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COINCIDENCE AND COMMON FIXED POINTS FOR MAPS OF ĆIRIĆ-BERINDE TYPE

MOHAMED AKKOUCHI

Department of Mathematics
Faculty of Sciences-Semlalia
University Cadi Ayyad
Av. Prince My. Abdellah, BP: 2390
Marrakesh (40.000-Marrakech)
Morocco (Maroc)
e-mail: akkm555@yahoo.fr

Abstract

Let (X, d) be a complete metric space. Let f be a given selfmap of X. A selfmap T of X is said to be a Ćirić strong almost contraction with respect to f if there exist constants $\delta \in [0, 1)$ and $L \geq 0$ such that the following condition is fulfilled:

$$d(Tx, Ty) \leq \delta M_f(x, y) + L \min\{d(fx, Ty), d(fy, Tx)\}, \qquad \text{(C-B)}$$

for all $x, y \in X$, where

$$M_f(x, y) := \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2}\}.$$

This concept extends and unifies many concepts already known in the literature like the Ćirić strong almost contraction, almost (or weak) contractions introduced by Berinde or the condition (B)' of Babu et al. [4] or its generalization due to Abbas et al. [2].

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In this paper, we investigate coincidence points of a pair (T, f) satisfying the condition (C-B) and look for conditions ensuring the existence of common fixed points. The results obtained in this line provide generalizations of several published results. We provide examples supporting our results. We end this work by discussing a problem concerining the Ćirić almost contractions introduced by Berinde in 2009.

1. Introduction and Preliminaries

Throughout this paper, (X, d) designates a metric space. Let $f, T : X \to X$ be selfmappings. For all $x, y \in X$, we consider the following function:

$$M_1(x, y) := \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}.$$

Berinde [8] defined a selfmapping T of X to be a strong Ćirić almost contraction if T satisfies the following condition:

There exist two constants $\alpha \in [0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \alpha M_1(x, y) + Ld(y, Tx), \quad \text{for all} \quad x, y \in X. \tag{1.1}$$

It is clear that the condition (1.1) is equivalent to the following condition:

$$d(Tx, Ty) \leq \alpha M_1(x, y) + L\min\{d(x, Ty), d(y, Tx)\}, \quad \text{for all} \quad x, y \in X. \tag{1.2}$$

We recall that the notion of strong Ćirić almost contraction is a generalization of the notion of *almost contraction* which was introduced by Berinde in [6] and [7].

Berinde [8] established the following result.

Theorem 1.1 (Theorem 2.2, [8]). *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a strong Ćirić almost contraction with parameters* $\alpha \in [0, 1)$ *and* $L \geq 0$. *Then*

(1) $Fix(T) := \{x \in X : Tx = x\} \neq \emptyset$;

(2) *for any* $x_0 \in X$, *the Picard iteration*

$$x_{n+1} = Tx_n, \; n = 0, 1, 2, \cdots \tag{1.3}$$

converges to some $x^* \in Fix(T)$;

(3) *the following estimates*

$$d(x_{n+i-1}, x^*) \leq \frac{\alpha^i}{1-\delta} d(x_n, x_{n-1}), \quad n = 0, 1, 2, \cdots; \; i = 1, 2, \cdots. \tag{1.4}$$

Babu et al. introduced in [4] the class of mappings that satisfy 'condition (B)'.

A map $T : X \to X$ is said to satisfy '*condition* (*B*)' if there exist a constant $\delta \in \,]0, 1[$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \tag{B}$$

for all $x, y \in X$.

We observe that if T satisfies (B), then it is a strong Ćirić almost contraction.

The following fixed point theorem was proved in [4].

Theorem 1.2 (Babu et al. [4], Theorem 2.3)**.** *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a map satisfying condition* (B). *Then* T *has a unique fixed point.*

Theorem 1.2 has been generalized by Berinde in [8].

Theorem 1.3 (Theorem 2.4, [8])**.** *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a selfmap of X for which there exist* $\alpha \in [0, 1)$ *and* $L \geq 0$ *such that for all* $x, y \in X$,

$$d(Tx, Ty) \leq \alpha M_1(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \tag{1.5}$$

Then

(1) *T has a unique fixed point, i.e.,* $Fix(T) := \{x^*\}$;

(2) *for any* $x_0 \in X$, *the Picard iteration*

$$x_{n+1} = Tx_n, \; n = 0, 1, 2, \cdots$$

converges to x^*;

(3) *the following estimates*

$$d(x_{n+i-1}, x^*) \leq \frac{\alpha^i}{1-\delta} d(x_n, x_{n-1}), \quad n = 0, 1, 2, \cdots; \; i = 1, 2, \cdots.$$

Abbas et al. (see [2]) introduced the almost contraction property to a pair of selfmaps as follows:

Definition 1.1. Let (X, d) be a metric space. A map $T : X \to X$ is called an *almost contraction* with respect to a mapping $f : X \to X$ if there exist a constant $\delta \in \,]0, 1[$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(fx, fy) + Ld(fy, Tx), \qquad \text{(A.C} - f)$$

for all $x, y \in X$.

