 2(2n_3+1) \qquad (18)$$

as it can be proved easily.

Now, note also that the numbers $C_k = \frac{A^{2^k}-B^{2^k}}{2^{k+1}}$ and $\frac{A^{2^k}+B^{2^k}}{2}$ are relatively prime since

$$\frac{A^{2^k}+B^{2^k}}{2} - 2^k C_k = \frac{A^{2^k}+B^{2^k}}{2} + \frac{A^{2^k}-B^{2^k}}{2} = A^{2^k}$$

$$\frac{A^{2^k}+B^{2^k}}{2} - 2^k C_k = \frac{A^{2^k}+B^{2^k}}{2} - \frac{A^{2^k}-B^{2^k}}{2} = B^{2^k}$$

and by Equation (18) the number $\frac{A^{2^k}+B^{2^k}}{2}$ is odd.

This fact and Equations (16) and (17) imply that

$$\left(\frac{A-B}{2}\right)\left(\frac{A+B}{2}\right)\left(\frac{A^2+B^2}{2}\right)\left(\frac{A^4+B^4}{2}\right)\cdots\left(\frac{A^{2^{k-1}}+B^{2^{k-1}}}{2}\right), \left(\frac{A^{2^k}+B^{2^k}}{2}\right)$$

are pairwise relatively prime and the theorem is proved.

Corollary 2.6. *The number*

$$A^{2^k s_1} - B^{2^k s_2}$$

have at least k distinct prime factors.

Corollary 2.7. *If* $A^{2^k s_1} - B^{2^k s_2}$ *is a r-th perfect power* (*that is*, $A^{2^k s_1} - B^{2^k s_2} = C^r$), *then the odd* $k-1$ *numbers*

$$\frac{A^{2s_1} + B^{2s_2}}{2}, \frac{A^{4s_1} + B^{4s_2}}{2}, \ldots, \frac{A^{2^{k-1}s_1} + B^{2^{k-1}s_2}}{2} \tag{19}$$

are also all simultaneously r-th perfect powers.

3. An Application to the Diophantine Equations $x^4 - y^4 = z^p$ and $x^{(2^k)} - y^{(2^k)} = 2^{k+1}z^p$

We are interested in the Diophantine equation $x^4 - y^4 = z^s$ where $x > y > 0$, $z > 0$ and $s \geq 3$ denotes a positive odd integer. It is well-known that if s is an even positive integer this equation has no solutions (x, y, z) such that $xyz \neq 0$ (Fermat's theorem) (see [1], Volume 1, Chapter 6, page 391). We shall prove that if s is odd then the equation has infinitely many solutions (x, y, z) where $x > y > 0$ and $z > 0$. The proof is simple, we have that the linear Diophantine equation $sn_2 - 4n_1 = 1$ has infinitely many solutions (n_1, n_2), where n_1 and n_2 are positive integers. If $s = 4k+1$ then we can take $(n_1, n_2) = (k, 1)$ and if $s = 4k-1$ then we can take $(n_1, n_2) = (3k\text{-}1, 3)$. Therefore, we have the identities

$$\left(a\left(a^4 - b^4\right)^k\right)^4 - \left(b\left(a^4 - b^4\right)^k\right)^4 = \left(\left(a^4 - b^4\right)^1\right)^{4k+1},$$

$$\left(a\left(a^4 - b^4\right)^{3k-1}\right)^4 - \left(b\left(a^4 - b^4\right)^{3k-1}\right)^4 = \left(\left(a^4 - b^4\right)^3\right)^{4k-1}.$$

These identities give us infinitely many solutions (x, y, z), where $x > y > 0$ and $z > 0$, of the equation $x^4 - y^4 = z^s (s \geq 3 \text{ odd})$. If $s = 3$, then we have the following theorem:

Theorem 3.1. *Let us consider the Diophantine equation* $x^4 - y^4 = z^3$, *where* $x > y > 0$ *and* $z > 0$. *This equation has not solutions* (x, y, z), *where* x *and* y *are relatively prime. Therefore for each solution* (x, y, z), *we have* $\gcd(x, y, z) > 1$.

Proof. Suppose that there exists a solution (x, y, z) such that x and y are relatively prime.

(1) Suppose that either x is even and y is odd or x is odd and y is even. Then we have the factorization

$$x^4 - y^4 = (x^2 + y^2)(x + y)(x - y) = z^3, \tag{20}$$

where the three factors are odd and pairwise relatively prime. Therefore (20) gives the system of three equations

$$x - y = u^3, \tag{21}$$

$$x + y = v^3, \tag{22}$$

$$x^2 + y^2 = w^3, \tag{23}$$

where $u, v,$ and w are positive integers pairwise relatively prime.

Equations (21) and (22) give

$$x = \frac{v^3 + u^3}{2}, \quad y = \frac{v^3 - u^3}{2}. \tag{24}$$

Substituting Equations (24) into (23) we find that

$$x^2 + y^2 = \frac{u^6 + v^6}{2} = w^3.$$

That is,

$$u^6 + v^6 = 2w^3, \tag{25}$$

where $u \neq v$. Now, it is well-known (see [1], Chapter 6, page 373) that all solutions of the equation $a^3 + b^3 = 2c^3$ where $a > 0, b > 0$ and $c > 0$ satisfy $a = b$. Therefore Equation (25) is impossible.

(2) Suppose that x and y are odd. Note that $\frac{x+y}{2}$ and $\frac{x-y}{2}$ are one even and the other odd. Since

$$\frac{x+y}{2} + \frac{x-y}{2} = x, \quad \frac{x+y}{2} - \frac{x-y}{2} = y.$$

Suppose that $\frac{x-y}{2}$ is even.

