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STRUCTURES INDUCED ON TRANSVERSALS TO A SUBGROUP OF A GROUP AND HYPERGROUPS OVER THE GROUP

S. H. DALALYAN

Yerevan State University
Armenia
e-mail: dalalyan@ysu.am

Abstract

For a group, arbitrary its subgroup and a transversal to this subgroup, using the binary operation of the group, by a standard construction four induced mappings are determined, some their properties are established and a concept of hypergroups over the group is introduced, which generalizes the concept of quotient-groups. By a construction, called the exact product, associated with a hypergroup over the group (which generalizes the semidirect product of two groups), a universal property of the standard construction is proved. It is proved that the hypergroups over the group generalize also the fields and the linear spaces over the field. A necessary and sufficient condition for a group to be the multiplicative group of a field is established in the language of hypergroups over the group.

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1. Introduction

In this article a new algebraic object is introduced, which is called a hypergroup over the group. The notion of a hypergroup over the group generalizes and unifies the concepts of a group, quotient-group, field, linear space over the field. We give a *standard construction* of a hypergroup over the group, which generalizes the construction of a quotient-group for an arbitrary subgroup of a group. In this case, the (left or right) transversals to the subgroup of the group are used. The binary operation of the group induces four structures on the corresponding transversal:

(1) A one-sided quasigroup with a one-sided neutral element;

(2) An action of the subgroup on this quasigroup;

(3) An action of the quasigroup on the subgroup;

(4) A scalar product on the quasigroup with values in the subgroup.

In Theorem 1, we prove some relations connecting these structures, which are used as axioms for the definition of hypergroups over the group.

In Theorem 2, we prove a universal property of the standard construction: up to isomorphism, any hypergroup over the group can be obtained by this construction.

Then we give some applications of hypergroups over the group. It is proved that, up to isomorphism, all fields and all linear spaces over the field can be obtained as hypergroups over the group.

Finally, a necessary and sufficient condition for a group to be the multiplicative group of a field is established.

2. Main Results

1. Let (G, H, M) be a triple consisting of a group G, of an arbitrary subgroup H of G and of a section M of the partition (right coset) $\{Ha, a \in G\}$ of G by H. Such a section M is called a (*right*) *transversal* to the subgroup H of the group G. Further, we use lowercase (small) letters of the Greek alphabet for denotation of elements of H and use small letters of the Latin alphabet for denotation of elements of M.

Any (right) transversal M to the subgroup H of the group G is a (*right*) *complementary set* to the subgroup H of G and conversely, ([1]): any element x of G is uniquely represented as a product $x = \alpha \cdot a$, where $\alpha \in H, a \in M$. Therefore for any elements $\alpha \in H, a, b \in M$, the elements

$$^{a}\alpha,\ (a, b) \in H,\ a^{\alpha},\ [a, b] \in M$$

are uniquely determined by conditions

$$\text{(F0)} \quad a \cdot \alpha = {}^{a}\alpha \cdot a^{\alpha}, \quad a \cdot b = (a, b) \cdot [a, b].$$

A system of mappings $\Omega = (\Phi, \Psi, \Xi, \Lambda)$ is associated with any triple (G, H, M), where

(**B1**) $\quad \Phi : M \times H \to M, \quad \Phi(a, \alpha) := a^{\alpha},$

(**B2**) $\quad \Psi : M \times H \to H, \quad \Psi(a, \alpha) := {}^{a}\alpha,$

(**B3**) $\quad \Xi : M \times M \to M, \quad \Xi(a, b) := [a, b],$

(**B4**) $\quad \Lambda : M \times M \to H, \quad \Lambda(a, b) := (a, b).$

Theorem 1. *The system of mappings* $\Omega = (\Phi, \Psi, \Xi, \Lambda)$ *has the following basic properties*:

(P1) *The mapping* Ξ *determines on* M *a structure of a right quasigroup with a left neutral element*, *which is unique and is denoted* o.

(P2) *The mapping* Φ *is a* (*right*) *action of the group* H *on the set* M.

(P3) *The mapping* Ψ *sends the subset* $\{o\} \times H$ *on* H.