We observe that if we choose $f = I_X$ where I_X is the identity map on X, then we obtain the definition of *almost contraction* which was introduced by Berinde in [6] and [7].

This concept was first introduced by Berinde as 'weak contraction' in [6]. But Berinde renamed this concept in [7] as 'almost contraction'.

Let f and T be two selfmaps of a metric space (X, d). T is said to be *f-contraction* if there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(fx, fy)$ for all $x, y \in E$.

In 2006, Al-Thagafi and Shahzad [3] proved the following theorem.

Theorem 1.4 (Al-Thagafi and Shahzad [3], Theorem 2.1)**.** *Let E be a subset of a metric space* (X, d), *and f and T be selfmaps of E and* $T(E) \subseteq f(E)$. *Suppose that f and T are weakly compatible, T is f-contraction and* $T(E)$ *is complete. Then f and T have a unique common fixed point in E.*

We observe that if T is an f-contraction, then T is almost contraction with respect to f.

To extend Theorem 1.4, Abbas et al. (see [2]) introduced a generalization of 'condition (B)' for a pair of selfmaps.

Definition 1.2 ([2])**.** A selfmap T on a metric space X is said to satisfy '*generalized condition* (*B*)' associated with a selfmap f of X if there exists $\delta \in \,]0, 1[$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta M_f(x, y) + L \min\{d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}, \tag{G.B}$$

for all $x, y \in X$, where

$$M_f(x, y) := \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2}\}.$$

If $f = I_X$, then we say that T satisfies 'generalized condition (B)'.

It was observed in [2] that 'condition (B)' implies 'generalized condition (B)'. But its converse need not be true.

To state the main result of [2], we need to recall the following.

Definition 1.3. A pair (f, T) of selfmappings on X is said to be *weakly compatible* if f and T commute at their coincidence point (i.e., $fTx = Tfx, x \in X$ whenever $fx = Tx$).

A point $y \in X$ is called a *point of coincidence* of two selfmappings f and T on X if there exists a point $x \in X$ such that $y = Tx = fx$.

Also, for the sequel we need to recall the following lemma which is stated in Proposition 1.4 of [1].

Lemma 1.1. *Let X be a non-empty set and the mappings $f, T : X \to X$ have a unique point of coincidence v in X. If the pair (f, T) is weakly compatible, then f and T have a unique common fixed point.*

In [2], the following extension of Theorem 1.1 was established.

Theorem 1.5 ([2]). *Let (X, d) be a metric space. Let $f, T : X \to X$ be such that $T(X) \subseteq f(X)$. Assume that T satisfies generalized condition* (B) *associated with f. Assume that either $f(X)$ or $T(X)$ is a complete subspace of X, then f and T have a unique point of coincidence.*

If in addition, the pair $\{T, f\}$ is weakly compatible then f and T have a unique fixed point.

We observe that Theorem 1.5 extends properly and unifies Theorems 1.1, 1.2, 1.3, and 1.4.

The aim of this paper is to give another proper a common generalization of all theorems quoted above. To this end, we introduce the concept of Ćirić-Berinde type pairs as described in the Section 2.

This paper is organized as follows.

In Section 2, we introduce some new concepts concerning Ćirić-Berinde type maps and pairs of Ćirić-Berinde type.

In Section 3, we establish our main results. In the first result (see Theorem 3.1), we investigate the existence of coincidence points of Ćirić-Berinde pairs of selfmaps of type (C-B). Contrary to Theorem 1.5, Ćirić-Berinde pairs of selfmaps of type (C-B) may have more than one point of coincidence. This is in contrast with Theorem 1.5. This holds, because the condition (C-B) is not strong enough to guarantee the uniqueness of coincidence points. We give some consequences and corollaries of our results. Also, we provide examples to support our results.

We end this work by discussing a problem concerining the Ćirić almost contractions introduced and studied by Berinde in the papers [7] and [8].

