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LINDELOFNESS WITH RESPECT TO IDEAL VIA β-OPEN SETS

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Abstract

In this paper, we present and study new forms of Lindelofness in ideal topological spaces viz; $\beta\rho I$-Lindelofness and by utilizing the concept of β-open sets. Also, we study Lindelofness of subset of ideal topological spaces.

1. Introduction

Research in the field of ideal topological spaces extensively developed in the last few decades. In 1990, Jankovic and Hamlett [9] obtained new topologies using old ones and introduced the notion of ideal topological spaces. Hamlett [7] firstly defined Lindelof spaces with respect to an ideal and investigated the basic properties of the concept, its relation to known concepts, and its preservation by functions, subspaces, pre-images and

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products. Recently, Qahis [14] introduced and studied almost Lindelof modulo an ideal spaces. He discussed their properties and studied the effects of functions on them. In 1983, Abd El-Monsef et al. [1] initiated the study of β-open sets in topological spaces and the analogous notion of semi-preopen set was given exclusively by Andrijevic [2] and additionally examined by Ganster and Andrijevic [6]. More application and background of β-open sets was discussed by Caldas and Jafari [3]. Recently, Catalan et al. [4] produced a study on β-open sets and ideals in topological spaces and contributed on the concepts of β-compactiness and β-connectedness. An ideal I on a topological space (X, τ) is a non empty collection of subsets of X which satisfies (i) $A \in I$ and $B \subseteq A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given an ideal topological space (X, τ, I) and a subset A of X, a set operator $(.)^* : \mathscr{P}(X) \to \mathscr{P}(X)$ called a local function of A with respect to τ and I is defined to be the set $A^*(I, \tau) = \{x \in X : (A \cap U) \notin I,$ for every $U \in U_x\}$, where $U_x = \{U \in \tau : x \in U\}$. The Kuratowski closure operator $Cl^*(.)$ for a topology $\tau^*(I, \tau)$, called the $*$-topology, finer than τ, is defined by $Cl^*(A) = A \cup A^*(I, \tau)$. Where no confusion will arise we write A^* for $A^*(I, \tau)$ and τ^* for $\tau^*(I, \tau)$. For an ideal topological space (X, τ, I) the collection $\{U \setminus A : U \in \tau \text{ and } A \in I\}$ is a base for τ^*. The aim of this paper is to apply the concept of β-open sets to introduce and examine some types of Lindelofness modulo ideal called $\beta\rho I$-Lindelofness. Throughout this paper, (X, τ, I) denotes a topological space (X, τ) with ideal I on (X, τ). For a subset A of a space (X, τ), $cl(A)$, $\text{int}(A)$ and $(X \setminus A)$ denote respectively the closure, interior and the complement of A.

2. Preliminaries

Definition 2.1. An ideal I on a topological space (X, τ) is a collection of subsets of X such that if

(i) $A \in I$ and $B \subseteq A$, then $B \in I$ (heredity).

(ii) $A \in I$ and $B \in I$, then $A \cup B \in I$ (finite additivity).

Definition 2.2. An ideal I in a topological space (X, τ) is said to be [9] σ-*ideal* if it is countably additive. That is if $\{A_\alpha : \alpha \in \mathbb{N}\}$ is any countable collection of members I, then $\bigcup_{\alpha=1}^{\infty} U_\alpha \in I$.

Lemma 2.3. *If* I *and* J *are ideals on a topological space* (X, τ), *then the following hold*:

(a) [9] $I \cap J = \{A : A \in I$ *and* $A \in J\}$ *is an ideal on* X.

(b) [9] $I \vee J = \{A \cup B : A \in I$ *and* $B \in J\}$ *is an ideal on* X.

Definition 2.4. A subset A of a space (X, τ) is said to be β-*open* [1] (*semi-preopen* [2]) if $A \subseteq cl(\text{int}(cl(A)))$. The complement of a β-open set is called a β-closed set. The collection of all β-open sets in a topological space (X, τ) is denoted as $\beta O(X, \tau)$. The β-closure of A is the intersection over all β-closed sets containing A and is denoted by $\beta cl(A)$. The β-interior of A is the union over all β-open sets contained in A and is denoted by $\beta\,\text{int}(A)$.

Remark 2.5. By Definition 2.4 it is clear that given any topological space (X, τ), if $A \in \tau$, then $A \subseteq cl(\text{int}(cl(A)))$. Thus if $A \in \tau$, then $A \in \beta O(X, \tau)$. The complement of a β-open set is called a β-closed set and since every open set is β-open, the complement of every open set is β-closed. Therefore, every closed set is β-closed. Also $\beta O(X, \tau) \subseteq \beta O$ (X, τ^*) since τ^* is finer than τ. And $\beta O(X, \tau)$ is closed under arbitrary union.

Definition 2.6 ([4]). A subset A of an ideal topological space (X, τ, I) is said to β-*open with respect to* I (or βI-*open*) if there exists an open set U such that

(i) $(U \setminus A) \in I$, and

(ii) $(A \setminus cl(\text{int}(cl(U)))) \in I$.

Definition 2.7 ([4]). A subset A of an ideal topological space (x, τ, I) is said to be *generalized closed with respect to ideal* I (briefly Ig-*closed*) if $(cl(A) \setminus U) \in I$, whenever $A \subseteq U$ and $U \in \tau$.

Definition 2.8. A subset A of an ideal topological space (X, τ, I) is said to be $Ig\beta$-*closed* if $(\beta cl(A) \setminus U) \in I$, whenever $A \subseteq U$ and $U \in \beta\ O(X, \tau)$.

Definition 2.9. A subset A of a space (X, τ) is said to be *generalized*-β-*closed* ($g\beta$-*closed*) if $\beta cl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \beta O(X, \tau)$.