(P4) *The system of mappings* $\Omega = (\Phi, \Psi, \Xi, \Lambda)$ *satisfies the following relations*:

(A1) $$^{a}(\alpha \cdot \beta) = {}^{a}\alpha \cdot {}^{a^{\alpha}}\beta,$$

(A2) $$[a, b]^{\alpha} = [a^{{}^{b}\alpha}, b^{\alpha}],$$

(A3) $$(a, b) \cdot {}^{[a,b]}\alpha = {}^{a}({}^{b}\alpha) \cdot (a^{{}^{b}\alpha}, b^{\alpha}),$$

(A4) $$[[a, b], c] = [a^{(b,c)}, [b, c]],$$

(A5) $$(a, b) \cdot ([a, b], c) = {}^{a}(b, c) \cdot (a^{(b,c)}, [b, c]).$$

Definition 1. A (*right*) *hypergroup over the group* (abbreviated to HOG) is a pair (M, H), consisting of a set M and of a group H, equipped with a *system of structural mappings* $\Omega = (\Phi, \Psi, \Xi, \Lambda)$, satisfying the conditions (P1)-(P4). Such a hypergroup over a group is denoted M_H.

Initially, the definition of the notion of a hypergroup over the group appeared in [5], then it was improved on (for example, in [6] and [7]). This concept was already used to obtain: A theorem of type of the Cayley's theorem for right quasigroups ([6]), a generalization of the Schreier's theorem on extensions of groups ([7]), and some results, describing hypergroups over the group ([8], [9], [10]).

In this paper by hypergroups over the group a description of multiplicative groups of fields is given.

Remark 1. It can be proved that the conditions (P1), (P2), (P3), (A1), (A2), (A3), (A4), (A5) are independent.

Remark 2. In algebra the term "hypergroup" is already used for an other object (see Marty [2] and Wall [3], as well as [4]). We propose completely different term "hypergroup over the group". It is consonant with the term "linear space over the field" and generalizes this concept. Since we never use the concept "hypergroup" in the sense of [2] and [3], sometimes the hypergroups over the group M_H are called shortly hypergroups.

Remark 3. By considering the left coset $G / H = \{aH, a \in G\}$ and its section M a left hypergroup ${}_H M$ over the group can be defined and a dual theory of left hypergroups over the group can be developed. Further in this paper only right HOG's are considered and the word "right" is often omitted.

2. The method described in item 1 for constructing a hypergroup over the group, starting with a triple (G, H, M), is called the *standard construction* of an HOG. This construction has the following *universal property*: up to isomorphism, any hypergroup over the group can be obtained by the standard construction. More exactly, the following theorem holds.

Theorem 2. *Let* M_H *be a hypergroup over the group*, ε *be the neutral element of the group* H *and* $\theta = \Lambda(o, o)^{-1}$. *Consider the two-letter words*

$$\alpha a, \quad \alpha \in H, a \in M,$$

and the multiplication operation

$$\text{(F1)} \qquad \alpha a \cdot \beta b = (\alpha \cdot {}^{a}\beta \cdot (a^{\beta}, b))[a^{\beta}, b].$$

Then

(a) *the set of all two-letter words together with their multiplication operation is a group* (*which we denoted further by* G);

(b) *the maps*

$$f_0 : H \to G, \quad \alpha \mapsto \overline{\alpha} := (\alpha \cdot \theta)o, \quad f_1 : M \to G, \quad a \mapsto \overline{a} := \varepsilon a$$

are, *respectively*, *a monomorphism of groups and an injection of sets*, *moreover*, *the set* $\overline{M} := f_1(M)$ *is a* (*right*) *complementary set to the subgroup* $\overline{H} := f_0(H)$;

(c) *the pair* $\bar{f} = (\bar{f}_0, \bar{f}_1)$ *with*

$$\bar{f}_0 : H \to \overline{H}, \quad \bar{f}_1 : M \to \overline{M},$$

induced, *respectively*, *by* f_0 *and* f_1, *gives an isomorphism*

$$M_H \to \overline{M}_{\overline{H}},$$

where $\overline{M}_{\overline{H}}$ *is the HOG*, *obtained by the standard construction from the triple* $(G, \overline{H}, \overline{M})$.

Definition 2. For any hypergroup M_H over the group, the constructed triple $(G, \overline{H}, \overline{M})$ is called an *exact product*, associated with M_H.

Definition 3. Let M_H and $M'_{H'}$ be HOG's with systems of structural mappings $\Omega = (\Phi, \Psi, \Xi, \Lambda)$ and $\Omega' = (\Phi', \Psi', \Xi', \Lambda')$, respectively. A *morphism of HOG's*

$$f : M_H \to M'_{H'}$$

is a pair $f = (f_0, f_1)$, consisting of a homomorphism of groups $f_0 : H \to H'$, and of a map of sets $f_1 : M \to M'$ such that the following diagrams commute:

$$\begin{array}{ccc} M \times H & \xrightarrow{\Phi} & M \\ {\scriptstyle f_1 \times f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ M' \times H' & \xrightarrow{\Phi'} & M' \end{array} \qquad \begin{array}{ccc} M \times H & \xrightarrow{\Psi} & H \\ {\scriptstyle f_1 \times f_0}\downarrow & & \downarrow{\scriptstyle f_0} \\ M' \times H' & \xrightarrow{\Psi'} & H' \end{array}$$

