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INVERSE PROBLEM FOR STOCHASTIC SYSTEMS AND OPTIMAL CHOICE OF VECTOR FIELDS CONTROLLING DRIFT, DIFFUSION AND JUMP PROCESSES

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Abstract

In this paper, we consider an inverse problem for a general class of nonlinear stochastic differential equations on finite dimensional spaces whose generating operators (drift, diffusion and Jump kernel) are unknown. We introduce a class of function spaces with a suitable topology and prove existence of optimal generating operators from these spaces. We present also necessary conditions of optimality including an algorithm and its convergence whereby one can construct the optimal generators (drift, diffusion and jump kernel). Also we present briefly an alternative approach giving the Hamilton-Jacobi-Bellman (HJB) equation and discuss the merits and demerits of the two methods. This paper is an extension of our previous studies on similar inverse problem for continuous diffusion.

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1. Introduction

Stochastic differential equations have been extensively used in modelling physical and engineering systems, including biological processes, finance, management, and many more [1, 2, 3, 4]. One can find stochastic models for population dynamics, computer communication networks, internet traffic control and many more, see [2]. In the literature, one can also find some stochastic models to capture the dynamics of stock market used for portfolio management and control [3, 4, 2]. One of the most well-known model is given by the celebrated Black-Scholes equation [3] described by a scalar stochastic differential equation of the form,

$$dx = \nu x dt + \sigma x dW(t),\ x(0) = x_0,\ t \geq 0.$$

Its solution $\{x(t),\ t \geq 0\}$ is the so called geometric Brownian motion given by

$$x(t) = x_0 \exp\{(\nu - (1/2)\sigma^2)t + \sigma W(t)\},\ t \geq 0.$$

This model is used as the basis for option pricing. For valuation of the option price one can easily derive the well known Black-Scholes partial differential equation,

$$\frac{\partial V}{\partial t} + (1/2)\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + rx\frac{\partial V}{\partial x} - rV = 0,\ (t,\ x) \in (0,\ T) \times R^+,$$

subject to the initial boundary conditions given by

$$V(0,\ x) = \max\{x - s,\ 0\},\ V(t,\ 0) = 0,\ V(t,\ x) = x,\ x \to \infty,$$

where x is the current stock price, s is the strike price of the option and r is the annualized risk free interest rate continuously compounded and T is the expiration date. For detailed assumptions and derivation, see [3].

This and similar models are based on market experience, intuition, and many apparently reasonable assumptions as seen in [4] and the extensive references therein. The validity of this simple model for stock price dynamics is probably questionable. Therefore, to avoid significant errors, it is necessary to identify the drift, diffusion, jump parameters as accurately as possible using available historical data. Recently, we considered inverse problems for continuous drift-diffusion processes [6]. Our objective here is to generalize our previous work and develop a method for solving stochastic inverse problem designed to determine all the infinitesimal generators controlling the drift-diffusion-jump processes. Inclusion of Poisson jump process in the dynamics with appropriate Lévy measure can model jumps in the stock price as often seen in practice. We hope the method presented here can be used to identify or develop stochastic models on the basis of available historical data, for example, the stock price, population process, information flow on communication network, traffic flow, both vehicular and internet, and many more. This has to be based on rigorous stochastic analysis. Here we consider the general stochastic inverse problem and develop the necessary conditions of optimality designed to determine the system model from available data. Thus the theory developed here can be used to construct the mathematical model for any finite dimensional stochastic dynamic system.

2. Proposed System Model and Problem Formulation

Let $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$ be a complete filtered probability space where $\{\mathcal{F}_t, t \geq 0\}$ is an increasing family of sub-sigma algebras of the sigma algebra $\mathcal{F}$, continuous from the right and having limits from the left. Let $\{W(t), t \geq 0\}$ be an $\mathcal{F}_t$-adapted $R^m (m \in N)$ valued standard Brownian motion, and $p(d\xi \times dt)$ denote a random measure defined on the sigma algebra of subsets of the set $V_\delta \times I$, where $I = [0, T]$ is the time interval and $V_\delta \equiv R^d \backslash B_\delta$ with B_δ denoting the open ball in R^d of radius $\delta > 0$ and centered at the origin. The measure p is said to be a Poisson random measure or a counting measure if for each time interval $\Gamma \subset I$ and any Borel set $S \subset V_\delta$ the probability that there are exactly n jumps of sizes (or with range) confined in the set S is given by

$$P\{p(S \times \Gamma) = n\} = \frac{(\pi(S)\lambda(\Gamma))^n}{n!} \exp - \{\pi(S)\lambda(\Gamma)\},$$

where λ denotes the Lebesgue measure on I and π denotes the Lévy (jump) measure on the sigma algebra of Borel subsets of the set V_δ. The term $\pi(S)$ (the Lévy measure of the set S) denotes the mean rate of jumps of sizes confined in the set S. We note that the measure π can be chosen according to the specific needs of applications. Define the random measure

$$q(S \times \Gamma) \equiv p(S \times \Gamma) - \pi(S)\lambda(\Gamma)$$

with mean zero and variance $\pi(S)\lambda(\Gamma)$. The process $q(d\xi \times dt)$ is called the compensated Poisson random measure.

For illustration, let us consider n different interacting randomly fluctuating entities defined on the time interval $I = [0, T]$ and taking values in R^n and denoted by $\{x(t), t \in I\}$. For example, the process x may represent the price fluctuation of n different stocks around a reference vector such as the mean flow.

In general, the dynamics of the process x can be modelled by a stochastic differential equation of the form,

$$dx = a(x)dt + b(x)dW(t) + \int_{V_\delta} c(x, \xi)q(d\xi \times dt), \; x(0) = x_0, \; t \in I = [0, T], \tag{1}$$

given that the generating operators (or the elements of the vector field) $\{a, b, c\}$ are known. Generally, the infinitesimal mean vector

$a : R^n \longrightarrow R^n,$ the diffusion matrix $b : R^n \longrightarrow \mathcal{L}(R^m, R^n)$ and the jump kernel $c : R^n \times R^d \longrightarrow R^n$ are unknown. These are the fundamental parameters or generators that determine the dynamics of the system. In many physical problems the triple $\{a, b, c\}$ is unknown and may be determined on the basis of basic sciences. For stock market there is no such basic science available. It is almost impossible to determine the dynamics of individual or even the mass phycology and use it to determine the drift-diffusion-jump triple $\{a, b, c\}.$ So one must use available market data $\{y(t), t \in I\}$ and find a way to determine the generating triple $\{a, b, c\}$ so that the solution of Equation (1) is as close as possible to the observed data $\{y(t), t \in I\}$ in some sense. This is the essence of inverse problem. Note that the Black-Scholes model is a scalar stochastic differential equation with the assumption that $a(x) = \iota x$ and $b(x) = \sigma x$ and $c = 0.$ This appears to be a very simple model for a rather complex system. In order to consider the general problem we have to be more precise; we must specify the function space from which we can choose the triple, and also we must specify the measure of closeness. Let $\mathcal{P}_{ad}$ denote the admissible class of drift-diffusion-jump triples and let the measure of closeness be given by mean square error. So we introduce the functional

$$J(a, b, c) = (1/2)\mathbf{E}\left\{\int_0^T \|x(t) - y(t)\|^2_{R^n}\, dt + \|x(T) - y(T)\|^2_{R^n}\right\},$$

and find $\{a^o, b^o, c^o\} \in \mathcal{P}_{ad}$ that minimizes the functional $J.$ In fact, we can consider a more general performance functional such as

$$J(a, b, c) = \mathbf{E}\left\{\int_0^T \ell(t, x(t))dt + \Phi(x(T))\right\}, \tag{2}$$

where $\ell : I \times R^n \longrightarrow R$ and $\Phi : R^n \longrightarrow R.$ Since the data y is fixed, it is contained in the definition of the functions ℓ and Φ though not shown explicitly. The objective is to find a generating triple $\{a^o, b^o, c^o\} \in \mathcal{P}_{ad}$ at which J attains its minimum. The model so obtained may be useful to capture the dynamics of unknown stochastic systems.