Definition 2.10. A topological space (X, τ) is said to be:

(a) [11] β-*Hausdorff* or (βT_2) *space* if given any two distinct points $x, y \in X$ there exist disjoint β-open sets U and V such that $x \in U$, $y \in V$.

(b) β-*Urysohn* or ($\beta T_{2\frac{1}{2}}$) *space* if given any two distinct points $x, y \in X$ there exist disjoint β-open sets U and V such that $x \in U$, $y \in V$ and $\beta cl(U) \cap \beta cl(V) = \emptyset$.

(c) [11] β-*normal* if given any two disjoint closed sets A and B in X there exist disjoint β-open sets U and V such that $A \subseteq U$, $B \subseteq V$.

(d) *Completely* β-*normal* if given any two disjoint sets A and B in X there exist disjoint β-open sets U and V such that $A \subseteq U$, $B \subseteq V$.

Definition 2.11 ([11])**.** A function $f : (X, \tau) \to (Y, \delta)$ is said to be:

(a) β-*irresolute* if $f^{-1}(U)$ is β-open in X whenever U is β-open in Y.

(b) *M*β-*open* if $f(U)$ is β-open in Y whenever U is β-open in X.

Lemma 2.12 ([12])**.** *If* $f : (X, \tau) \to (Y, \delta)$ *is* β-*irresolute surjection and* I *is an ideal on* X, *then*

(a) $J = \{A \subseteq Y : f^{-1}(A) \in I\}$ *is ideal on* Y.

(b) $f(I) = \{f(A) : A \in I\}$ *is ideal on* Y.

Lemma 2.13 ([11])**.** *If* $f : (X, \tau) \to (Y, \delta)$ *is* *M*β-*open injection and* J *is an ideal on* Y, *then* $f^{-1}(J) = \{f^{-1}(A) : A \in J\}$ *is ideal on* X.

Definition 2.14. Let $\mathscr{U} = \{U_\alpha : \alpha \in \Delta\}$ be a family of subsets of a topological space X and $A \subseteq X$. The family $\mathscr{U}$ is said to be

(i) *a cover for* A if $A \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$;

(ii) *a finite* (resp., *countable*) *cover for* A if $A \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$ and Δ is finite (resp., countable);

(iii) *an open* (resp., *closed*) *cover for* A if $A \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$ and U_α is open (resp., closed) $\forall \alpha \in \Delta$.

Definition 2.15. Let $\mathscr{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover for $A \subseteq X$, then $\mathscr{V} = \{V_\lambda : \lambda \in \Lambda\}$ is said to be a *subcover* of $\mathscr{U}$ if $\mathscr{V}$ is a cover for A and $V_\lambda \in \mathscr{U}$, $\forall \lambda \in \Lambda$. More generally $\mathscr{V}$ is said to be a *refinement* of $\mathscr{U}$ if $\mathscr{V}$ is a cover for A and $\forall \lambda \in \Lambda$, $\exists \alpha_\lambda \in \Delta$ such that $V_\lambda \subseteq U_{\alpha_\lambda}$. Every cover is a subcover for itself.

Definition 2.16 ([8]). An ideal topological space (X, τ, I) is said to be *I-compact* if for every open cover $\{U_\alpha : \alpha \in \Delta\}$ for X, there exists a finite subset Δ_0 of Δ such that $(X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha) \in I$.

Definition 2.17. An ideal topological space (X, τ, I) is said to be *countably I-compact* if for every countable open cover $\{U_\alpha : \alpha \in \mathbb{N}\}$ for X, there exists $n \in \mathbb{N}$ such that $X \setminus \bigcup_{\alpha=1}^{n} U_\alpha \in I$.

Definition 2.18 ([11]). A topological space (X, τ) is said to be β-*compact* (resp., *countably* β-*compact*) if for every β-open cover $\{U_\alpha : \alpha \in \Delta\}$ (resp., countable β-open cover $\{U_\alpha : \alpha \in \mathbb{N}\}$) for X, there exists a finite subset Δ_0 of Δ (resp., $n \in \mathbb{N}$) such that $(X \subseteq \bigcup_{\alpha \in \Delta_o} U_\alpha)$ (resp., $X \subseteq \bigcup_{\alpha=1}^{n} U_\alpha$).

Lemma 2.19. *Every finite discrete space is* β-*compact and countably* β-*compact.*

Definition 2.20. A topological space (X, τ) is said to be β-*Lindelof* if for every β-open cover $\{U_\alpha : \alpha \in \Delta\}$ for X, there exists a countable subset Δ_0 of Δ such that $(X \subseteq \bigcup_{\alpha \in \Delta_o} U_\alpha)$.

Lemma 2.21. *Every* β-*compact space is countably* β-*compact.*

Lemma 2.22. *Every* β-*compact space is* β-*Lindelof*.

Definition 2.23. An ideal topological space (X, τ, I) is said to be *I-Lindelof* if for every open cover $\{U_\alpha : \alpha \in \Delta\}$ for X, there exists a countable subset Δ_0 of Δ such that $(X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha) \in I$.

Lemma 2.24. *Every I-compact space is countably I-compact.*

Lemma 2.25. *Every I-compact space is I-Lindelof.*

Definition 2.26 ([11]). An ideal topological space (X, τ, I) is said to be βI-*compact* (resp., *countably* βI-*compact*) if for every β-open cover $\{U_\alpha : \alpha \in \Delta\}$ (resp., countable β-open cover $\{U_\alpha : \alpha \in \mathbb{N}\}$) for X, there exist a finite subset Δ_0 of Δ (resp., $n \in \mathbb{N}$) such that $(X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha) \in I$ (resp., $X \setminus \bigcup_{\alpha=1}^{n} U_\alpha \in I$).