$$\begin{array}{ccc} M \times M & \xrightarrow{\Xi} & M \\ {\scriptstyle f_1 \times f_1}\downarrow & & \downarrow{\scriptstyle f_1} \\ M' \times M' & \xrightarrow{\Xi'} & M' \end{array} \qquad \begin{array}{ccc} M \times M & \xrightarrow{\Lambda} & H \\ {\scriptstyle f_1 \times f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ M' \times M' & \xrightarrow{\Lambda'} & H' \end{array}$$

In this case, it is said that the pair $f = (f_0, f_1)$ preserves (respects) the system of structural mappings.

An *isomorphism* is a reversible morphism.

Now Theorem 2 can be reformulated as follows.

Any hypergroup M_H *over the group is isomorphic to the hypergroup* $\overline{M}_{\overline{H}}$ *over the group, obtained by the standard construction from the exact product* $(G, \overline{H}, \overline{M})$, *associated with* M_H.

3. The groups, quotient-groups, fields and linear spaces over the field can be obtained as special cases of hypergroups over the group.

Definition 4. Let M_H be an HOG, $\Omega = (\Phi, \Psi, \Xi, \Lambda)$ be its system of structural mappings. The structural mappings Φ, Ψ, Λ are called *trivial*, if, respectively,

$$a^{\alpha} = a, \quad {}^{a}\alpha = \alpha, \quad (a, b) = \varepsilon,$$

for any $a, b \in M$, $\alpha \in H$.

For example, a $\Psi\Lambda$-*trivial HOG* is a hypergroup over the group with trivial structural mappings Ψ and Λ.

An HOG M_H is reduced to a group (M, Ξ) and trivial mappings $\Phi, \Psi, \Lambda,$ if H is a trivial group.

In Proposition 1, some well-known results of group theory are expressed in the language of hypergroups over the group. In this proposition the HOGs are described, which are reduced to a quotient-group.

Proposition 1. *The subgroup H of a group G is normal if and only if for a transversal (consequently, for any transversal) M to H the structural mapping Φ is trivial. In this case the sets of both left and right cosets of G by H are equal and for any section M of the corresponding partition the structural mapping Ξ determines on M isomorphic groups. This group (M, Ξ) is a subgroup of G if and only if the structural mapping Λ is trivial. This subgroup is a normal subgroup of G if and only if the structural mapping Ψ is trivial.*

We emphasize that in general case, when the subgroup H is not normal, for two transversals M and M' to H in G the quasigroups (M, Ξ) and (M', Ξ') are not isomorphic (examples in [8], [9]).

There exists a canonical way to associate an HOG with any field k.

Let k be an arbitrary field, $H = k^*$ be its multiplicative group, M be the basis set of the field k. Consider the pair (M, H) together with a system of mappings $\Omega = (\Phi, \Psi, \Xi, \Lambda)$, where Ξ is given by the addition operation of the field k, Ψ and Λ are trivial mappings and Φ is defined by the multiplication operation of $k : \Phi(a, \alpha) = a^{\alpha} := a \cdot \alpha$. Then we get a $\Psi\Lambda$-trivial hypergroup M_H over the group with an abelian group (M, Ξ). This HOG and the field k are called *canonically associated*. A necessary and sufficient condition, when a $\Psi\Lambda$-trivial HOG with an abelian group (M, Ξ) is canonically associated with a field, is given in Proposition 2.

Definition 5. Let H be a multiplicative group (in general, non-abelian). A *monoid* H_ω, *associated with* H *and with a zero element* $\omega \notin H$, is the set $H_\omega := H \cup \{\omega\}$ together with a binary operation $\alpha \cdot \beta$, as in H for $\alpha, \beta \in H$ and $\alpha \cdot \omega = \omega = \omega \cdot \alpha$, for any $\alpha \in H_\omega$.

Let $\Phi : M \times H \to M$ be an action of the group H on a set M. An *extension of* Φ *by connecting the zero element* ω is a mapping $\Phi_\omega : M \times H_\omega \to M$ which is defined, using the mapping Φ and a condition $a^\omega = o$, for any $a \in M$.

The proof of the following proposition is evident.

Proposition 2. *Let* M_H *be a* $\Psi\Lambda$*-trivial HOG with an abelian group* (M, Ξ). *Then* M_H *is canonically associated with a field if and only if for an element* $a \in M$ *(consequently, for any element* $a \in M$, $a \neq o$*) by* $\alpha \to a^\alpha$ *a bijection is determined between the orbit* $O_H(a) := \{a^\alpha, \alpha \in H\}$ *and the set* $M^* = M \setminus \{o\}$.