3. Admissible Drift-Diffusion-Jump Triples

To solve the above inverse problem it is necessary to give a more precise characterization of the admissible set $\mathcal{P}_{ad}$. Let $L_2(V_\delta, \pi) = L_2(\pi)$ denote the class of real valued Borel measurable functions defined on V_δ which are square integrable with respect to the Lévy measure π and let $L_2^+(\pi)$ denote class of nonnegative members of $L_2(\pi)$. Let $C(V_\delta)$ denote the space continuous real valued functions defined on V_δ. Let $\{\alpha, K\}$ be any pair of positive numbers and $(\beta, \gamma) \in L_2^+(\pi) \cap C(V_\delta)$. We introduce the class of functions $\mathcal{A}_{\alpha, K}$, $\mathcal{B}_{\alpha, K}$ and $\mathcal{C}_{\beta, \gamma}$ given by

(A1)

$$\mathcal{A}_{\alpha, K} \equiv \{a : R^n \to R^n \mid \text{ (i): } \|a(0)\|_{R^n} \leq \alpha, \text{ and}$$

$$\text{(ii)}: \|a(x_1) - a(x_2)\|_{R^n} \leq K\|x_1 - x_2\|_{R^n} \,\forall\, x_1, x_2 \in R^n\}, \tag{3}$$

(A2)

$$\mathcal{B}_{\alpha, K} \equiv \{b : R^n \to \mathcal{L}(R^m, R^n) \mid \text{ (i)}: \|b(0)\|_{\mathcal{L}(R^m, R^n)} \leq \alpha, \text{ and}$$

$$\text{(ii)}: \|b(x_1) - b(x_2)\|_{\mathcal{L}(R^m, R^n)} \leq K\|x_1 - x_2\|_{R^n} \,\forall\, x_1, x_2 \in R^n\}, \tag{4}$$

(A3)

$$\mathcal{C}_{\beta, \gamma} \equiv \{c : R^n \times V_\delta \longrightarrow R^n (\text{continuous}) \mid$$

$$\text{(i)}: \|c(0, \xi)\|_{R^n} \leq \beta(\xi), \xi \in V_\delta, \text{ and}$$

$$\text{(ii)}: \|c(x, \xi) - c(y, \xi)\|_{R^n} \leq \gamma(\xi)\|x - y\|_{R^n}\}, \forall\, \xi \in V_\delta, x, y \in R^n. \tag{5}$$

Clearly, the class of maps $\{\mathcal{A}_{\alpha, K}, \mathcal{B}_{\alpha, K}\}$ are Lipschitz whose values at zero vector do not exceed the number α, and the Lipschitz coefficients do not exceed K. The larger the parameters $\{\alpha, K\}$ are, the larger are these classes $\{\mathcal{A}_{\alpha, K}, \mathcal{B}_{\alpha, K}\}$. Similarly, if the pair $\{\tilde{\beta}, \tilde{\gamma}\} \in L_2^+(\pi) \cap C(V_\delta)$ and we have $\beta(\xi) \leq \tilde{\beta}(\xi)$, $\gamma(\xi) \leq \tilde{\gamma}(\xi)$ for $\pi - a.e\ \xi \in V_\delta$, then $\mathcal{C}_{\beta, \gamma} \subset \mathcal{C}_{\tilde{\beta}, \tilde{\gamma}}$.

Let $Z \equiv R^n \times V_\delta$ and $Y \equiv R^n \times \mathcal{L}(R^m, R^n) \times R^n$ with their natural metric topologies. Consider the function space $F(Z, Y) = Y^Z$, denoting the set of all maps from Z to Y. The natural topology on this function space is the Tychonoff product topology. For any $z = (x, \xi) \in Z$, let $\prod_{x, \xi}$ denote the projection map from Y^Z to Y in the sense that for any $(a, b, c) \in Y^Z$,

$$\prod\nolimits_{x, \xi}(a, b, c) = \{a(x), b(x), c(x, \xi)\}.$$

Clearly, for each $x \in R^n$, the x-projections $\prod_x$ of $\mathcal{A}_{\alpha, K}$ and $\mathcal{B}_{\alpha, K}$ are bounded convex subsets of the finite dimensional (topological) vector spaces R^n and $\mathcal{L}(R^m, R^n)$, respectively and hence relatively compact. Similarly, the (x, ξ)-projection of the set $\mathcal{C}_{\beta, \gamma}$ is a bounded convex subset of R^n and hence also relatively compact. We define $\mathcal{P}_{ad} \equiv \mathcal{A}_{\alpha, K} \times \mathcal{B}_{\alpha, K} \times \mathcal{C}_{\beta, \gamma}$ and endow this with the product topology. We use the notation of Willard [5] and consider the function space $Y^Z = F(Z, Y)$. The set $\mathcal{P}_{ad}$ is a subset of the function space Y^Z, and it is given the point wise topology τ_p (topology of point wise convergence), see Willard [5].

Lemma 3.1. *The set* $\mathcal{P}_{ad}$, *a subset of the function space* Y^Z, *is compact in the* τ_p *topology.*

Proof. This is a special case of Theorem 42.3, [5, p.278]. Since Y is a metric space, it is clearly Hausdorff (uniform) space. So according to this theorem, it suffices to verify that (i) the set $\mathcal{P}_{ad}$ is point wise closed and (ii) each $z = (x, \xi)$ projection of $\mathcal{P}_{ad}$, denoted by $\prod_{x,\xi}(\mathcal{P}_{ad})$, has compact closure in Y. To prove that the set is point wise closed, it suffices to show that the point wise limit of any generalized sequence (net) from $\mathcal{P}_{ad}$ belongs to $\mathcal{P}_{ad}$. We verify that each component of the product space is point wise closed. Consider the component $\mathcal{A}_{\alpha, K}$ and the sequence $\{a^k\}_{k\geq 1} \in \mathcal{A}_{\alpha, K}$ and suppose it converges point wise to a^o. We show that $a^o \in \mathcal{A}_{\alpha, K}$. Clearly, $\|a^o(0)\| \leq \|a^o(0) - a^k(0)\| + \|a^k(0)\|$. Hence for any $\varepsilon > 0$ there exists $n_\varepsilon \in N$ such that

$$\|a^o(0)\| \leq \|a^o(0) - a^k(0)\| + \left\|a^k(0)\right\| \leq \varepsilon + \alpha \ \forall \ k > n_\varepsilon.$$

Since this holds true for arbitrary $\varepsilon > 0$, we have $\|a^o(0)\| \leq \alpha$. Similarly, for any pair $\{x_1, x_2\} \in R^n$, using triangle inequality we obtain

$$\|a^o(x_1) - a^o(x_2)\| \leq \|a^o(x_1) - a^k(x_1)\| + \|a^k(x_1) - a^k(x_2)\| + \|a^k(x_2) - a^o(x_2)\|.$$

Again, for any $\varepsilon > 0$, there exists $n_1, n_2 \in N$ such that $\|a^o(x_1) - a^k(x_1)\| \leq \varepsilon/2$ for all $k \geq n_1$ and $\|a^o(x_2) - a^k(x_2)\| \leq \varepsilon/2$ for all $k \geq n_2$. Thus for all $k \geq \max\{n_1, n_2\}$, we have

$$\|a^o(x_1) - a^o(x_2)\| \leq \varepsilon + \|a^k(x_1) - a^k(x_2)\| \leq \varepsilon + K\|x_1 - x_2\|.$$

Since $\varepsilon > 0$ is otherwise arbitrary we conclude that a^o is Lipschitz. Thus we have proved that $a^o \in \mathcal{A}_{\alpha, K}$ and hence $\mathcal{A}_{\alpha, K}$ is point wise closed. Following similar procedure one can easily verify that any point wise limit b^o of any generalized sequence $\{b^k\}_{k \geq 1} \subset \mathcal{B}_{\alpha, K}$ is an element of $\mathcal{B}_{\alpha, K}$. Similarly one can verify that the set $C_{\beta, \gamma}$ is also point wise closed. Thus the set $\mathcal{P}_{ad}$ is point wise closed. To prove (ii), let $(x, \xi) \in Z$ and consider the projection

$$\prod_{x, \xi} (\mathcal{P}_{ad}) = \{(a(x), b(x), c(x, \xi)) \big| (a, b, c) \in \mathcal{P}_{ad}\}.$$

Using the properties (3)-(5) characterizing the set $\mathcal{P}_{ad}$ one can easily verify that for all $(a, b, c) \in \mathcal{P}_{ad}$, we have

$$\|a(x)\|_{R^n} \leq \alpha + K\|x\|_{R^n}, \quad \|b(x)\|_{\mathcal{L}(R^m, R^n)} \leq \alpha + K\|x\|_{R^n},$$

$$\|c(x, \xi)\|_{R^n} \leq \beta(\xi) + \gamma(\xi)\|x\|_{R^n}.$$

Since this holds uniformly with respect to the set $\mathcal{P}_{ad}$, and also the functions $\beta, \gamma \in C(V_\delta) \cap L_2^+(\pi)$, we conclude that the set $\prod_{x, \xi} (\mathcal{P}_{ad})$ is a bounded subset of Y. Since Y is a finite dimensional Hausdorff space, the closure of $\prod_{x, \xi} (\mathcal{P}_{ad})$ is compact. This completes the proof. □

Throughout the rest of the paper we assume, without further notice, that the initial state x_0, the Wiener process W, and the compensated Poisson random measure q are stochastically independent.

4. Existence of Optimal Drift-Diffusion-Jump Triples

Consider the system (1) with $(a, b, c) \in \mathcal{P}_{ad}$ and the objective functional (2). First, we present the following result on existence, uniqueness, and regularity properties of solutions of Equation (1). For this we introduce the following spaces of random processes. Throughout the rest of the paper, we let $I \equiv [0, T]$ denote the closed bounded interval and $B_\infty(I, R^n)$ the Banach space of R^n valued bounded measurable functions endowed with the sup-norm topology. In the study of stochastic differential equations subject to both Wiener process and Poisson random process (or Lévy process) we expect the solution trajectories to have discontinuities at most of the first kind. In order to include such processes we may introduce the space $B_\infty^a(I, R^n)$ consisting of $\mathcal{F}_t$ adapted R^n valued random processes having bounded second moments. Here we introduce the norm topology given by

$$\|x\| \equiv \sup\{(\mathbf{E}\|x(t)\|_{R^n}^2)^{1/2}, t \in I\}.$$

With respect to this norm topology, $B_\infty^a(I, R^n)$ is a Banach space.

We consider the system governed by the stochastic differential equation (1) with the infinitesimal generators given by $(a, b, c) \in \mathcal{P}_{ad}$. We present briefly a proof of existence and uniqueness of solutions.