Definition 2.27. An ideal topological space (X, τ, I) is said to be βI-*Lindelof* if for every β-open cover $\{U_\alpha : \alpha \in \Delta\}$ for X, there exists a countable subset Δ_0 of Δ such that $(X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha) \in I$.

Lemma 2.28. *Every βI-compact space is countably βI-compact.*

Lemma 2.29. *Every βI-compact space is βI-Lindelof.*

Definition 2.30 ([13]). An ideal topological space (X, τ, I) is said to be ρI-*compact* (resp., *countably* ρI-*compact*) if for every family $\{U_\alpha : \alpha \in \Delta\}$ (resp., countable family $\{U_\alpha : \alpha \in \mathbb{N}\}$) of open sets in X with $(X \setminus \bigcup_{\alpha \in \Delta} U_\alpha) \in I$, then there exists a finite subset Δ_0 of Δ (resp., $n \in \mathbb{N}$) such that $(X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha) \in I$(resp., $X \setminus \bigcup_{\alpha=1}^{n} U_\alpha \in I$).

Lemma 2.31. *Every ρI-compact space is countably ρI-compact.*

Definition 2.32. An ideal topological space (X, τ, I) is said to be $\beta\rho I$*-compact* (resp., *countably* $\beta\rho I$*-compact*) if for every family $\{U_\alpha : \alpha \in \Delta\}$ (resp., countable family $\{U_\alpha : \alpha \in \mathbb{N}\}$) of β-open sets in X with $(X \setminus \bigcup_{\alpha \in \Delta} U_\alpha) \in I$ (resp., $X \setminus \bigcup_{\alpha=1}^{\infty} U_\alpha \in I$), then there exists a finite subset Δ_0 of Δ (resp., $n \in \mathbb{N}$) such that $(X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha) \in I$(resp., $X \setminus \bigcup_{\alpha=1}^{n} U_\alpha \in I$).

Lemma 2.33. *Every* $\beta\rho I$*-compact space is countably* $\beta\rho I$*-compact.*

Definition 2.34. An ideal topological space (X, τ, I) is said to be ρI*-Lindelof* (resp., $\beta\rho I$*-Lindelof*) if for every family $\{U_\alpha : \alpha \in \Delta\}$ of open sets (resp., for every family $\{U_\alpha : \alpha \in \Delta\}$ of β-open sets) in X with $(X \setminus \bigcup_{\alpha \in \Delta} U_\alpha) \in I$, then there exists a countable subset Δ_0 of Δ such that $(X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha) \in I$.

Lemma 2.35. *Every* ρI*-compact space is* ρI*-Lindelof.*

Lemma 2.36. *Every* $\beta\rho I$*-compact space is* $\beta\rho I$*-Lindelof.*

3. β-Lindelof and βρ*I*-Lindelof Sets

Theorem 3.1. *If* A *and* B *are* β*-Lindelof subsets of a topological space* (X, τ), *then* $A \cup B$ *is* β*-Lindelof.*

Proof. Let A and B be β-Lindelof subsets of a topological space (X, τ) and $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-open sets such that $(A \cup B) \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$. This implies $A \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$ and $B \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$. A and B are β-Lindelof implies there exist countable subsets Δ_1 and Δ_2 of Δ such that $A \subseteq \bigcup_{\alpha \in \Delta_1} U_\alpha$ and $B \subseteq \bigcup_{\alpha \in \Delta_2} U_\alpha$. Taking $\Delta_3 = \Delta_1 \cup \Delta_2$, then Δ_3 is countable and $(A \cup B) \subseteq \bigcup_{\alpha \in \Delta_3} U_\alpha$. Hence, $A \cup B$ is β-Lindelof. □

Corollary 3.2. *Finite union of* β-*Lindelof sets is* β-*Lindelof*.

Theorem 3.3. *If* A *and* B *are* $\beta\rho I$-*Lindelof subsets of an ideal topological space* (X, τ, I), *then* $A \cup B$ *is* $\beta\rho I$-*Lindelof*.

Proof. Let A and B be $\beta\rho I$-Lindelof subsets of an ideal topological space (X, τ, I) and $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-open sets such that $(A \cup B) \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I$. This implies $A \setminus \bigcup_{\alpha \in \Delta} U_\alpha \subseteq (A \cup B) \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I$ and similarly $B \setminus \bigcup_{\alpha \in \Delta} U_\alpha \subseteq (A \cup B) \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I$. Since A and B are $\beta\rho I$-Lindelof, there exist countable subsets Δ_1 and Δ_2 of Δ such that $A \setminus \bigcup_{\alpha \in \Delta_1} U_\alpha \in I$ and $B \setminus \bigcup_{\alpha \in \Delta_2} U_\alpha \in I$. Taking $\Delta_3 = \Delta_1 \cup \Delta_2$, then Δ_3 is countable and $A \setminus \bigcup_{\alpha \in \Delta_3} U_\alpha \in I$ and $B \setminus \bigcup_{\alpha \in \Delta_3} U_\alpha \in I$. Hence, $(A \cup B) \setminus \bigcup_{\alpha \in \Delta_2} U_\alpha \in I$. Therefore, $A \cup B$ is $\beta\rho I$-Lindelof. □

Lemma 3.4. *If* A *and* B *are* ρI-*Lindelof subsets of an ideal topological space* (X, τ, I), *then* $A \cup B$ *is* ρI-*Lindelof*.

Corollary 3.5. *Finite union of* $\beta\rho I$-*Lindelof sets is* $\beta\rho I$-*Lindelof*.

Corollary 3.6. *Finite union of* ρI-*Lindelof sets is* ρI-*Lindelof*.

Theorem 3.7. *Let* A *be a* $\beta\rho I$-*Lindelof subset of an ideal topological space* (X, τ, I). *If* B *is a* β-*open set contained in* A, *then* $A \setminus B$ *is* $\beta\rho I$-*Lindelof*.