Then by considering such a bijection $H \to M^*$ *and its extension* $H_\omega \to M, \omega \to o$, *the abelian group operation* Ξ *of* M *is transferred on* H_ω, *and the multiplication operation is transferred from* H_ω *to* M. *Thus the isomorphic fields* $(M, \Xi, \cdot)$ *and* $(H_\omega, \Xi_\omega, \cdot)$ *are obtained, for which the associated hypergroups over the group are isomorphic to* M_H.

Corollary 1. *A group* H *is the multiplicative group of a field* k *if and only if there exists an abelian group* M *with an effective action*

$$\Phi : M \times H \to M, \quad \Phi(a, \alpha) := a^\alpha$$

of the multiplicative group H *on the additive group* M *by automorphisms, such that for a non-trivial element* $a \in M$ *the following condition is satisfied: the map* $\alpha \to a^\alpha$ *determines a bijection between* H *and non-trivial elements of the group M.*

Proof. Indeed, we can define a $\Psi\Lambda$-trivial HOG with an abelian group (M, Ξ), satisfying the condition of Proposition 2, and H will be isomorphic to the multiplicative groups of fields $(M, \Xi, \cdot) \cong (H_\omega, \Xi_\omega, \cdot)$.

□

Remark 4. One can obtain the results similar to the results of Proposition 2, if consider instead of a field k a linear space L over the field.

Remark 5. This article was already written when the work [11] became known to the author. As in the present paper, the author of the paper [11] use the transversals to the subgroup of a group to introduce in the consideration a new algebraic object, which he calls c-groupoids. iA c-groupoid is a particular case of a hypergroup over the group, so called *unitary* hypergroup over the group (see [10]). On the other hand, the homomorphism of c-groupoids are defined more generally, than the morphisms of hypergroups over the group. Thus, the categories of c-groupoids and hypergroups over the group are completely different. Note also that we use a new system of denotations, which, in our opinion, is much more convenient.

3. Proofs of the Main Results

Proof of Theorem 1

We use the associative law of group G in the following particular forms:

$$(a \cdot \alpha) \cdot \beta = a \cdot (\alpha \cdot \beta), \quad (a \cdot b) \cdot \alpha = a \cdot (b \cdot \alpha), \quad (a \cdot b) \cdot c = a \cdot (b \cdot c),$$

where $a, b, c \in M$ and $\alpha, \beta \in H$. Applying the relations (F0) to both sides of these equalities, we obtain

$$(a \cdot \alpha) \cdot \beta = {}^{a}\alpha \cdot a^{\alpha} \cdot \beta = ({}^{a}\alpha \cdot {}^{a^{\alpha}}\beta) \cdot (a^{\alpha})^{\beta},$$

$$a \cdot (\alpha \cdot \beta) = {}^{a}(\alpha \cdot \beta) \cdot a^{\alpha \cdot \beta},$$

$$(a \cdot b) \cdot \alpha = (a, b) \cdot [a, b] \cdot \alpha = ((a, b) \cdot {}^{[a, b]}\alpha) \cdot [a, b]^{\alpha},$$

$$a \cdot (b \cdot \alpha) = a \cdot {}^{b}\alpha \cdot b^{\alpha} = {}^{a}({}^{b}\alpha) \cdot a^{{}^{b}\alpha} \cdot b^{\alpha} = ({}^{a}({}^{b}\alpha) \cdot (a^{{}^{b}\alpha} \cdot b^{\alpha})) \cdot [a^{{}^{b}\alpha} \cdot b^{\alpha}],$$

$$(a \cdot b) \cdot c = (a, b) \cdot [a, b] \cdot c = ((a, b) \cdot ([a, b], c)) \cdot [[a, b], c],$$

$$a \cdot (b \cdot c) = a \cdot (b, c) \cdot [b, c] = {}^{a}(b, c) \cdot a^{(b, c)} \cdot [b, c] = ({}^{a}(b, c) \cdot (a^{(b, c)}, [b, c])) \cdot [a^{(b, c)}, [b, c]].$$

Since M is a right complementary set to the subgroup H, we get, respectively, the pairs of relations $(a^{\alpha})^{\beta} = a^{\alpha \cdot \beta}$ (a part of the property (P2)) and (A1), (A2), (A3), (A4) and (A5).