Theorem 4.1. *For each initial state* $x_0 \in L_2(\mathcal{F}_0, R^n)$ (*having finite second moment*) *and each drift-diffusion-jump triple* $(a, b, c) \in \mathcal{P}_{ad}$, *the stochastic differential equation* (1) *has a unique solution* $x \in B_\infty^a(I, R^n)$.

Proof. The proof is standard see [1, 2]. We present a brief outline. Define the operator Λ given by

$$(\Lambda x)(t) = x_0 + \int_0^t a(x(s))ds + \int_0^t b(x(s))dW(s)$$

$$+ \int_0^t \int_{V_\delta} c(x(s), \xi)q(d\xi \times ds), \; t \in I. \tag{6}$$

We show that $\Lambda : B_\infty^a(I, R^n) \longrightarrow B_\infty^a(I, R^n)$. Taking the expected value of the norm square and using Fubini's theorem and the properties of Itô integrals we obtain

$$\mathbf{E}\|(\Lambda x)(t)\|_{R^n}^2 \leq 4\mathbf{E}\|x_0\|_{R^n}^2 + 4t\int_0^t \mathbf{E}\|a(x(s))\|_{R^n}^2 ds$$

$$+ 4\int_0^t \mathbf{E}\|b(x(s))\|_{\mathcal{L}(R^m, R^n)}^2 ds$$

$$+ 4\int_0^t \int_{V_\delta} \mathbf{E}\|c(x(s), \xi)\|_{R^n}^2 \pi(d\xi)ds, \ t \in I. \quad (7)$$

Using the properties (3)-(5) of the triple $(a, b, c) \in \mathcal{P}_{ad}$, in the above inequality one can easily verify that

$$\mathbf{E}\|(\Lambda x)(t)\|_{R^n}^2 \leq C_1(t) + C_2(t)\sup\{\mathbf{E}\|x(s)\|_{R^n}^2, \ 0 \leq s \leq t\}, \quad (8)$$

where

$$C_1(t) = (4\mathbf{E}\|x_0\|_{R^n}^2 + 8\alpha^2 t(1+t) + 8t\|\beta\|_{L_2(\pi)}^2),$$

and

$$C_2(t) = (8K^2 t(1+t) + 8t\|\gamma\|_{L_2(\pi)}^2).$$

Thus, for $x \in B_\infty^a(I, R^n)$, we have

$$\sup\{\mathbf{E}\|(\Lambda x)(t)\|_{R^n}^2, \ t \in I\} \leq C_1(T) + C_2(T)\sup\{\mathbf{E}\|x(s)\|_{R^n}^2, \ 0 \leq s \leq T\}, \quad (9)$$

and hence we conclude that $\Lambda x \in B_\infty^a(I, R^n)$ whenever $x \in B_\infty^a(I, R^n)$ proving that Λ maps $B_\infty^a(I, R^n)$ in to itself. Following similar steps, one can verify that

$$\mathbf{E}\|(\Lambda x)(t) - (\Lambda y)(t)\|^2_{R^n} \le 4\{K^2(t+1)$$

$$+ \|\gamma\|^2_{L_2(\pi)}\}\int_0^t \mathbf{E}\,\|x(s) - y(s)\|^2_{R^n}\,ds,\ t \in I. \quad (10)$$

Define $\rho_t^2(x, y) \equiv \sup\{\mathbf{E}\|x(s) - y(s)\|^2, 0 \le s \le t\}$, and $C = 4\{K^2(T+1) + \|\gamma\|^2_{L_2(\pi)}\}$, and denote $\rho_t^2(x, y)$ by $\varrho_t(x, y)$ for all $t \in I$. Using these notations in the above inequality we obtain

$$\varrho_t(\Lambda x, \Lambda y) \le C\int_0^t \varrho_s(x, y)ds,\ t \in I. \quad (11)$$

Let Λ^m denote the m-fold composition of the operator Λ. For $m = 2$, it follows from the above expression that

$$\varrho_t(\Lambda^2 x, \Lambda^2 y) \le C\int_0^t \varrho_s(\Lambda x, \Lambda y)ds \le C^2\int_0^t s\varrho_s(x, y)ds$$

$$\le C^2(t^2/2!)\varrho_t(x, y),\ t \in I. \quad (12)$$

Repeating this iterative process m-times one finds that

$$\varrho_t(\Lambda^m x, \Lambda^m y) \le C^m(t^m/m!)\varrho_t(x, y),\ t \in I,$$

and hence $\rho_T(\Lambda^m x, \Lambda^m y) \le \sqrt{C^m(T^m/m!)}\rho_T(x, y)$. In terms of the norm of the Banach space $B^a_\infty(I, R^n)$, this inequality is equivalent to the following inequality:

$$\left\|\Lambda^m x - \Lambda^m y\right\|_{B^a_\infty(I, R^n)} \le \gamma_m\|x - y\|_{B^a_\infty(I, R^n)},$$

where $\gamma_m \equiv \sqrt{C^m(T^m/m!)}$. It is clear that for $m_0 \in N$ sufficiently large, $\gamma_{m_0} < 1$. Thus Λ^{m_0} is a contraction and hence it follows from Banach fixed point theorem that it has a unique fixed point

$x^o \in B_\infty^a(I, R^n)$. This implies that the operator Λ itself has x^o as the unique fixed point. Hence the system (1) has a unique solution in $B_\infty^a(I, R^n)$. This completes the outline of our proof. □

As indicated earlier, our objective is to solve the inverse problem. The problem is to find a drift-diffusion-jump triple $(a^o, b^o, c^o) \in \mathcal{P}_{ad}$ for system (1) that minimizes the functional (2). For this we must be assured that an optimal triple exists. Before we consider the question of existence, we prove a result on continuity of the solution map $(a, b, c) \longrightarrow x(a, b, c)$ of the stochastic differential equation (1). We present this in the following theorem.

Theorem 4.2. *Consider the system* (1) *with the admissible set of drift-diffusion-jump triples* $\mathcal{P}_{ad}$, *and suppose the assumptions of Theorem* 4.1 *hold. Then the solution map* $(a, b, c) \longrightarrow x(a, b, c)$ *is continuous with respect to the* τ_p *topology on* $\mathcal{P}_{ad}$ *and norm topology on the space* $B_\infty^a(I, R^n)$.

Proof. Let $(a^k, b^k, c^k) \in \mathcal{P}_{ad}$ be any sequence converging to $(a^o, b^o, c^o) \in \mathcal{P}_{ad}$ in the τ_p topology. Let $x^k \in B_\infty^a(I, R^n)$ be the solution of Equation (1) corresponding to the triple (a^k, b^k, c^k), and $x^o \in B_\infty^a(I, R^n)$ the solution corresponding to the triple (a^o, b^o, c^o). We show that $x^k \xrightarrow{s} x^o$ in the Banach space $B_\infty^a(I, R^n)$ as $(a^k, b^k, c^k) \xrightarrow{\tau_p}$ (a^o, b^o, c^o). Clearly the pair (x^k, x^o) satisfies the following stochastic integral equations:

$$x^k(t) = x_0 + \int_0^t a^k(x^k(s))ds + \int_0^t b^k(x^k(s))dW(s)$$

$$+ \int_0^t \int_{V_\delta} c^k(x^k(s), \xi)q(d\xi \times ds), \; t \in I, \tag{13}$$

$$x^o(t) = x_0 + \int_0^t a^o(x^o(s))ds + \int_0^t b^o(x^o(s))dW(s)$$

$$+ \int_0^t \int_{V_\delta} c^o(x^o(s), \xi)q(d\xi \times ds), \; t \in I. \tag{14}$$

Subtracting Equation (14) from Equation (13) term by term we obtain the following identity:

$$\begin{aligned} x^k(t) - x^o(t) = &\int_0^t [a^k(x^k(s)) - a^k(x^o(s))]ds \\ &+ \int_0^t [a^k(x^0(s)) - a^o(x^o(s))]ds \\ &+ \int_0^t [b^k(x^k(s)) - b^k(x^o(s))]dW(s) \\ &+ \int_0^t [b^k(x^o(s)) - b^o(x^o(s))]dW(s) \\ &+ \int_0^t \int_{V_\delta} [c^k(x^k(s), \xi) - c^k(x^o(s), \xi)]q(d\xi \times ds) \\ &+ \int_0^t \int_{V_\delta} [c^k(x^o(s), \xi) - c^o(x^o(s), \xi)]q(d\xi \times ds), \; t \in I. \end{aligned} \tag{15}$$

Computing the expected value of the norm square of the fifth term on the right hand side of the above expression using the Lipschitz property of the elements of the set $\mathcal{C}_{\beta,\gamma}$ and the properties of the compensated Poisson random measure q and Fubini's theorem we obtain

$$\mathbf{E} \left\| \int_0^t \int_{V_\delta} [c^k(x^k(s), \xi) - c^k(x^o(s), \xi)]q(d\xi \times ds) \right\|_{R^n}^2$$

$$\leq \|\gamma\|_{L_2(\pi)}^2 \int_0^t \mathbf{E} \|x^k(s) - x^o(s)\|_{R^n}^2 \, ds, \; t \in I.$$