Proof. Let A be a $\beta\rho I$-Lindelof subset of an ideal topological space (X, τ, I) and B be a β-open set such that $B \subseteq A$. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-open sets in X such that $(A \setminus B) \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I$. This implies

$A \setminus \bigcup_{\alpha \in \Delta} (B \cup U_\alpha) \in I$. Given A is $\beta\rho I$-Lindelof and B is a β-open set, there exists a countable subset Δ_o of Δ such that $A \setminus \bigcup_{\alpha \in \Delta_o} (B \cup U_\alpha) \in I$. This implies $(A \setminus B) \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha \in I$. Hence, $A \setminus B$ is $\beta\rho I$-Lindelof. □

Theorem 3.8. *Let* (X, τ, I) *be an ideal topological space and* $A \subseteq X$. *If for every collection* $\{U_\alpha : \alpha \in \Delta\}$ *of* β-*open sets, if* $(A \setminus \bigcup_{\alpha \in \Delta} U_\alpha) \in I$ *there exists a* $\beta\rho I$-*Lindelof set* B *containing* A *such that* $(B \setminus \bigcup_{\alpha \in \Delta} U_\alpha) \in I$, *then* A *is* $\beta\rho I$-*Lindelof*.

Proof. Let the condition holds, $A \subseteq X$ and $\{U_\alpha : \alpha \in \Delta\}$ be a family β-open sets such that $(A \setminus \bigcup_{\alpha \in \Delta} U_\alpha) \in I$ and B be $\beta\rho I$-Lindelof such that $A \subseteq B$ and $(B \setminus \bigcup_{\alpha \in \Delta} U_\alpha) \in I$. Now B is $\beta\rho I$-Lindelof implies that there exists a countable subset Δ_o of Δ such that $B \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha \in I$. And $A \subseteq B \Rightarrow A \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha \in I \Rightarrow A$ is $\beta\rho I$-Lindelof. □

Theorem 3.9. *If* $(X, \tau, \{\emptyset\})$ *is* $\beta\rho\{\emptyset\}$-*Lindelof, then every* $\{\emptyset\}g\beta$-*closed subset of* $(X, \tau, \{\emptyset\})$ *is* β-*Lindelof*.

Proof. Let A be a $\{\emptyset\}g\beta$-closed subset of $\beta\rho\{\emptyset\}$-Lindelof space $(X, \tau, \{\emptyset\})$ and $\{U_\alpha : \alpha \in \Delta\}$ be a β-open sets such that $A \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$. Now A is $\{\emptyset\}g\beta$-closed $\Rightarrow \beta cl(A) \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$. Therefore, $X = (X \setminus \beta Cl(A))$

$\cup(\bigcup_{\alpha\in\Delta} U_\alpha) \Rightarrow X\setminus[(X\setminus\beta cl(A))\cup(\bigcup_{\alpha\in\Delta} U_\alpha)] = \emptyset \in \{\emptyset\}$. Since, $(X\setminus\beta cl(A))$ is β-open and (X, τ, I) be $\beta\rho I$-Lindelof, there exists a countable subset Δ_o of Δ such that $X\setminus[(X\setminus\beta cl(A))\cup(\bigcup_{\alpha\in\Delta_o} U_\alpha)] = \emptyset$.

But $X\setminus[(X\setminus\beta cl(A))\cup(\bigcup_{\alpha\in\Delta_o} U_\alpha)] = \beta cl(A))\setminus\bigcup_{\alpha\in\Delta_o} U_\alpha$. Hence, $\beta cl(A))\setminus\bigcup_{\alpha\in\Delta_o} U_\alpha = \emptyset \in \{\emptyset\}$. This implies $A\subseteq\bigcup_{\alpha\in\Delta_o} U_\alpha$. Hence, $A\beta$-Lindelof. □

Theorem 3.10. *If A and B are subsets of an ideal topological space $(X, \tau, \{\emptyset\})$ such that $A\subseteq B$, $B\subseteq\beta cl(A)$ and A is $\{\emptyset\}g\beta$-closed, then A is $\beta\rho\{\emptyset\}$-Lindelof if and only if B is $\beta\rho\{\emptyset\}$-Lindelof.*

Proof. Suppose the condition holds and A is $\beta\rho\{\emptyset\}$-Lindelof. Let $\{U_\alpha : \alpha\in\Delta\}$ be a family of β-open sets such that $B\setminus\bigcup_{\alpha\in\Delta} U_\alpha = \emptyset\in\{\emptyset\}$. Now $A\subseteq B \Rightarrow A\setminus\bigcup_{\alpha\in\Delta} U_\alpha \subseteq B\setminus\bigcup_{\alpha\in\Delta} U_\alpha = \emptyset \Rightarrow A\setminus\bigcup_{\alpha\in\Delta} U_\alpha = \emptyset$. Given that A is $\{\emptyset\}g\beta$-closed and $\beta\rho\{\emptyset\}$-Lindelof, there exists a countable subset Δ_o of Δ such that $A\setminus\bigcup_{\alpha\in\Delta_o} U_\alpha = \emptyset$ and $\beta cl(A)\subseteq\bigcup_{\alpha\in\Delta_o} U_\alpha$. This implies $B\setminus\bigcup_{\alpha\in\Delta_o} U_\alpha \subseteq \beta cl(A)\setminus\bigcup_{\alpha\in\Delta_o} U_\alpha = \emptyset \Rightarrow B\setminus\bigcup_{\alpha\in\Delta_o} U_\alpha = \emptyset\in\{\emptyset\}$. Hence, B is $\beta\rho\{\emptyset\}$-Lindelof.