Let ε be the neutral element of the group H and

$$\text{(F2)} \qquad \varepsilon = \theta \cdot o$$

be its decomposition into the product of elements $\theta \in H$ and $o \in M$. Then we have for any $a \in M$ the following relations:

$$\varepsilon \cdot a = a \cdot \varepsilon = {}^{a}\varepsilon \cdot a^{\varepsilon},$$

$$(\alpha \cdot \theta) \cdot o = \theta \cdot o \cdot \alpha = (\theta \cdot {}^{o}\alpha) \cdot o^{\alpha},$$

$$(o, a) \cdot [o, a] = o \cdot a = \theta^{-1} \cdot a,$$

$$(a, o) \cdot [a, o] = a \cdot o = a \cdot \theta^{-1} = {}^{a}(\theta^{-1}) \cdot a^{\theta^{-1}}.$$

Consequently, $a^{\varepsilon} = a$ (the second part of the property (P2)), $[o, a] = a$ (a part of the property (P1)), and also we get the relations that we collected in Proposition 3.

Proposition 3. *The relations* (A6)-(A11) *hold for structural mappings of HOG's, arising by standard construction*:

(A6) $$^{a}\varepsilon = \varepsilon,$$

(A7) $$o^{\alpha} = o,$$

(A8) $$^{o}\alpha = \theta^{-1} \cdot \alpha \cdot \theta,$$

(A9) $$(o,\ a) = \theta^{-1},$$

(A10) $$[a,\ o] = a^{\theta-1},$$

(A11) $$(a,\ o) = {}^{a}(\theta^{-1}).$$

The property (P3) follows immediately from relation (A8).

To complete the proof of Theorem 1 it remains to verify that $(M,\ \Xi)$ is a right quasigroup, i.e., the equation $[x,\ a] = b$ has a unique solution in M for any $a,\ b \in M$.

Lemma 1. *For elements* $x,\ a,\ b \in M$ *the relations* $[x,\ a] = b$ *and* $x \in M \cap H(b \cdot a^{-1})$ *are equivalent.*

Proof. Indeed

$$[x,\ a] = b \Leftrightarrow x \cdot a = (x,\ a) \cdot [x,\ a] = (x,\ a) \cdot b \Leftrightarrow x = (x,\ a) \cdot b \cdot a^{-1}$$

$$\Leftrightarrow x \in M \cap H(b \cdot a^{-1}).$$

□

Finally, note that $M \cap H(b \cdot a^{-1})$ has a unique element, since M is a right transversal to H. Theorem 1 is completely proved.

In the proof of Theorem 2 the properties (A6)-(A11) are used in the case of an arbitrary hypergroup over the group. Therefore, it is necessary to check that they can be derived directly from the conditions (P1)-(P4).

Proposition 4. *The properties* (A6)-(A11) *follow from the properties* (P1)-(P4), *if define* $\theta = (o, o)^{-1}$. *Thus, any HOG has the properties* (A6)-(A11).

Proof. (A6) Substituting $\alpha = \varepsilon$ in (A1) and using (P2), we get ${}^{a}(\varepsilon \cdot \beta) = {}^{a}\varepsilon \cdot {}^{a^{\varepsilon}}\beta$ and ${}^{a}\beta = {}^{a}\varepsilon \cdot {}^{a}\beta$. Consequently, ${}^{a}\varepsilon = \varepsilon$.

(A7) Substituting $a = o$ in (A2), using (P1) and (P3), we get $[o, b]^{\alpha} = [o^{{}^{b}\alpha}, b^{\alpha}]$ and $b^{\alpha} = [o^{{}^{b}\alpha}, b^{\alpha}]$. Hence $o^{\beta} = o$.

(A8) Substituting $a = b = o$ in (A3), using (P1), (A7), (P3), (A9), we obtain

$$(o, o) \cdot {}^{[o, o]}\alpha = {}^{o}({}^{o}\alpha) \cdot (o^{{}^{o}\alpha}, o^{\alpha})$$

and $\theta^{-1} \cdot \beta = {}^{o}\beta \cdot \theta^{-1}$.

(A9) Substituting $a = b = o$ in (A5), using (A8), (A7), (P1), we get

$$(o, o) \cdot ([o, o], c) = {}^{o}(o, c) \cdot (o^{(o, c)}, [o, c])$$

and $\theta^{-1} = \theta^{-1} \cdot (o, c) \cdot \theta$.

(A10) Substituting $b = o$ in (A4), using (P1) and (A9), we get $[[a, o], c] = [a^{(o, c)}, [o, c]]$ and $[a, o] = a^{\theta^{-1}}$.