Using this fact and computing the norm square of $[x^k(t) - x^o(t)]$ given by Equation (15) and keeping in mind the properties of stochastic integrals and Fubini's theorem and using the Lipschitz property and triangle inequality, we obtain the following inequality:

$$\mathbf{E}\|x^k(t) - x^o(t)\|^2 \leq 8([T+1]K^2 + \|\gamma\|^2_{L_2(\pi)})\int_0^t \mathbf{E}\|x^k(s) - x^o(s)\|^2 ds$$

$$+ 8T\int_0^t \mathbf{E}\| a^k(x^o(s)) - a^o(x^o(s))\|^2_{R^n} ds$$

$$+ 8\int_0^t \mathbf{E}\| b^k(x^o(s)) - b^o(x^o(s))\|^2_{\mathcal{L}(R^m,\, R^n)} ds$$

$$+ 8\int_0^t \int_{V_\delta} \mathbf{E}\| c^k(x^o(s), \xi) - c^o(x^o(s), \xi)\|^2_{R^n} \pi(d\xi)ds,\ t \in I. \tag{16}$$

Define

$$e_1^k(t) \equiv 8T\int_0^t \mathbf{E}\| a^k(x^o(s)) - a^o(x^o(s))\|^2_{R^n} ds,\ t \in I, \tag{17}$$

$$e_2^k(t) \equiv 8\int_0^t \mathbf{E}\| b^k(x^o(s)) - b^o(x^o(s))\|^2_{\mathcal{L}(R^m,\, R^n)} ds,\ t \in I, \tag{18}$$

$$e_3^k(t) \equiv 8\int_0^t \int_{V_\delta} \mathbf{E}\| c^k(x^o(s), \xi) - c^o(x^o(s), \xi)\|^2_{R^n} \pi(d\xi)ds,\ t \in I, \tag{19}$$

and

$$\varphi^k(t) \equiv \mathbf{E}\|x^k(t) - x^o(t)\|^2_{R^n},\ t \in I, \tag{20}$$

$$e^k(t) \equiv e_1^k(t) + e_2^k(t) + e_3^k(t),\ t \in I. \tag{21}$$

Using the expressions (20) and (21) in the inequality (16), we obtain the following inequality:

$$\varphi^k(t) \leq C \int_0^t \varphi^k(s) ds + e^k(t),\ t \in I, \tag{22}$$

where $C = 8((T+1)K^2 + \|\gamma\|^2_{L_2(\pi)})$. It follows from Grönwall inequality applied to (22) that

$$\varphi^k(t) \leq e^k(t) + C \int_0^t \{\exp C(t-s)\} e^k(s) ds,\ t \in I. \tag{23}$$

Since $a^k \longrightarrow a^o$ in $\mathcal{A}_{\alpha, K}$ and $b^k \longrightarrow b^o$ in $\mathcal{B}_{\alpha, K}$ point wise, it is clear that $a^k(x^o(s)) \to a^o(x^o(s))$ in R^n, and $b^k(x^o(s)) \to b^o(x^o(s))$ in $\mathcal{L}(R^m, R^n)$ for almost all $s \in I$, $P - a.s.$ Further, it follows from the Lipschitz and growth properties of the elements of $\mathcal{P}_{ad}$ that both $a^k(x^o(s))$ and $b^k(x^o(s))$ are dominated in their respective norms by norm square integrable random processes. Hence, by virtue of Lebesgue dominated convergence theorem, it follows from the expressions (17)-(18) that both $e_1^k(t) \longrightarrow 0$ and $e_2^k(t) \longrightarrow 0$ for each $t \in I$. Considering the third component $e_3^k(t)$ for any $t \in I$, we recall that $c^k \longrightarrow c^o$ in $\mathcal{C}_{\beta, \gamma}$ point wise. Hence it is clear that $c^k(x^o(s), \xi) \longrightarrow c^o(x^o(s), \xi)$ in R^n for $dt \times d\pi$ almost all $(s, \xi) \in I \times V_\delta$, $P - a.s.$ Further, it follows from the properties (i) and (ii) of $\mathcal{C}_{\beta, \gamma}$ that

$$\|c^k(x^o(s), \xi)\|^2_{R^n} \leq 2(\beta^2(\xi) + \gamma^2(\xi)\|x^o(s)\|^2_{R^n}),\ \forall\ k \in N,\ (s, \xi) \in I \times V_\delta.$$

Since $x^o \in B^a_\infty(I, R^n)$, it follows from the above inequality that the sequence $\{c^k(x^o, \xi)\}$ is dominated by norm square integrable random process. Hence it follows from the expression (19) and Lebesgue dominated convergence theorem that $e_3^k(t) \longrightarrow 0$ for each $t \in I$. Thus, it follows from the expression (21) that $e^k(t) \to 0$ for each $t \in I$ and, by virtue of the estimates as seen above, it is uniformly bounded on I. Hence by virtue of Lebesgue bounded convergence theorem it follows from the inequality (23) that $\varphi^k(t) \to 0$ for all $t \in I$. This proves that the map $(a, b, c) \longrightarrow x(a, b, c)$ from $\mathcal{P}_{ad}$ to $B_\infty^a(I, R^n)$ is continuous with respect to their distinct topologies. This completes the proof. □

Now we are prepared to prove existence of optimal drift-diffusion pair.

Theorem 4.3. *Consider the system* (1) *with the objective functional* (2) *and admissible set of drift-diffusion-jump triples* $\mathcal{P}_{ad}$. *Suppose the assumptions of Theorem* 4.2 *hold and that* ℓ *is a real valued Borel measurable function on* $I \times R^n$ *and lower semi-continuous on* R^n (*the state variable*), *and* Φ *is also a Borel measurable real valued function and lower semi-continuous on* R^n *satisfying the following growth properties*:

$$|\ell(t, x)| \leq \alpha_1(t) + \alpha_2\|x\|^2, \tag{24}$$

$$|\Phi(x)| \leq \alpha_3 + \alpha_4\|x\|^2, \tag{25}$$

for $\alpha_1 \in L_1^+(I)$, *and* $\alpha_2, \alpha_3, \alpha_4 > 0$. *Then there exists an optimal triple* $(a^o, b^o, c^o) \in \mathcal{P}_{ad}$ *that minimizes the cost functional* (2).

Proof. Since the set $\mathcal{P}_{ad}$ is compact in the τ_p topology, it suffices to show that J is lower semicontinuous in this topology. Let $(a^k, b^k, c^k) \in \mathcal{P}_{ad}$ be a generalized sequence converging to $(a^o, b^o, c^o) \in \mathcal{P}_{ad}$ in the τ_p topology. Let $(x^k, x^o) \in B_\infty^a(I, R^n)$ denote the corresponding solutions of

Equation (1). It follows from Theorem 4.2 that $x^k \xrightarrow{s} x^o$ in $B_\infty^a(I, R^n)$. Since ℓ is lower semicontinuous in the state variable it is clear that

$$\ell(t, x^o(t)) \leq \varliminf_{k\to\infty} \ell(t, x^k(t)) \text{ for a.e. } t \in I, \ P - a.s. \tag{26}$$

The set I is a finite interval and the elements of $\mathcal{P}_{ad}$ have at most linear growth and therefore the solutions $\{(x^k, x^o)\}$ are contained in a bounded subset of $B_\infty^a(I, R^n)$. Thus it follows from the growth property of ℓ as described by the inequality (24), that $\ell(t, x^k(t)), t \in I,$ is dominated from bellow by an integrable random process. Thus it follows generalized Fatou's Lemma that

$$\mathbf{E}\int_0^T \ell(t, x^o(t))dt \leq \mathbf{E}\int_0^T \varliminf_{k\to\infty} \ell(t, x^k(t))dt \leq \varliminf_{k\to\infty} \mathbf{E}\int_0^T \ell(t, x^k(t))dt. \tag{27}$$

Since Φ is also lower semicontinuous on R^n and has the growth property as represented by the inequality (25), it follows from similar argument that

$$\mathbf{E}\Phi(x^o(T)) \leq \mathbf{E} \varliminf_{k\to\infty} \Phi(x^k(T)) \leq \varliminf_{k\to\infty} \mathbf{E}\Phi(x^k(T)). \tag{28}$$

It is well known that sum of lower semi continuous functionals is lower semi continuous. Thus by adding (27) and (28) we obtain

$$J(a^o, b^o, c^o) \leq \varliminf_{k\to\infty} J(a^k, b^k, c^k),$$

proving lower semicontinuity of J on $\mathcal{P}_{ad}$ in the τ_p topology. Hence it follows from compactness of the set $\mathcal{P}_{ad}$ in this topology that J attains its minimum on $\mathcal{P}_{ad}$. This completes the proof of existence of an optimal drift-diffusion-jump triple. □

5. Necessary Conditions of Optimality

In this section, we present the necessary conditions of optimality characterizing the optimal drift-diffusion-jump triple whereby one can determine the optimal triple from the admissible set $\mathcal{P}_{ad}$ and hence determine the stochastic dynamic model. We recall that $B_\infty^a(I, R^n) \subset L_\infty^a(I, L_2(\Omega, R^n))$ denotes the space of $\mathcal{F}_t$-adapted $L_2(\Omega, R^n)$ valued norm bounded measurable processes defined on I. Similarly, $B_\infty^a(I, \mathcal{L}(R^m, R^n)) \subset L_\infty^a(I, L_2(\Omega, \mathcal{L}(R^m, R^n)))$ denotes the space of $\mathcal{F}_t$-adapted $L_2(\Omega, \mathcal{L}(R^m, R^n))$ valued norm bounded measurable processes on I. Let $L_2(\pi, R^n)$ denote the Hilbert space of measurable functions defined on the set V_δ with values in R^n having norms square integrable with respect to the L'evy measure π, that is, for $f \in L_2(\pi, R^n)$ we have

$$\int_{V_\delta} \|f(\xi)\|_{R^n}^2 \pi(d\xi) < \infty.$$

For convenience of notation we use $\{Da, Db, Dc\}$ to denote respectively, the Gâteaux differentials (directional derivatives) of $\{a, b, c\}$ in the state variable $x \in R^n$.