Conversely, suppose the condition holds, B is $\beta\rho\{\emptyset\}$-Lindelof and $\{U_\alpha : \alpha\in\Delta\}$ be a family of β-open sets such that $A\setminus\bigcup_{\alpha\in\Delta} U_\alpha = \emptyset\in\{\emptyset\}$ and A is $\{\emptyset\}g\beta$-closed set in X. This implies $\beta cl(A)\subseteq\bigcup_{\alpha\in\Delta} U_\alpha \Rightarrow \beta cl(A)\setminus\bigcup_{\alpha\in\Delta} U_\alpha = \emptyset$. $B\subseteq\beta cl(A) \Rightarrow B\setminus\bigcup_{\alpha\in\Delta} U_\alpha \subseteq \beta cl(A)\setminus\bigcup_{\alpha\in\Delta} U_\alpha = \emptyset \Rightarrow B\setminus\bigcup_{\alpha\in\Delta} U_\alpha$

$= \emptyset$. B is $\beta\rho\{\emptyset\}$-Lindelof implies that there exists a countable subset Δ_o of Δ such that $B \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha = \emptyset \in \{\emptyset\}$. Now $A \subseteq B \Rightarrow A \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha \subseteq B \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha = \emptyset \in \{\emptyset\} \Rightarrow A \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha = \emptyset \in \{\emptyset\}$. Hence, A is $\beta\rho\{\emptyset\}$-Lindelof. □

Lemma 3.11. *Let A and B be subsets of a topological space (X, τ) with $A \subseteq B$, $B \subseteq \beta cl(A)$, and A is $g\beta$-closed. Then A is β-compact if and only if B is β-compact.*

Theorem 3.12. *A β-closed subset of a β-Lindelof space is β-Lindelof.*

Proof. Let B be a β-closed subset of a β-Lindelof space (X, τ) and $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-open sets such that $B \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$. Then $(X \setminus B)$ is β-open and $X \subseteq \bigcup_{\alpha \in \Delta} U_\alpha \cup (X \setminus B)$. Since X is β-Lindelof, there exists a countable subset Δ_o of Δ such that $X \subseteq [\bigcup_{\alpha \in \Delta_o} U_\alpha \cup (X \setminus B)]$. Hence, $B \subseteq \bigcup_{\alpha \in \Delta_o} U_\alpha$. Therefore, B is β-Lindelof. □

Theorem 3.13. *If B is β-Lindelof subset of a β-Hausdorff space (X, τ) and $x \notin B$, then there exist β-open sets U and V such that $x \in U$, $B \subseteq V$ and $U \cap V = \emptyset$.*

Proof. Let B be β-Lindelof, (X, τ) be β-Hausdorff and $x \notin B$. Now $\forall b \in B$, $x \neq b$, therefore there exist β-open, sets U_x and V_b such that $x \in U_x$, $b \in V_b$ and $U_x \cap V_b = \emptyset$. The family $\{V_b : b \in B\}$ forms a β-open cover for B. Since B is β-Lindelof, there exists a countable subcollection $\{V_i : i = 1, 2, \ldots\}$ such that $B \subseteq \bigcup_{i \in \mathbb{N}} V_i$. Let $V = \bigcup_{i \in \mathbb{N}} V_i$, then V is β-open and $B \subseteq V$. For each open V_i, $i \in \mathbb{N}$ there exists a corresponding U_i such that $x \in U_i$ and $V_i \cap U_i = \emptyset$. Let $U = \beta\,\mathrm{int}(\bigcap_{i \in \mathbb{N}} U_i)$, then U is β-open, $x \in U$ and $U \cap V = \emptyset$. □

Theorem 3.14. *A β-Lindelof subset of a β-Hausdorff space is β-closed.*

Proof. Let X be β-Hausdorff and $B \subseteq X$ be β-Lindelof. Let $x \notin B$, then by Theorem 3.13, there exist β-open sets U and V such that $x \in U$, $B \subseteq V$ and $U \cap V = \emptyset$. This implies $\forall x \notin B$ there exists β-open set U_x such that $x \in U_x$ and $U_x \cap B = \emptyset$. Let $H = \bigcup_{x \notin B} U_x$, then H is β-open and $(X \setminus H) = B$. Hence, B is β-closed. □

4. β-Lindelof and βρ*I*-Lindelof Spaces

Theorem 4.1. *Let I and J be ideals in (X, τ) and $K = I \cap J$. If (X, τ, I) is βρI-Lindelof and (X, τ, J) is βρJ-Lindelof, then (X, τ, K) is βρI-Lindelof.*

Proof. Let the condition holds and $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-open sets in X such that $X \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in K$. This implies $X \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I$ and $X \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in J$. Now (X, τ, I) and (X, τ, J) are βρJ-Lindelof implies that there exist countable subsets Δ_1 and Δ_2 of Δ such that $X \setminus \bigcup_{\alpha \in \Delta_1} U_\alpha \in I$ and $X \setminus \bigcup_{\alpha \in \Delta_2} U_\alpha \in J$. Taking $\Delta_3 = \Delta_1 \cap \Delta_2$, then Δ_3 is countable and $X \setminus \bigcup_{\alpha \in \Delta_3} U_\alpha \in I$ and $X \setminus \bigcup_{\alpha \in \Delta_3} U_\alpha \in J$. Therefore, $X \setminus \bigcup_{\alpha \in \Delta_3} U_\alpha \in (I \cap J)$. This implies $X \setminus \bigcup_{\alpha \in \Delta_3} U_\alpha \in K$. Hence, (X, τ, K) is βρI-Lindelof. □

Theorem 4.2. *Let* $f:(X, \tau, I) \to (Y, \delta)$ *be* β-*irresolute and surjective. If* (X, τ, I) *is* $\beta\rho I$-*Lindelof and* $J = \{B \subseteq Y : f^{-1}(B) \in I\}$, *then* (Y, δ, J) *is* $\beta\rho I$-*Lindelof*.