(A11) Substituting $b = c = o$ in (A5) and using (A10), we obtain

$$(a, o) \cdot ([a, o], o) = {}^{a}(o, o) \cdot (a^{(o, o)}, [o, o])$$

and $(a, o) = {}^{a}(\theta^{-1})$. □

Proof of Theorem 2

(a) The multiplication operation of two-letter words is associative, because

$$(\alpha a\cdot\beta b)\cdot\gamma c=(\alpha\cdot{}^{a}\beta\cdot(a^{\beta},\,b))[a^{\beta},\,b]\cdot\gamma c=(\alpha\cdot{}^{a}\beta\cdot(a^{\beta},\,b)\cdot{}^{[a^{\beta},b]}\gamma\cdot$$

$$([a^{\beta},\,b]^{\gamma},\,c)[[a^{\beta},\,b]^{\gamma},\,c],$$

$$\alpha a\cdot(\beta b\cdot\gamma c)=\alpha a\cdot(\beta\cdot{}^{b}\gamma\cdot(b^{\gamma},\,c))[b^{\gamma},\,c]$$

$$=(\alpha\cdot{}^{a}(\beta\cdot{}^{b}\gamma\cdot(b^{\gamma},\,c))\cdot(a^{\beta\cdot{}^{b}\gamma\cdot(b^{\gamma},c)},\,[b^{\gamma},\,c]))[a^{\beta\cdot{}^{b}\gamma\cdot(b^{\gamma},c)},\,[b^{\gamma},\,c]].$$

and here

$$(\alpha\cdot{}^{a}(\beta\cdot{}^{b}\gamma\cdot(b^{\gamma},\,c))\cdot(a^{\beta\cdot{}^{b}\gamma\cdot(b^{\gamma},c)},\,[b^{\gamma},\,c])$$

$$=\alpha\cdot{}^{a}\beta\cdot{}^{a^{\beta}}({}^{b}\gamma)\cdot{}^{a^{\beta\cdot{}^{b}\gamma}}(b^{\gamma},\,c)\cdot(a^{\beta\cdot{}^{b}\gamma\cdot(b^{\gamma},c)},\,[b^{\gamma},\,c])$$

$$=\alpha\cdot{}^{a}\beta\cdot(a^{\beta},\,b)\cdot{}^{[a^{\beta},b]}\gamma\cdot((a^{\beta})^{{}^{b}\gamma},\,b^{\gamma})^{-1}\cdot{}^{a^{\beta\cdot{}^{b}\gamma}}(b^{\gamma},\,c)\cdot(a^{\beta\cdot{}^{b}\gamma\cdot(b^{\gamma},c)},\,[b^{\gamma},\,c])$$

$$=\alpha\cdot{}^{a}\beta\cdot(a^{\beta},\,b)\cdot{}^{[a^{\beta},b]}\gamma\cdot([a^{\beta\cdot{}^{b}\gamma},\,b^{\gamma}],\,c)$$

$$=((\alpha\cdot{}^{a}\beta\cdot(a^{\beta},\,b))\cdot{}^{[a^{\beta},b]}\gamma\cdot([a^{\beta},\,b]^{\gamma},\,c))$$

according to (A1), (A3), (A5), (A2), and

$$[[a^{\beta},\,b]^{\gamma},\,c]=[a^{\beta\cdot{}^{b}\gamma\cdot(b^{\gamma},c)},\,[b^{\gamma},\,c]]$$

according to (A2), (A4).

This operation has a left neutral element θo, since

$$\theta o\cdot\alpha a=(\theta\cdot{}^{o}\alpha\cdot(o^{\alpha},\,a))[o^{\alpha},\,a]=\alpha a$$

according to (A8), (A7), (A9) and (P1).

Finally, we check that for any $\alpha a, \beta b$ there exists a unique ξx such that $\xi x \cdot \alpha a = \beta b(\alpha, \beta, \xi \in H, a, b, x \in M)$, i.e., that $(G, \cdot)$ is a right quasigroup. The above equation is equivalent to the relation

$$(\xi \cdot {}^{x}\alpha \cdot (x^{\alpha}, a))[x^{\alpha}, a] = \beta b,$$

and, consequently, to the pair of relations

$$\xi \cdot {}^{x}\alpha \cdot (x^{\alpha}, a) = \beta, \quad [x^{\alpha}, a] = b.$$

By using a denotation b / a for unique solution of equation $[y, a] = b$, we get that the last system of two equations has exactly one solution

$$x = (b / a)^{\alpha^{-1}}, \quad \xi = \beta \cdot ({}^{x}\alpha \cdot (b / a, a))^{-1}.$$

Thus, G is an associative right quasigroup with a left neutral element, consequently, it is a group.