Theorem 5.1. *Consider the system given by Equation* (1) *with* $(a, b, c) \in \mathcal{P}_{ad}$ *and the cost functional given by* (2). *Suppose the assumptions of Theorem* 4.3 *hold and that the elements of* $\mathcal{P}_{ad}$ *are once continuously Gâteaux differentiable in the state variable with the Gâteaux derivatives uniformly bounded. Then, in order for the triple* $(a^o, b^o, c^o) \in \mathcal{P}_{ad}$ *with the corresponding solution* $x^o \in B_\infty^a(I, R^n)$ *to be optimal, it is necessary that there exists a triple* $(\psi, Q, \varphi) \in B_\infty^a(I, R^n) \times L_\infty^a(I, \mathcal{L}(R^m, R^n)) \times$

$L_\infty^a(I, L_2(\pi, R^n))$ *satisfying the inequality* (29) *and the stochastic adjoint and state differential equations* (30)-(31) *as presented below*:

$$\mathbf{E}\int_I < a(x^o) - a^o(x^o),\ \psi > dt + \mathbf{E}\int_I Tr[(b(x^o) - b^o(x^o))Q^*]dt$$

$$+ \mathbf{E}\int_I \int_{V_\delta} < c(x^o, \xi) - c^o(x^o, \xi),\ \varphi > \pi(d\xi)dt \geq 0\ \forall\ (a, b, c) \in \mathcal{P}_{ad}, \tag{29}$$

where $Q(t) \equiv - Db^o(x^o(t); \psi(t))$, $\varphi(t, \xi) \equiv -(Dc^o(x^o(t), \xi))^* \psi(t)$, $(t, \xi) \in I \times V_\delta$, *with* $\{Db^o, Dc^o\}$ *denoting the Gâteaux derivatives of* $\{b^o, c^o\}$ *with respect to the state variable evaluated at* x^o. *The function* ψ *denotes the solution of the following adjoint equation*:

$$- d\psi = (Da^o(x^o(t)))^* \psi dt + V_1(x^o(t))\psi dt + V_2(x^o(t))\psi dt + \ell_x(t, x^o(t))dt$$

$$+ Db^o(x^o(t); \psi(t))dW + \int_{V_\delta} (Dc^o(x^o, \xi))^* \psi q(d\xi \times dt),\ t \in I,$$

$$\psi(T) = \Phi_x(x^o(T)); \tag{30}$$

where $V_1(x^o(t)), t \in I$, *is a symmetric* $n \times n$ *negative semi-definite matrix valued random process given by the bilinear form* $-Tr(Db^o(x^o; y)(Db^o(x^o; \psi))^*) \equiv < V_1(x^o)y, \psi >$; *and* $V_2(x^o(t)), t \in I$, *is another symmetric* $n \times n$ *negative semi-definite matrix valued random process given by the bilinear form*

$$- \int_{V_\delta} < Dc^o(x^o, \xi)y, (Dc^o(x^o, \xi))^* \psi > \pi(d\xi) \equiv < V_2(x^o(t))y, \psi >,$$

with $x^o \in B_\infty^a(I, R^n)$ *being the solution of the state equation*

$$dx^o = a^o(x^o)dt + b^o(x^o)dW$$

$$+ \int_{V_\delta} c^o(x^o, \xi)q(d\xi \times dt),\ x^o(0) = x_0,\ t \in I. \tag{31}$$

Proof. Let $(a^o, b^o, c^o) \in \mathcal{P}_{ad}$ be the optimal drift-diffusion-jump triple with the corresponding solution of Equation (1) denoted by $x^o \in B_\infty^a(I, R^n)$. Let $(a, b, c) \in \mathcal{P}_{ad}$ be an arbitrary element and $\varepsilon \in [0, 1]$. Define the triple $(a^\varepsilon, b^\varepsilon, c^\varepsilon)$ as follows

$$a^\varepsilon \equiv a^o + \varepsilon(a - a^o),\ b^\varepsilon = b^o + \varepsilon(b - b^o),\ c^\varepsilon = c^o + \varepsilon(c - c^o),\ \varepsilon \in [0, 1].$$

It follows from convexity of the set $\mathcal{P}_{ad}$ that $(a^\varepsilon, b^\varepsilon, c^\varepsilon) \in \mathcal{P}_{ad}$, and by virtue of optimality of the triple (a^o, b^o, c^o), we have

$$J(a^\varepsilon, b^\varepsilon, c^\varepsilon) \geq J(a^o, b^o, c^o)\ \forall\, \varepsilon \in [0, 1] \text{ and } (a, b, c) \in \mathcal{P}_{ad}. \tag{32}$$

Let $x^\varepsilon \in B_\infty^a(I, R^n)$ denote the solution of Equation (1) corresponding to the triple $(a^\varepsilon, b^\varepsilon, c^\varepsilon)$. Clearly, the processes $\{x^\varepsilon, x^o\}$ satisfy respectively, the following stochastic integral equations:

$$x^\varepsilon(t) = x_0 + \int_0^t a^\varepsilon(x^\varepsilon(s))ds + \int_0^t b^\varepsilon(x^\varepsilon(s))dW(s)$$

$$+ \int_0^t \int_{V_\delta} c^\varepsilon(x^\varepsilon(s), \xi)q(d\xi \times ds),\ t \in I, \tag{33}$$

$$x^o(t) = x_0 + \int_0^t a^o(x^o(s))ds + \int_0^t b^o(x^o(s))dW(s)$$

$$+ \int_0^t \int_{V_\delta} c^o(x^o(s), \xi)q(d\xi \times ds),\ t \in I, \tag{34}$$

and they are elements of $B_\infty^a(I, R^n)$. Clearly $(a^\varepsilon, b^\varepsilon, c^\varepsilon) \xrightarrow{\tau_p} (a^o, b^o, c^o)$ and hence it follows from Theorem 4.2 that $x^\varepsilon \xrightarrow{s} x^o$ in $B_\infty^a(I, R^n)$. As justified below, the process y given by the following limit:

$$y(t) = \lim_{\varepsilon \downarrow 0}(1/\varepsilon)(x^\varepsilon(t) - x^o(t)),\ t \in I,$$

exists. Subtracting Equation (34) from Equation (33) term by term and dividing by ε and letting $\varepsilon \downarrow 0$ one can easily verify that y satisfies the following stochastic integral equation:

$$y(t) = \int_0^t Da^o(x^o(s))y(s)ds + \int_0^t [a(x^o(s)) - a^o(x^o(s))]ds$$

$$+ \int_0^t Db^o(x^o(s);\ y(s))dW(s) + \int_0^t [b(x^o(s)) - b^o(x^o(s))]dW(s)$$

$$+ \int_0^t \int_{V_\delta} Dc^o(x^o(s), \xi)y(s)q(d\xi \times ds)$$

$$+ \int_0^t \int_{V_\delta} [c(x^o(s), \xi) - c^o(x^o(s), \xi)]q(d\xi \times ds),\ t \in I, \tag{35}$$

where for any $u, v \in R^n$, $Da^o(u)v$ denotes the Gâteaux differential of a^o evaluated at u in the direction v with $Da^o(u) \in \mathcal{L}(R^n)$, and $Db^o(u; v)$ denotes the Gâteaux differential of b^o evaluated at u in the direction v, with $Db^o(u; \cdot) \in \mathcal{L}(R^n, \mathcal{L}(R^m, R^n))$; and for any $\xi \in R^d$, $Dc^o(u, \xi))v$ denotes the Gâteaux differential of c^o evaluated at u in the direction v. Note that for each fixed $u \in R^n$, $v \longrightarrow Db(u; v)$ is a continuous linear map from R^n to $\mathcal{L}(R^m, R^n)$. It follows from the integral equation (35) that y is the solution (if one exists) of the following linear stochastic differential equation (SDE):

$$dy = Da^o(x^o(t))y(t)dt + Db^o(x^o(t);\ y(t))dW(t)$$

$$+ \int_{V_\delta} Dc^o(x^o(t), \xi))y(t)q(d\xi \times dt) + dM_t^{a,b,c},\ y(0) = 0,\ t \in I. \tag{36}$$

It is driven by the process $M^{a,b,c} \equiv \{M_t^{a,b,c}, t \in I\}$ which is given by

$$dM_t^{a,b,c} = [a(x^o(t)) - a^o(x^o(t))]dt + [b(x^o(t)) - b^o(x^o(t))]dW(t)$$

$$+ \int_{V_\delta} [c(x^o(t), \xi) - c^o(x^o(t), \xi)]q(d\xi \times dt), \; t \in I. \tag{37}$$