Proof. Let $f: X \to Y$ be β-irresolute, surjective and (X, τ, I) be $\beta\rho I$-Lindelof. By (a) of Lemma 2.12, J is an ideal on Y. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-open sets in Y such that $Y \setminus \bigcup_{\alpha\in\Delta} U_\alpha \in J$. This implies $f^{-1}(Y \setminus \bigcup_{\alpha\in\Delta} U_\alpha) \in I$ where $\{f^{-1}(U_\alpha) : \alpha \in \Delta\}$ is a family of β-open sets in X.

Now $f^{-1}(Y \setminus \bigcup_{\alpha\in\Delta} U_\alpha) = f^{-1}(Y) \setminus f^{-1}(\bigcup_{\alpha\in\Delta} U_\alpha) = X \setminus \bigcup_{\alpha\in\Delta} f^{-1}(U_\alpha) \in I$ and (X, τ, I) is $\beta\rho I$-Lindelof implies there exists a countable subset Δ_o of Δ such that $X \setminus \bigcup_{\alpha\in\Delta_o} f^{-1}(U_\alpha) \in I$.

Hence, $f(X \setminus \bigcup_{\alpha\in\Delta_o} f^{-1}(U_\alpha)) = f(X) \setminus f(\bigcup_{\alpha\in\Delta_o} f^{-1}(U_\alpha)) = Y \setminus \bigcup_{\alpha\in\Delta_o} f(f^{-1}(U_\alpha)) = Y \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha \in J$. Therefore, (Y, δ, J) is $\beta\rho J$-Lindelof. □

Theorem 4.3. *Let* $f:(X, \tau, I) \to (Y, \delta)$ *be* β-*irresolute and bijective. If* (X, τ, I) *is* $\beta\rho I$-*Lindelof and* $f(I) = \{f(A) : A \in I\}$, *then* $(Y, \delta, f(I))$ *is* $\beta\rho f(I)$-*Lindelof*.

Proof. Let $f: X \to Y$ be β-irresolute, bijective and (X, τ, I) be $\beta\rho I$-Lindelof. By (b) of Lemma 2.12, $f(I)$ is an ideal on Y. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-open sets in Y such that $Y \setminus \bigcup_{\alpha\in\Delta} U_\alpha \in f(I)$. This implies there exists $A \in I$ such that $Y \setminus \bigcup_{\alpha\in\Delta} U_\alpha = f(A)$.

Hence, $A = f^{-1}(Y \setminus \bigcup_{\alpha\in\Delta} U_\alpha) = f^{-1}(Y) \setminus f^{-1}(\bigcup_{\alpha\in\Delta} U_\alpha) = X \setminus \bigcup_{\alpha\in\Delta} f^{-1}(U_\alpha) \in I$. Given that (X, τ, I) is $\beta\rho I$-Lindelof there exists a countable subset Δ_o of Δ such that $X \setminus \bigcup_{\alpha\in\Delta_o} f^{-1}(U_\alpha) \in I$. Therefore, $f(X \setminus \bigcup_{\alpha\in\Delta_o} f^{-1}(U_\alpha)) = f(X) \setminus f(\bigcup_{\alpha\in\Delta_o} f^{-1}(U_\alpha)) = Y \setminus \bigcup_{\alpha\in\Delta_o} f(f^{-1}(U_\alpha)) = Y \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha \in f(I)$. Hence, $(Y, \delta, f(I))$ is $\beta\rho f(I)$-Lindelof. □

Theorem 4.4. *Let $f : (X, \tau) \to (Y, \delta, J)$ be $M\beta$-open and bijective. If (Y, δ, J) is $\beta\rho J$-Lindelof and $f^{-1}(J) = \{f^{-1}(C) : C \in J\}$, then $(X, \tau, f^{-1}(J)$ is $\beta\rho f^{-1}(J)$-Lindelof.*

Proof. Let $f : (X, \tau) \to (Y, \delta, J)$ be an $M\beta$-open bijection and (Y, δ, J) be $\beta\rho J$-Lindelof. By Lemma 2.13, $f^{-1}(J)$ is an ideal on X. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-open sets in X such that $X \setminus \bigcup_{\alpha\in\Delta} U_\alpha \in f^{-1}(J)$. Then there exists $Z \in J$ such that $X \setminus \bigcup_{\alpha\in\Delta} U_\alpha = f^{-1}(Z)$.

Therefore, we have $f(X \setminus \bigcup_{\alpha\in\Delta} U_\alpha) = f(X) \setminus f(\bigcup_{\alpha\in\Delta} U_\alpha) = Y \setminus \bigcup_{\alpha\in\Delta} f(U_\alpha) = Z \in J$. Given that (Y, δ, J) be $\beta\rho J$-Lindelof, there exists a countable subset Δ_o of Δ such that $Y \setminus \bigcup_{\alpha\in\Delta_o} f(U_\alpha) \in J$.

Therefore, $f^{-1}(Y \setminus \bigcup_{\alpha\in\Delta_o} f(U_\alpha)) = f^{-1}(Y) \setminus f^{-1}(\bigcup_{\alpha\in\Delta_o} f(U_\alpha)) = X \setminus \bigcup_{\alpha\in\Delta_o} f^{-1}(f(U_\alpha)) = X \setminus \bigcup_{\alpha\in\Delta_o} U_\alpha \in f^{-1}(J)$. Hence, $(X, \tau, f^{-1}(J)$ is $\beta\rho f^{-1}(J)$-Lindelof.

□

Lemma 4.5. *If* $(X, \tau, \{\emptyset\})$ *is* $\beta\rho\{\emptyset\}$-*Lindelof and* B *is* β-*closed in* (X, τ), *then* B *is* $\beta\rho\{\emptyset\}$-*Lindelof*.