(b) Evidently, f_0 and f_1 are injective maps. Moreover, f_0 is a homomorphism of groups, since

$$(\alpha \cdot \theta)o \cdot (\beta \cdot \theta)o = (\alpha \cdot \theta \cdot {}^{o}(\beta \cdot \theta) \cdot (o^{\beta \cdot \theta}, o))[o^{\beta \cdot \theta}, o] = ((\alpha \cdot \beta) \cdot \theta)o$$

according to (F1), (A8), (A7), (P1). Thus, f_0 is a monomorphism of groups and $\overline{H} = f_0(H)$ is a subgroup of G, isomorphic to H.

The set $\overline{M} = f_1(M)$ is a right complementary set to the subgroup $\overline{H}$ of G, since for every two-letter word $\alpha a \in G$, there is a decomposition

$$(\alpha \cdot \theta)o \cdot \varepsilon a = (\alpha \cdot \theta \cdot {}^{o}\varepsilon \cdot (o^{\varepsilon}, a))[o^{\varepsilon}, a] = \alpha a$$

according to (F1), (A8), (A7), (A9), (P1), and here $(\alpha \cdot \theta)o \in \overline{H}, \varepsilon a \in \overline{M}$.

Such a decomposition is unique, because $\overline{H} \cap \overline{M}$ has a unique element, namely, $\theta^{-1}o$.

(c) Consider the HOG $\overline{M}_{\overline{H}}$, associated with the triple $(G, \overline{H}, \overline{M})$. The system of structural mappings $\overline{\Omega} = (\overline{\Phi}, \overline{\Psi}, \overline{\Xi}, \overline{\Lambda})$

(B1) $\overline{\Phi} : \overline{M} \times \overline{H} \to \overline{M}, \quad \overline{\Phi}(\overline{a}, \overline{\alpha}) := \overline{a}^{\overline{\alpha}},$

(B2) $\overline{\Psi} : \overline{M} \times \overline{H} \to \overline{H}, \quad \overline{\Psi}(\overline{a}, \overline{\alpha}) := {}^{\overline{a}}\overline{\alpha},$

(B3) $\overline{\Xi} : \overline{M} \times \overline{M} \to \overline{M}, \quad \overline{\Xi}(\overline{a}, \overline{b}) := [\overline{a}, \overline{b}],$

(B4) $\overline{\Lambda} : \overline{M} \times \overline{M} \to \overline{H}, \quad \overline{\Lambda}(\overline{a}, \overline{b}) := (\overline{a}, \overline{b})$

is determined from the relations

$$(\overline{\text{F0}}) \qquad \overline{a} \cdot \overline{\alpha} = {}^{\overline{a}}(\overline{\alpha}) \cdot (\overline{a})^{\overline{\alpha}}, \quad \overline{a} \cdot \overline{b} = (\overline{a}, \overline{b}) \cdot [\overline{a}, \overline{b}],$$

where $\overline{a}, \overline{b}, (\overline{a})^{\overline{\alpha}}, [\overline{a}, \overline{b}] \in \overline{M}, \overline{\alpha}, {}^{\overline{a}}(\overline{\alpha}), (\overline{a}, \overline{b}) \in \overline{H}.$

Since $\bar{f}_0$ is a group monomorphism, $\bar{f}_1$ is an injective map, the proof of the item (c) is reduced to checking that the pair $\bar{f} = (\bar{f}_0, \bar{f}_1)$ respects the structural mappings, that is four diagrams, corresponding to the four structural mappings, in Definition 2 commute. This condition is reduced to the relations

$$\text{(F3)} \qquad \overline{a^{\alpha}} = \overline{a}^{\overline{\alpha}}, \quad \overline{{}^{a}\alpha} = {}^{\overline{a}}\overline{\alpha}. \quad \overline{[a, b]} = [\overline{a}, \overline{b}], \quad \overline{(a, b)} = (\overline{a}, \overline{b}).$$

Lemma 2. *For arbitrary* $\alpha \in H$, $a, b \in M$

$$\overline{a} \cdot \overline{\alpha} = \overline{{}^{a}\alpha} \cdot \overline{a^{\alpha}}, \quad \overline{a} \cdot \overline{b} = \overline{(a, b)} \cdot \overline{[a, b]}.$$