Then we have the factorization

$$x^4 - y^4 = \left(\frac{x^2+y^2}{2}\right)\left(\frac{x+y}{2}\right)\left(2^3\,\frac{x-y}{2}\right) = z^3, \tag{26}$$

where the three factors are pairwise relatively prime. Therefore (26) give us the system of three equations

$$2^3\,\frac{x-y}{2} = u^3, \tag{27}$$

$$\frac{x+y}{2} = v^3, \tag{28}$$

$$\frac{x^2+y^2}{2} = w^3, \tag{29}$$

where $u, v,$ and w are positive integers pairwise relatively prime.

Equations (27) and (28) give

$$x = v^3 + \frac{u^3}{8}, \quad y = v^3 - \frac{u^3}{8}. \tag{30}$$

Substituting Equations (30) into (29) we find that

$$\frac{1}{2}(x^2 + y^2) = \frac{u^6 + 2^6 v^6}{2^6} = w^3.$$

That is,

$$u^6 + 2^6 v^6 = 2^6 w^3. \tag{31}$$

Now, it is well-known (see [1], Chapter 6, page 373) that the equation $a^3 + b^3 = c^3$ has no solutions (a, b, c) where $a > 0$, $b > 0$ and $c > 0$. Therefore Equation (31) is impossible.

(3) Suppose that $\frac{x+y}{2}$ is even. Then the same proof as in (2) gives the same equation as in (2), namely

$$2^6 u^6 + v^6 = 2^6 w^3.$$

This equation is impossible. The theorem is proved.

Remark 3.2. Let $p > 3$ be a prime. If we consider solutions (x, y, z) such that $x > y > 0$ and $z > 0$, then as in case (1) of Theorem 3.1, we obtain the Diophantine equation

$$u^{2p} + v^{2p} = 2w^p.$$

On the other hand in case (2) of Theorem 3.1, we obtain the Diophantine equation

$$u^{2p} + 2^6 v^{2p} = 2^6 w^p,$$

and in case (3) of Theorem 3.1, we obtain the Diophantine equation

$$2^6 u^{2p} + v^{2p} = 2^6 w^p.$$

If the behaviour of these equations is as the case $p = 3$, then Theorem 3.1 holds for all odd prime p and consequently for all positive odd integer $s \geq 3$. The author does not know the behaviour of these equations.

Now, we are interested in the Diophantine equation $x^{(2^k)} - y^{(2^k)} = 2^{k+1}z^s$, where $k \geq 2$ is an arbitrary but fixed positive integer, $s \geq 3$ is an arbitrary but fixed odd positive integer and where $x > y > 0$ and $z > 0$. For example, if $k = 2$ then we obtain the equation $x^4 - y^4 = 8z^s$. We shall prove that the equation has infinitely many solutions (x, y, z) where $x > y > 0$ and $z > 0$. The proof is simple, we have that the linear Diophantine equation $sn_2 - 2^k n_1 = 1$ has infinitely many solutions (n_1, n_2), where n_1 and n_2 are positive integers. If we choose a pair (n_1, n_2), then we have the identity

$$\left(ha\left(h^{t-1}a^t - h^{t-1}b^t\right)^{n_1}\right)^t - \left(hb\left(h^{t-1}a^t - h^{t-1}b^t\right)^{n_1}\right)^t$$

$$= h\left(\left(h^{t-1}a^t - h^{t-1}b^t\right)^{n_2}\right)^s,$$

where $t = 2^k$ and $h = 2^{k+1}$.

This identity give us infinitely many solutions (x, y, z), where $x > y > 0$ and $z > 0$, of the equation $x^{(2^k)} - y^{(2^k)} = 2^{k+1}z^s$. In the next theorem, we prove that there are not solutions (x, y, z) such that x and y are relatively prime.

Theorem 3.3. *Let $k \geq 2$ be an arbitrary but fixed positive integer and let us consider the Diophantine equation $x^{(2^k)} - y^{(2^k)} = 2^{k+1}z^s$, where $x > y > 0$, $z > 0$ and p is any odd prime. This equation has not solutions (x, y, z), where x and y are relatively prime.*

Proof. First, we examine the Diophantine system

$$\frac{x-y}{2} = u^p, \tag{32}$$

$$\frac{x+y}{2} = v^p, \tag{33}$$

$$\frac{x^2+y^2}{2} = w^p, \tag{34}$$

where $u, v,$ and w are positive integers. We are interested in solutions (x, y) such that $x > y > 0$. We shall prove that there are not solutions (x, y) such that $x > y > 0$. The proof is simple (as case 1) of Theorem 3.1) since Equations (32) and (33) give $x = v^p + u^p$ and $y = v^p - u^p$. Substituting these two equations in Equation (34) of the system we obtain the equation $u^{2p} + v^{2p} = w^p$. This equation is impossible for the Wiles-Taylor-Fermat theorem. This complete the proof.

Now, suppose that there is a solution (x, y, z) to the equation where $x > y > 0, z > 0$ and x and y are relatively prime odd numbers. We have

$$z^p = \frac{x^{2^k} - y^{2^k}}{2^{k+1}} = \left(\frac{x-y}{2}\right)\left(\frac{x+y}{2}\right)\left(\frac{x^2+y^2}{2}\right)\left(\frac{x^4+y^4}{2}\right)\cdots\left(\frac{x^{2^{k-1}}+y^{2^{k-1}}}{2}\right),$$

where the numbers in the right hand are pairwise relatively prime. Then the system of three equations (32), (33) and (34) has a solution. This is impossible. The theorem is proved.

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