Proof. Let $(X, \tau, \{\emptyset\})$ be $\beta\rho\{\emptyset\}$-Lindelof, B be β-closed in (X, τ) and $\{U_\alpha : \alpha \in \Delta\}$ be a family of β open sets such that $B \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I = \{\emptyset\}$.

This implies $B \setminus \bigcup_{\alpha \in \Delta} U_\alpha = \emptyset \Rightarrow B \subseteq \bigcup_{\alpha \in \Delta} U_\alpha \Rightarrow X \subseteq \bigcup_{\alpha \in \Delta} U_\alpha \cup (X \setminus B)$ $\Rightarrow X \setminus (\bigcup_{\alpha \in \Delta} U_\alpha \cup (X \setminus B)) = \emptyset \in \{\emptyset\}$. Now $(X, \tau, \{\emptyset\})$ is $\beta\rho\{\emptyset\}$-Lindelof implies that, there exists a countable subset Δ_o of Δ such that $X \setminus (\bigcup_{\alpha \in \Delta_o} U_\alpha \cup (X \setminus B)) \in \{\emptyset\}$. This implies $B \setminus (\bigcup_{\alpha \in \Delta_o} U_\alpha \cup (X \setminus B)) \in \{\emptyset\}$. Hence, $B \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha \in \{\emptyset\}$. Therefore, B is $\beta\rho\{\emptyset\}$-Lindelof. □

Theorem 4.6. *A* β-*Lindelof and* β-*Hausdorff space is* β-*normal.*

Proof. Let (X, τ) be β-Lindelof and β-Hausdorff and A and B be any disjoint closed sets in X. By Theorem 3.12 A and B are β-Lindelof and $\forall a \in A$, $a \notin B$. Therefore, by Theorem 3.13, there exist β-open sets U_a and V_a open with $a \in U_a$ and $B \subseteq V_a$ such that $U_a \cap V_a = \emptyset$. Now $A \subseteq \bigcup_{a \in A} U_a$ and $B \subseteq V_a$, $a \in A$. Since A is β-Lindelof, there exists a countable subcover say $\{U_i : i = 1, 2, \ldots\}$ such that $A \subseteq \bigcup_{i \in \mathbb{N}} U_i$. Let $U = \bigcup_{i \in \mathbb{N}} U_i$, then U is β-open and $A \subseteq U$. Now for each U_i there exists V_i such that $B \subseteq V_i$ and $U_i \cap V_i = \emptyset$. Let $V = \beta\, \mathrm{int}(\bigcap_{i \in \mathbb{N}} V_i)$, then V is β-open and $B \subseteq V$. Moreover $U \cap V = \emptyset$. Hence, X is β-normal. □

Lemma 4.7. *If* $(X, \tau, \{\emptyset\})$ *is* $\beta\rho\{\emptyset\}$*-Lindelof and* β*-Hausdorff, then* (X, τ) *is* β*-normal.*

Theorem 4.8. *A topological space* (X, τ) *is* β*-Lindelof if and only if* $(X, \tau, \{\emptyset\})$ *is* $\beta\rho\{\emptyset\}$*-Lindelof.*

Proof. Let (X, τ) be β-Lindelof and $\{U_\alpha : \alpha \in \Delta\}$ be a collection of β-open sets such that $X \setminus \bigcup_{\alpha \in \Delta} U_\alpha = \emptyset \in \{\emptyset\}$. This implies $X \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$. Since (X, τ) is β-Lindelof, there exists a countable subset Δ_o of Δ such that $X \subseteq \bigcup_{\alpha \in \Delta_o} U_\alpha$. This implies $X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha = \emptyset \in \{\emptyset\}$. Hence, $(X, \tau, \{\emptyset\})$ is $\beta\rho\{\emptyset\}$-Lindelof. Conversely suppose $(X, \tau, \{\emptyset\})$ is $\beta\rho\{\emptyset\}$-Lindelof and $\{U_\alpha : \alpha \in \Delta\}$ is a β-open cover for X. This implies $X \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$ and $X \setminus \bigcup_{\alpha \in \Delta} U_\alpha = \emptyset \in \{\emptyset\}$. Since, $(X, \tau, \{\emptyset\})$ is $\beta\rho\{\emptyset\}$-Lindelof, there exists a countable subset Δ_o of Δ such that there exists a countable subset Δ_o of Δ such that $(X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha) \in \{\emptyset\}$. This implies $X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha = \emptyset$ and $X \subseteq \bigcup_{\alpha \in \Delta_o} U_\alpha$. Hence, (X, τ) is β-Lindelof.

□

Theorem 4.9. *If* (X, τ, I) *is* $\beta\rho I$*-Lindelof, then* (X, τ, I) *is* ρI*-Lindelof.*

Proof. Let (X, τ, I) is $\beta\rho I$-Lindelof and $\{U_\alpha : \alpha \in \Delta\}$ be a family of open sets in X such that $X \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I$. By Remark 2.5, $U_\alpha \in \beta O(X, \tau)$, $\forall \alpha \in \Delta$. By our hypothesis, there exists a countable subset Δ_o of Δ such that $X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha \in I$. Hence, (X, τ, I) is ρI-Lindelof. □

Theorem 4.10. *An ideal topological space* (X, τ, I) *is* $\beta\rho I$*-Lindelof iff* (X, τ^*, I) *is* $\beta\rho I$*-Lindelof.*