Proof. Using the definitions of maps f_0 and f_1, (F1), (A1), (A11), (A10), (A1), (A6) (P2) to the left part of the first equality of Lemma 2, we obtain

$$\overline{a} \cdot \overline{\alpha} = \varepsilon a \cdot (\alpha \cdot \theta) o = \varepsilon \cdot {}^{a}(\alpha \cdot \theta) \cdot (a^{\alpha \cdot \theta}, o)[a^{\alpha \cdot \theta}, o] = {}^{a}\alpha \cdot {}^{a^{\alpha}}\theta \cdot {}^{a^{\alpha \cdot \theta}}(\theta^{-1})$$

$$(a^{\alpha \cdot \theta})^{\theta^{-1}} = {}^{a}\alpha a^{\alpha}.$$

Applying the definitions of maps f_0 and f_1, (F1), (A7), (A8), (A9), (P1) to the right part of the first equality of Lemma 2, we get

$$\overline{{}^{a}\alpha} \cdot \overline{a^{\alpha}} = ({}^{a}\alpha \cdot \theta)o \cdot \varepsilon a^{\alpha} = ({}^{a}\alpha \cdot \theta \cdot {}^{o}\varepsilon \cdot (o^{\varepsilon}, a^{\alpha}))[o^{\varepsilon}, a^{\alpha}] = {}^{a}\alpha a^{\alpha}.$$

Thus, the first equality of Lemma 2 is proved.

Consider the second equality of Lemma 2. In left part

$$\overline{a} \cdot \overline{b} = \varepsilon a \cdot \varepsilon b = (\varepsilon \cdot {}^{a}\varepsilon \cdot (a^{\varepsilon}, b))[a^{\varepsilon}, b] = (a, b)[a, b]$$

according to definition of maps f_0 and f_1, (F1), (A6), (P2). In right part

$$\overline{(a, b)} \cdot \overline{[a, b]} = ((a, b) \cdot \theta)o \cdot \varepsilon[a, b] = ((a, b) \cdot \theta \cdot {}^{o}\varepsilon \cdot (o^{\varepsilon}, [a, b]))[o^{\varepsilon}, [a, b]]$$
$$= (a, b)[a, b]$$

according to definition of maps f_0 and f_1, (F1), (A8), (A7), (A9), (P1). □

Now the relations (F3) are obtained by using (F0) and Lemma 2

$${}^{\overline{a}}\overline{\alpha} \cdot \overline{a}^{\overline{\alpha}} = \overline{a} \cdot \overline{\alpha} = \overline{{}^{a}\alpha} \cdot \overline{a^{\alpha}}, \quad (\overline{a}, \overline{b}) \cdot [\overline{a}, \overline{b}] = \overline{a} \cdot \overline{b} = \overline{(a, b)} \cdot \overline{[a, b]},$$

since

$${}^{\overline{a}}\overline{\alpha}, \overline{{}^{a}\alpha}, (\overline{a}, \overline{b}), \overline{(a, b)} \in \overline{H}, \quad \overline{a}^{\overline{\alpha}}, \overline{a^{\alpha}}, [\overline{a}, \overline{b}], \overline{[a, b]} \in \overline{M}.$$

Theorem 2 is proved.

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References

[1] J. J. Rotman, An Introduction to the Theory of Groups, Springer-Verlag, 1994.

[2] F. Marty, Sur une generalisation de la notion de group, 8'eme Course Math. Scandinaves Stockholm (1934), 45-49.

[3] H. S. Wall, Hypergroups, American Journal of Mathematics 59(1) (1937), 77-98.

DOI: https://doi.org/10.2307/2371563

[4] G. L. Litvinov, Hypergroups and hypergroup algebras, VINITI 26, Moscow, (1985), 57-106 (in Russian).

[5] S. H. Dalalyan, On hypergroups, prenormal subgroups and simplest groups, Conf. dedicated to 90-anniversary of M. M. Djrbashyan, Yerevan, (2008), 27-28 (in Russian).

[6] S. H. Dalalyan, Theory of reducibility of hypergroups over the group, Conf. of AMU, Mai 29 - Jun. 2 (2012), 22-24 (in Russian).

[7] S. H. Dalalyan, Hypergroups over the group and generalizations of Schreier's theorem on group extensions, arXiv.submit/09409000 [math.Gr] 24 Mar, 2014.

[8] P. Zolfaghari, The hypergroups of order 3, arising from symmetric group S_3, in Proceedings of the 4th International Group Theory Conference of Iran, Isfahan, (2012), 198-201.

[9] S. H. Dalalyan and P. Zolfaghari, The hypergroups of order 3 over a group arising from the dihedral group of order 18, Journal of Mathematical Sciences 209(3) (2015), 225-236.

DOI: https://doi.org/10.1007/s10958-015-2498-y

[10] S. H. Dalalyan and Sh. Navasardyan, On unitary hypergroups over the group, Lobachevskii Journal of Mathematics 40(8) (2019), 1045-1057.

DOI: https://doi.org/10.1134/S1995080219080079

[11] Ramji Lal, Transversals in groups, Journal of Algebra 181(1) (1996), 70-81.

DOI: https://doi.org/10.1006/jabr.1996.0109

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