Let $\mathcal{SM}_2$ denote the Hilbert space of norm square integrable R^n valued $\mathcal{F}_t$-adapted semi martingales starting from the origin, that is $M_0^{a,b,c} = 0$. Since $(a, b, c), (a^o, b^o, c^o) \in \mathcal{P}_{ad}$ and $x^o \in B_\infty^a(I, R^n)$, it is straightforward to verify that the drift (vector) $[a(x^o(\cdot)) - a^o(x^o(\cdot))]$, the diffusion (matrix) $[b(x^o(\cdot)) - b^o(x^o(\cdot))]$, and the jump kernel $[c(x^o(\cdot), \xi)) - c^o(x^o(\cdot), \xi)]$, $\xi \in R^d$, are $\mathcal{F}_t$-adapted random processes having square integrable norms. Hence $M^{a,b,c} \in \mathcal{SM}_2$ and therefore, as a special case, it follows from Theorem 4.1 that Equation (36) has a unique solution $y \in B_\infty^a(I, R^n)$. Thus $M^{a,b,c} \longrightarrow y$ is a bounded linear map, denoted by Υ, from the Hilbert space $\mathcal{SM}_2$ to the Banach space $B_\infty^a(I, R^n)$ and hence continuous. We denote this by $y = \Upsilon(M^{a,b,c})$. Using the inequality (32) and dividing the following expression:

$$J(a^\varepsilon, b^\varepsilon, c^\varepsilon) - J(a^o, b^o, c^o) \geq 0, \; \forall \, \varepsilon \in [0, 1],$$

by ε and letting $\varepsilon \downarrow 0$ we obtain the Gâteaux differential of J at $(a^o, b^o, c^o) \in \mathcal{P}_{ad}$ in the direction $(a - a^o, b - b^o, c - c^o)$ satisfying the following inequality:

$$\begin{aligned} &dJ((a^o, b^o, c^o), (a - a^o, b - b^o, c - c^o)) \\ &= \mathbf{E}\left\{\int_0^T < \ell_x(t, x^o(t)), \; y(t) >_{R^n} dt + < \Phi_x(x^o(T)), \; y(T) >_{R^n}\right\} \\ &\geq 0 \; \forall (a, b, c) \in \mathcal{P}_{ad}. \end{aligned} \tag{38}$$

For notational convenience, we introduce the following linear functional:

$$L(y) \equiv \mathbf{E}\left\{\int_0^T < \ell_x(t, x^o(t)), y(t) >_{R^n} dt + < \Phi_x(x^o(T)), y(T) >_{R^n}\right\}. \tag{39}$$

Since $\ell_x(\cdot, x^o(\cdot)) \in L_1^a(I, L_2(\Omega, R^n))$, $y \in B_\infty^a(I, R^n) \subset L_\infty^a(I, L_2(\Omega, R^n))$, $\Phi_x(x^o(T)) \in L_2(\Omega, \mathcal{F}_T, R^n)$ and $y(T) \in L_2(\Omega, \mathcal{F}_T, R^n)$, we conclude that $y \longrightarrow L(y)$ is a continuous linear functional on $B_\infty^a(I, R^n)$. Hence it follows from the above analysis that the functional $\tilde{L}$ given by the composition map,

$$M^{a,b,c} \longrightarrow y \longrightarrow L(y) = (L \circ \Upsilon)(M^{a,b,c}) \equiv \tilde{L}(M^{a,b,c}), \tag{40}$$

is a continuous linear functional on the Hilbert space of semi-martingales $\mathcal{SM}_2$. Thus it follows from representation of semi-martingales and Riesz representation theorem for Hilbert spaces that there exists a triple

$$\{\psi, Q, \varphi\} \in L_2^a(I, L_2(\Omega, R^n)) \times L_2^a(I, L_2(\Omega, \mathcal{L}(R^m, R^n))) \times L_2^a(I, L_2(\pi, L_2(\Omega, R^n)))$$

such that

$$\begin{aligned}\tilde{L}(M^{a,b,c}) &= \mathbf{E}\int_0^T < a(x^o(s)) - a^o(x^o(s)), \psi(s) > ds \\ &+ \mathbf{E}\int_0^T Tr\{(b(x^o(s)) - b^o(x^o(s)))Q^*(s))\} ds \\ &+ \mathbf{E}\int_0^T\int_{V_\delta} < c(x^o(s), \xi) - c^o(x^o(s), \xi), \varphi(s, \xi) > \pi(d\xi) ds.\end{aligned} \tag{41}$$

Hence, it follows from (38), (39), (40) and (41) that

$$dJ((a^o, b^o, c^o), (a - a^o, b - b^o, c - c^o))$$

$$= \mathbf{E}\int_0^T < a(x^o(s)) - a^o(x^o(s)), \psi(s) > ds$$

$$+ \mathbf{E}\int_0^T Tr[(b(x^o(s)) - b^o(x^o(s)))Q^*(s))]ds$$

$$+ \mathbf{E}\int_0^T \int_{V_\delta} < c(x^o(s), \xi) - c^o(x^o(s), \xi), \varphi(s, \xi) > \pi(d\xi)ds$$

$$\geq 0, \forall (a, b, c) \in \mathcal{P}_{ad}. \tag{42}$$

This proves the necessary condition (29). We show that the triple (ψ, Q, φ) is given by the solution of the adjoint equation (30). Since $y \in B_\infty^a(I, R^n) \subset L_\infty^a(I, L_2(\Omega, R^n))$ and $\psi \in L_2^a(I, L_2(\Omega, R^n)) \subset L_1^a$ $(I, L_2(\Omega, R^n))$, the scalar product $< y, \psi >$ is well defined for almost all $t \in I$, $P - a.s.$ Computing the Itô differential of this scalar product we have

$$d < y, \psi > = < dy, \psi > + < y, d\psi > + << dy, d\psi >>, \tag{43}$$

where the third term on the right hand side of the above equation denotes the quadratic variation. Using the stochastic variational equation (36) in the first term on the right hand side of the above expression we obtain

$$< dy, \psi > + < y, d\psi > = \{< Da^o(x^o)ydt + Db^o(x^o; y)dW$$

$$+ \int_{V_\delta} Dc^o(x^o, \xi)yq(d\xi \times dt), \psi >\} + < dM^{a,b,c}, \psi > + < y, d\psi >$$

$$= < y, d\psi + (Da^o(x^o))^* \psi dt + Db^o(x^o; \psi)dW$$

$$+ \int_{V_\delta} (Dc^o(x^o, \xi))^* \psi q(d\xi \times dt) > + < \psi, dM^{a,b,c} >. \tag{44}$$

In order to consider the quadratic variation term $<< dy, d\psi >>$, let us note that the variational equation for y given by (36)-(37) contains (the sum of) four martingale terms as follows,

$$Db^o(x^o;\, y)dW + [b(x^o) - b^o(x^o)]dW$$

$$+ \int_{V_\delta} Dc^o(x^o,\, \xi)yq(d\xi \times dt) + \int_{V_\delta} [c(x^o,\, \xi) - c^o(x^o,\, \xi)]q(d\xi \times dt).$$

In contrast, it follows from the expression (44) that the equation for ψ contains the sum of only two martingale terms given by $-\ Db^o(x^o;\, \psi)dW - \int_{V_\delta}(Dc^o(x^o,\, \xi))^* \psi q(d\xi \times dt)$. Hence it follows from stochastic independence, as stated at the end of Section 3, that the quadratic variation term is given by

$$<< dy,\, d\psi >> = << dy,\, d\psi >>_1 + << dy,\, d\psi >>_2,$$

where $<< dy,\, d\psi >>_1 = - << Db^o(x^o;\, y)dW + [b(x^o) - b^o(x^o)]dW,\, Db^o(x^o,\, \psi)dW >>$. Integrating this we obtain

$$\mathbf{E}\int_I << dy,\, d\psi >>_1 = \mathbf{E}\int_I - Tr\{(Db^o(x^o;\, y))^* Db^o(x^o;\, \psi)\}dt$$

$$- \mathbf{E}\int_I Tr\{[b(x^o) - b^o(x^o)]^*(Db^o(x^o,\, \psi))\}dt$$

$$\equiv \mathbf{E}\int_I <y,\, V_1(x^o)\psi> dt - \mathbf{E}\int_I Tr\{[b(x^o) - b^o(x^o)]^*(Db^o(x^o;\, \psi))\}dt. \tag{45}$$

Note that $V_1(x^o(t)),\, t \in I$, is a negative semi definite symmetric $n \times n$ matrix valued essentially norm bounded random process following from the first component. Considering the second quadratic variation we obtain

$$<< dy,\ d\psi >>_2 = - << \int_{V_\delta} Dc^o(x^o,\ \xi)\, y q(d\xi \times dt),$$

$$\int_{V_\delta} (Dc^o(x^o,\ \xi))^* \psi q(d\xi \times dt) >> - << \int_{V_\delta} [c(x^o,\ \xi)$$

$$- c^o(x^o,\ \xi)] q(d\xi \times dt),\ \int_{V_\delta} (Dc^o(x^o,\ \xi))^* \psi q(d\xi \times dt) >>.$$