Proof. Suppose (X, τ, I) is $\beta\rho I$-Lindelof and $\{U_\alpha : \alpha \in \Delta\}$ is a collection of β-open sets in (X, τ^*, I) such that $X \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I$. Now $U_\alpha = V_\alpha \setminus W_\alpha$ such that $V_\alpha \in \beta O(X, \tau)$, $W_\alpha \in I$, $\forall \alpha \in \Delta$. This implies $\{V_\alpha : \alpha \in \Delta\}$ is a collection of β-open sets in (X, τ, I) and $\bigcup_{\alpha \in \Delta} U_\alpha \subseteq \bigcup_{\alpha \in \Delta} V_\alpha$. Therefore, we have $X \setminus \bigcup_{\alpha \in \Delta} V_\alpha \subseteq X \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I \Rightarrow X \setminus \bigcup_{\alpha \in \Delta} V_\alpha \in I$. By our hypothesis there exists a countable subset Δ_o of Δ such that $X \setminus \bigcup_{\alpha \in \Delta_o} V_\alpha \in I$. Now $X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha \subseteq [(X \setminus \bigcup_{\alpha \in \Delta_o} V_\alpha) \cup (\bigcup_{\alpha \in \Delta_o} W_\alpha)] \in I \Rightarrow X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha \in I \Rightarrow (X, \tau^*, I)$ is $\beta\rho I$-Lindelof. Conversely, suppose (X, τ^*, I) is $\beta\rho I$-Lindelof and $\{U_\alpha : \alpha \in \Delta\}$ is a collection of β-open sets in (X, τ, I) such that $X \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I$. By Remark 2.5 and our hypothesis $\{U_\alpha : \alpha \in \Delta\}$ is a collection of β-open sets in (X, τ^*, I) and there exists a countable subset Δ_o of Δ such that $X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha \in I$. Hence, (X, τ, I) is $\beta\rho I$-Lindelof. □

Theorem 4.11. *An ideal topological space* (X, τ, I) *is* $\beta\rho I$*-Lindelof iff for any family* $\{V_\alpha : \alpha \in \Delta\}$ *of* β*-closed sets in* X*, if* $\bigcap_{\alpha \in \Delta} V_\alpha \in I$*, then there exists countable subset* Δ_o *of* Δ *such that* $\bigcap_{\alpha \in \Delta_o} V_\alpha \in I$.

Proof. Let (X, τ, I) be a $\beta\rho I$-Lindelof and $\{V_\alpha : \alpha \in \Delta\}$ be a family of β-closed sets in X such that $\bigcap_{\alpha \in \Delta} V_\alpha \in I$. Now if $U_\alpha = (X \setminus V_\alpha)$, $\forall \alpha \in \Delta$, then $\{U_\alpha : \alpha \in \Delta\}$ is a family of β-open sets in X and we have $\bigcap_{\alpha \in \Delta} V_\alpha = \bigcap_{\alpha \in \Delta} (X \setminus U_\alpha) = X \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I$. By our hypothesis there exists a countable subset Δ_o of Δ such that $X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha \in I$. Hence, $X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha = \bigcap_{\alpha \in \Delta_o} (X \setminus U_\alpha) = \bigcap_{\alpha \in \Delta_o} V_\alpha \in I$.

Conversely, suppose the condition holds and $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-open sets in X such that $X \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I$. Then $\{(X \setminus U_\alpha) : \alpha \in \Delta\}$ is a family of β-closed sets and $X \setminus \bigcup_{\alpha \in \Delta} U_\alpha = \bigcap_{\alpha \in \Delta} (X \setminus U_\alpha) \in I$. By our hypothesis there exists a countable subset Δ_o of Δ such that $\bigcap_{\alpha \in \Delta_o} (X \setminus U_\alpha) \in I$. Hence, there exists a countable subset Δ_o of Δ such that $\bigcap_{\alpha \in \Delta_o} (X \setminus U_\alpha) = X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha \in I$. Hence, (X, τ, I) is $\beta\rho I$-Lindelof.

□

Corollary 4.12. (X, τ^*, I) *is* $\beta\rho I$*-Lindelof iff for any family* $\{V_\alpha : \alpha \in \Delta\}$ *of* β*-closed sets in* (X, τ, I) *if* $\bigcap_{\alpha \in \Delta} V_\alpha \in I$, *then there exists a countable subset* Δ_o *of* Δ *such that* $\bigcap_{\alpha \in \Delta_o} V_\alpha \in I$.

Theorem 4.13. *A countably* β*-compact and* β*-Lindelof space is* β*-compact.*

Proof. Let X be countably β-compact and β-Lindelof and $\{U_\alpha : \alpha \in \Delta\}$ be a β-open cover X. The space is β-Lindelof implies that there exists a countable subset Δ_o of Δ such that $X \subseteq \bigcup_{\alpha \in \Delta_o} U_\alpha$ and X is countably β-compact implies that there exists a finite subset Δ_1 of Δ_o such that $X \subseteq \bigcup_{\alpha \in \Delta_1} U_\alpha$. Hence, X is β-compact. □

Theorem 4.14. *A countably* $\beta\rho I$*-compact and* $\beta\rho I$*-Lindelof ideal topological space is* $\beta\rho I$*-compact.*

Proof. Let (X, τ, I) be countably $\beta\rho I$-compact and $\beta\rho I$-Lindelof and $\{U_\alpha : \alpha \in \Delta\}$ be a family of β-open sets such that $X \setminus \bigcup_{\alpha \in \Delta} U_\alpha \in I$. The space (X, τ, I) is $\beta\rho I$-Lindelof implies that, there exists a countable subset Δ_o of Δ such that $X \setminus \bigcup_{\alpha \in \Delta_o} U_\alpha \in I$. The space (X, τ, I) is countably $\beta\rho I$-compact implies that, there exists a finite subset Δ_1 of Δ_o such that $X \setminus \bigcup_{\alpha \in \Delta_1} U_\alpha \in I$. Hence, (X, τ, I) is $\beta\rho I$-compact. □

Lemma 4.15. *A countably* ρI*-compact and* ρI*-Lindelof ideal topological space is* ρI*-compact.*

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