Integrating this we have

$$\mathbf{E} \int_I << dy,\ d\psi >>_2 = - \mathbf{E} \int_I \int_{V_\delta} < Dc^o(x^o,\ \xi) y,\ (Dc^o(x^o,\ \xi))^* \psi > \pi(d\xi) dt$$

$$- \mathbf{E} \int_I \int_{V_\delta} < c(x^o,\ \xi) - c^o(x^o,\ \xi),\ (Dc^o(x^o,\ \xi))^* \psi > \pi(d\xi) dt$$

$$\equiv \mathbf{E} \int_I < y,\ V_2(x^o(t)) \psi > dt$$

$$- \mathbf{E} \int_I \int_{V_\delta} < c(x^o,\ \xi) - c^o(x^o,\ \xi),\ (Dc^o(x^o,\ \xi))^* \psi > \pi(d\xi) dt, \qquad (46)$$

where $V_2(x^o(t)),\ t \in I,$ is a negative semi-definite symmetric $n \times n$ matrix valued essentially norm bounded random process following from the first component of the above quadratic variation term. Integrating the identity (43) and substituting the expressions (44), (45) and (46) we arrive at the following expression:

$$\mathbf{E} \int_I d < y,\ \psi > = \mathbf{E} \int_I \{< y,\ d\psi + (Da^o(x^o))^* \psi dt$$

$$+ V_1(x^o) \psi dt + V_2(x^o(t)) \psi dt >\}$$

$$+ \mathbf{E} \int_I \{< y,\ Db^o(x^o;\ \psi) dW + \int_{V_\delta} (Dc^o(x^o,\ \xi))^* \psi q(d\xi \times dt) >\}$$

$$+ \mathbf{E} \int_I \{< \psi,\ dM^{a,b,c} > - Tr\{[b(x^o) - b^o(x^o)] (Db^o(x^o;\ \psi))^*\} dt\}$$

$$- \mathbf{E} \int_I \int_{V_\delta} < c(x^o,\ \xi) - c^o(x^o,\ \xi),\ (Dc^o(x^o,\ \xi))^* \psi > \pi(d\xi) dt. \qquad (47)$$

Then setting

$$d\psi + (Da^o(x^o))^* \psi dt + V_1(x^o)\psi dt + V_2(x^o)\psi dt + Db^o(x^o;\ \psi)dW$$

$$+ \int_{V_\delta} (Dc^o(x^o,\ \xi))^* \psi q(d\xi,\ dt) = -\ \ell_x(t,\ x^o)dt,\ t \in I, \tag{48}$$

in the expression (47) we obtain

$$\mathbf{E}\int_I d < y,\ \psi > = \mathbf{E}\int_I - < y,\ \ell_x(t,\ x^o)dt >$$

$$+ \mathbf{E}\int_I \{< \psi,\ dM^{a,b,c} > - Tr\{[b(x^o) - b^o(x^o)](Db^o(x^o;\ \psi))^*\}dt\}$$

$$- \mathbf{E}\int_I \int_{V_\delta} < c(x^o,\ \xi) - c^o(x^o,\ \xi),\ (Dc^o(x^o,\ \xi))^* \psi > \pi(d\xi)dt. \tag{49}$$

Next, using the identity (37) (characterizing the semi martingale $M^{a,b,c}$) in the above expression we obtain

$$\mathbf{E} < y(T),\ \psi(T) > + \mathbf{E}\int_0^T < y(t),\ \ell_x(t,\ x^o(t)) > dt$$

$$= \mathbf{E}\int_0^T \{< \psi,\ [a(x^o) - a^o(x^o)] > - Tr\{[b(x^o) - b^o(x^o)](Db^o(x^o;\ \psi))^*\},$$

$$- \int_{V_\delta} < [c(x^o,\ \xi) - c^o(x^o,\ \xi)],\ (Dc^o(x^o,\ \xi))^* \psi > \pi(d\xi)\}dt$$

$$+ \mathbf{E}\int_0^T < \psi,\ [b(x^o(t)) - b^o(x^o(t))]dW(t) >$$

$$+ \mathbf{E}\int_0^T \int_{V_\delta} < \psi,\ [c(x^o,\ \xi) - c^o(x^o,\ \xi)] > q(d\xi \times dt). \tag{50}$$

Using stopping time argument one can verify that the last two stochastic integrals in Equation (50) vanish. Hence, for $\psi(T) = \Phi_x(x^o(T))$, the identity (50) reduces to

$$\mathbf{E} < y(T),\ \Phi_x(x^o(T)) > +\ \mathbf{E}\int_0^T < y(t),\ \ell_x(t,\ x^o(t)) > dt$$

$$= \mathbf{E}\int_0^T \{< \psi,\ [a(x^o) - a^o(x^o)] > -\ Tr\Big([b(x^o) - b^o(x^o)](Db^o(x^o;\ \psi))^*\Big),$$

$$-\int_{V_\delta} < [c(x^o,\ \xi) - c^o(x^o,\ \xi)],\ (Dc^o(x^o,\ \xi))^* \psi > \pi(d\xi)\}dt. \qquad (51)$$

Denoting $-\ Db^o(x^o(t);\ \psi(t)) \equiv Q(t),\ t \in I,$ and $-\ (Dc^o(x^o(t),\ \xi))^* \psi \equiv \varphi(t,\ \xi),$ $(t,\ \xi) \in I \times V_\delta,$ and using this in the above expression, one can observe that the right hand member equals $\tilde{L}(M^{a,\,b,\,c})$ as seen in equation (41), while the left hand member equals $L(y)$ as seen in (39), thereby satisfying the required identity (40). As seen above in equation (42), this gives the necessary condition (29). Thus equation (48) with the terminal condition $\psi(T) = \Phi_x(x^o(T))$ is a necessary condition. Hence equation (30), being identical to equation (48) with the terminal condition as stated above, is a necessary condition giving the adjoint equation. Necessary condition (31) needs no proof since it is the system equation (1) corresponding to the optimal drift-diffusion-jump triple $(a^o,\ b^o,\ c^o)$ with x^o being the corresponding solution. This proves all the necessary conditions of optimality. It remains to verify that the $\{\psi,\ Q,\ \varphi\} \in B_\infty^a(I,\ R^n) \times L_\infty^a(I,\ \mathcal{L}(R^m,\ R^n)) \times L_\infty^a(I,\ L_2(\pi,\ R^n)).$ This follows from the fact that the adjoint system (30) has a unique solution $\psi \in B_\infty^a(I,\ R^n)$ and that the Gâteaux differentials of $\{a,\ b,\ c\} \in \mathcal{P}_{ad}$ are uniformly bounded. Hence it follows readily from the expressions for Q and φ, as presented after Equation (29), that $Q \in L_\infty^a(I,\ \mathcal{L}(R^m,\ R^n))$ and $\varphi \in L_\infty^a(I,\ L_2(\pi,\ R^n)).$ This completes the proof. □

6. Convergence of Numerical Algorithm

Here we present an algorithm whereby one can construct the optimal drift-diffusion-jump triple.

Proposition 6.1. *Suppose the assumptions of Theorem* 5.1 *hold. Then there exists* (*and one can construct*) *a sequence* $\{(a^k, b^k, c^k)\} \in \mathcal{P}_{ad}$ *along which the sequence of cost functionals* $\{J(a^k, b^k, c^k)\}$ *converge monotonically to a* (*possibly*) *local minimum*.

Proof. Step 1: Choose a triple $(a^1, b^1, c^1) \in \mathcal{P}_{ad}$ and consider the system equation (31) with (a^o, b^o, c^o) replaced by the triple (a^1, b^1, c^1) and let x^1 denote the corresponding solution.

Step 2: Use the quadruple (a^1, b^1, c^1, x^1) in place of (a^o, b^o, c^o, x^o) in the adjoint equation (30) with $V_1(x^1(t))$ and $V_2(x^1(t))$ given by

$$< V_1(x^1(t))\eta_1, \eta_2 >= - Tr\left\{Db^1(x^1; \eta_1)(Db^1(x^1; \eta_2))^*\right\},$$

$$\forall\ \eta_1, \eta_2 \in R^n, t \in I,$$

$$< V_2(x^1(t))\eta_1, \eta_2 >= -\int_{V_\delta} < Dc^1(x^1, \xi)\eta_1, (Dc^1(x^1, \xi))^*\eta_2 > \pi(d\xi),$$

$$\forall\ \eta_1, \eta_2 \in R^n, t \in I,$$

and solve the adjoint equation giving ψ^1. Then define

$$Q(t) = Q^1(t) \equiv - Db^1(x^1(t); \psi^1(t)), t \in I,$$

$$\varphi(t, \xi) = \varphi^1(t, \xi) = - (Dc^1(x^1(t), \xi))^* \psi^1(t), (t, \xi) \in I \times V_\delta.$$

This step yields the septuple $(a^1, b^1, c^1, x^1, \psi^1, Q^1, \varphi^1)$.

Step 3: At this step, replace the septuple $(a^o, b^o, c^o, x^o, \psi, Q, \varphi)$ by the septuple $(a^1, b^1, c^1, x^1, \psi^1, Q^1, \varphi^1)$ in the inequality (29) giving

$$\mathbf{E}\int_0^T < a(x^1(t)) - a^1(x^1(t)), \psi^1(t) > dt$$

$$+ \mathbf{E}\int_0^T Tr\{(b(x^1(t)) - b^1(x^1(t)))Q^1(t)^*\}dt$$

$$+ \mathbf{E}\int_{I\times V_\delta} < [c(x^1(t), \xi) - c^1(x^1(t), \xi)], \varphi^1(t, \xi) > \pi(d\xi)dt$$

$$\geq 0 \ \forall (a, b, c) \in \mathcal{P}_{ad}. \tag{52}$$

If this inequality holds the septuple $(a^1, b^1, c^1, x^1, \psi^1, Q^1, \varphi^1)$ is optimal. Since an arbitrary choice of the triple (a^1, b^1, c^1) is not expected to be optimal we ignore this and proceed to the next step.

Step 4: Here we choose a new triple (a^2, b^2, c^2) as follows:

$$a^2 \equiv a^1 - \varepsilon\psi^1, \ b^2 \equiv b^1 - \varepsilon Q^1, \ c^2 = c^1 - \varepsilon\varphi^1, \tag{53}$$

where $\varepsilon > 0$ is chosen sufficiently small so that $(a^2, b^2, c^2) \in \mathcal{P}_{ad}$. Then the Gâteaux differential of the cost functional J evaluated at (a^1, b^1, c^1) in the direction $-(\psi^1, Q^1, \varphi^1)$ with step size $\varepsilon > 0$ is given by

$$dJ((a^1, b^1, c^1); -\varepsilon(\psi^1, Q^1, \varphi^1)) = -\varepsilon\mathbf{E}\int_0^T \|\psi^1(t)\|^2 dt$$

$$-\varepsilon\mathbf{E}\int_0^T Tr(Q^1(t)(Q^1(t))^*)dt - \varepsilon\,\mathbf{E}\int_{I\times V_\delta} \|\varphi^1(t, \xi)\|_{R^n}^2 \pi(d\xi)dt. \tag{54}$$

For notational convenience let us define

$$G(\psi^1, Q^1, \varphi^1) \equiv \mathbf{E}\int_0^T \|\psi^1(t)\|^2 dt + \mathbf{E}\int_0^T Tr(Q^1(t)(Q^1(t))^*)dt$$

$$+ \mathbf{E}\int_{I\times V_\delta} \|\varphi^1(t, \xi)\|^2 \pi(d\xi)dt. \tag{55}$$

Using Lagrange formula and the expressions (54) and (55), the cost functional evaluated at (a^2, b^2, c^2) can be approximated as follows:

$$J(a^2, b^2, c^2) = J(a^1, b^1, c^1) - \varepsilon G(\psi^1, Q^1, \varphi^1) + o(\varepsilon). \tag{56}$$

Since $G(\psi^1, Q^1, \varphi^1)$ is positive, it is clear from the above expressions that for $\varepsilon > 0$ sufficiently small

$$J(a^1, b^1, c^1) \geq J(a^2, b^2, c^2).$$

Step 5: Returning to (step1) with the triple (a^2, b^2, c^2) and repeating the process, one generates the sequence $\{(a^k, b^k, c^k)\}_{k\geq 1}$ including the corresponding sequence $\{J(a^k, b^k, c^k)\}_{k\geq 1}$ that satisfies the following train of inequalities:

$$J(a^1, b^1, c^1) \geq J(a^2, b^2, c^2) \geq \cdots \geq J(a^k, b^k, c^k)$$

$$\geq J(a^{k+1}, b^{k+1}, c^{k+1})\cdots.$$

Further, it follows from the assumptions of Theorem 4.3 that

$$\inf\{J(a, b, c), (a, b, c) \in \mathcal{P}_{ad}\} > -\infty.$$

Thus the sequence $\{J(a^k, b^k, c^k)\}_{k\geq 1}$ is a monotone decreasing sequence bounded away from $-\infty$ and hence it converges possibly to a local minimum. This completes the proof. □

Remark 6.2. The results presented in this paper also hold for drift-diffusion-jump triples which are functions of both time and space, $\{a(t, x), b(t, x), c(t, x, \xi)\}$, under the assumption that the family of functions $\{\mathcal{A}_{\alpha, K}, \mathcal{B}_{\alpha, K}, \mathcal{C}_{\beta, \gamma}\}$ satisfy the properties (3), (4) and (5) for each $t \in I$ and that they are also uniformly Hölder continuous exponent $0 < \theta \leq 1$ in $t \in I$.

7. An Alternative Approach

An alternative approach to the inverse problem considered in this paper is to follow Bellman's principle of optimality leading to HJB (Hamilton-Jacobi-Bellman) equation. We present this briefly and discuss the merits and demerits of the two approaches.

Choose any drift-diffusion-jump triple $(a, b, c) \in \mathcal{P}_{ad}$ and $(t, x) \in I \times R^n$ and let the process $\{\xi_{t,x}(s), s \in (t, T]\}$ denote the solution of the following SDE corresponding to the triple (a, b, c)

$$d\xi_{t,x}(s) = a(s, \xi_{t,x}(s))ds + b(s, \xi_{t,x}(s))dW(s)$$

$$+ \int_{V_\delta} c(s, \xi_{t,x}(s), \eta)q(d\eta \times ds), \; \xi_{t,x}(t) = x, \; s \in (t, T]. \tag{57}$$

Let $\Psi(t, x)$ denote the value function defined as follows:

$$\Psi(t, x) \equiv \inf_{\mathcal{P}_{ad}} \mathbf{E}\left\{\int_t^T \ell(s, \xi_{t,x}(s))ds + \Phi(\xi_{t,x}(T))\right\}, \tag{58}$$

and let $D\Psi \equiv \{(\partial_{x_i}\Psi), i = 1, 2, \cdots, n\}$ and $D^2\Psi \equiv \{(\partial^2_{x_i, x_j}\Psi), i, j = 1, 2, 3, \cdots, n\}$ denote the first and second partials of Ψ with respect to the state variables. Following the principle of optimality and using the properties of the Wiener process and the Poisson random measure one can verify that Ψ satisfies the following backward integro-partial differential equation,

$$-\frac{\partial}{\partial t}\Psi(t, x) = \mathcal{H}_{\pi}(t, x, \Psi, D\Psi, D^2\Psi), \ (t, x) \in [0, T) \times R^n$$

$$\Psi(T, x) = \Phi(x), \ x \in R^n, \tag{59}$$

where the function $\mathcal{H}_{\pi}$ is given by

$$\mathcal{H}_{\pi}(t, x, \Psi, D\Psi, D^2\Psi)$$

$$\equiv \inf_{(a,b,c)\in\mathcal{P}_{ad}} \Big\{ \ell(t, x) + \langle a(t, x), D\Psi(t, x)\rangle_{R^n}$$

$$+ (1/2)Tr(b^*(t, x)D^2\Psi(t, x)b(t, x)) + \int_{V_\delta} (\Psi(t, x + c(t, x, \eta))$$

$$- \Psi(t, x) - \langle D\Psi(t, x), c(t, x, \eta)\rangle_{R^n})\pi(d\eta)\Big\}. \tag{60}$$

It is very satisfying to note that a complex inverse problem is transformed into an elegant and apparently simple problem requiring only the solution of a partial differential equation. But, in fact it is a formidable problem requiring proof of existence of solution of a highly nonlinear second order integro-partial differential equation on unbounded domain $I \times R^n$. With substantial efforts one may be able to prove existence of a viscosity solution generalizing the notion of classical solution. Given that such a solution exists, the next problem is to solve this equation numerically on an unbounded domain $I \times R^n$ giving Ψ^o. This is also a very challenging numerical problem. Using this solution one has to determine the triple $(a_o, b_o, c_o) \in \mathcal{P}_{ad}$ that satisfies the following identity:

$$\ell(t, x) + \langle a_o, D\Psi^o\rangle_{R^n} + (Tr(b_o^* D^2\Psi^o b_o)) + L_{c_o}(\Psi^o)$$

$$= \mathcal{H}_{\pi}(t, x, \Psi^o, D\Psi^o, D^2\Psi^o), \ (t, x) \in I \times R^n, \tag{61}$$

where

$$L_{c_o}(\Psi^o) = \int_{V_\delta} (\Psi^o(t, x + c_o(t, x, \eta)) - \Psi^o(t, x)$$

$$- < D\Psi^o(t, x), c_o(t, x, \eta) >_{R^n})\pi(d\eta).$$

With little reflection, one will notice that this last step is also not so trivial. The optimal cost is given by $\Psi^o(0, x)$ provided the initial state x is deterministic. In case the initial state is random with probability measure μ^0, the optimal cost is given by

$$J(a_o, b_o, c_o) = \int_{R^n} \Psi^o(0, x)\mu^0(dx).$$

Briefly this is the HJB approach presenting significant theoretical and numerical challenges.

In contrast, based on the theory developed in this paper one can determine the optimal (a_o, b_o, c_o) following the numerical steps presented in Proposition 6.1. This is a successive approximation technique and not too difficult to program on computers using Matlab. The necessary steps are clearly outlined in the Proposition mentioned above. While carrying out these steps one has to solve a pair of stochastic differential equations forward and backward in time and use this solution to determine the optimal direction of decent which is given by the directional derivative of the cost functional evaluated at the preceding choice of $(a, b, c) \in \mathcal{P}_{ad}$. Using this direction one constructs a new element (a, b, c) which guarantees reduction of the cost functional. This new element is then used to repeat the steps from the beginning thereby reducing the cost functional at every stage. Numerical techniques based on Monte Carlo scheme for solving stochastic differential equations are well known. This technique requires programs generating standard Brownian motion and the Poisson random process having Lèvy measure

π. The method is computationally intensive as it requires solving the state and the adjoint system repeatedly for a great number of times to achieve a desired accuracy. Though numerically intensive it is practically feasible and relatively simpler.

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