2. Maps and Pairs of Ćirić-Berinde Type

The purpose of this section is to introduce some new definitions.

Definition 2.1. A pair (T, f) of selfmaps on a metric space X is said to be of Ćirić-Berinde pair of type (C-B) if there exists $\delta \in \,]0, 1[$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta M_f(x, y) + L \min\{d(fx, Ty), d(fy, Tx)\}, \qquad \text{(C-B)}$$

for all $x, y \in X$, where

$$M_f(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2}\}.$$

We say also that the T is a Ćirić-Berinde map of type (C-B) with respect to f or that (T, f) is a Ćirić-Berinde pair of type (C-B) with parameters (δ, L).

If $f = I_X$, then we say that T is a Ćirić-Berinde map of type (C-B).

Definition 2.2. A pair (T, f) of selfmaps on a metric space X is said to be of Ćirić-Berinde pair of type (K) if there exists $\delta \in \,]0, 1[$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta M_f(x, y) + L \min\{d(fx, Tx), d(fy, Ty)\}, \qquad \text{(K)}$$

for all $x, y \in X$, where

$$M_f(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2}\}.$$

We also say that the T is of Ćirić-Berinde map of type (K) with respect to f or that (T, f) is a Ćirić-Berinde pair of type (K) with parameters (δ, L).

If $f = I_X$, then we say that T is a Ćirić-Berinde map of type (K).

Definition 2.3. A pair (T, f) of selfmaps on a metric space X is said to be a Ćirić-Berinde pair if it satisfies the condition (C-B) or the condition (K).

If $f = I_X$, then we say that T is a Ćirić-Berinde map or Ćirić-Berinde type map.

Remark. Let (T, f) of selfmaps on a metric space X. Then the following asserions are equivalent:

(i) (T, f) satisfies both conditions (C-B) and (K).

(ii) (T, f) satisfies the generalized condition (G.B).

Next we fix some notations and make some observations.

Let (X, d) be a metric space and $f, T : X \to X$ be selfmaps of X. The set of coincidence points of the mappings f and T will be denoted by $C_{oin}\{f, T\}$. That is

$$C_{oin}\{f, T\} := \{u \in X : fu = Tu\}.$$

The set of points of coincidence of f and T will be denoted by $P_{oc}\{f, T\}$.

So, by definition, we have $P_{oc}\{f, T\} = f(C_{oin}\{f, T\}) = T(C_{oin}\{f, T\})$.

We observe that if $f = I_X$, then we have $C_{oin}\{I_X, T\} := \{u \in X : u = Tu\} = Fix(T)$, where $Fix(T)$ is the set of fixed points of T.

Remarks.

(a) Any Banach contraction (see Banach) on (X, d) is a Ćirić-Berinde map of type (C-B).

(b) Any Kannan mapping (see [12]) on (X, d) is a Ćirić-Berinde map of type (C-B).

(c) Any Zamfirescu mapping (see [14]) is a Ćirić-Berinde map of type (C-B).

(d) Any almost contraction (see [6] and [7]) a Ćirić-Berinde map of type (C-B).

(e) Any mapping T satisfying the condition (B) is a Ćirić-Berinde map of type (C-B).

(f) Any pair (T, f) of selfmaps of (X, d) satisfying the generalized condition (G.B) is is a Ćirić-Berinde pair of type (C-B). In particular, if T is an f-contraction on (X, d), then (T, f) is a Ćirić-Berinde pair of type (C-B).

Hence, the contractive condition (C-B) is more general than all the previous contractive conditions presented before.

We observe that if a selfmap T satisfies the generalized condition (G.B) with respect to a mapping f, then (T, f) is a Ćirić-Berinde pair of type (C-B). Through the next example, we prove that the converse is not true.

Example 2.1. Let $X = \{0, \frac{1}{2}, 1\}$ with the usual metric. We define a mapping $f, T : X \to X$ by

$$f(x) = Tx := \begin{cases} 1, & \text{if } x = 0, \\ \frac{1}{2}, & \text{if } x = \frac{1}{2}, \\ 0, & \text{if } x = 1. \end{cases}$$

Then the pair (f, T) satisfies generalized condition (C-B) with $\delta = \frac{1}{2}$ and $L = 1$. But (f, T) does not satisfy condition (G.B), for by taking $x = 0$ and $y = \frac{1}{2}$; condition (G.B) fails to hold for any $\delta \in \,]0, 1[$ and any $L \geq 0$.

So the conditions (G.B) and (C-B) are independent and the condition (C-B) is strictly weaker that the condition (G.B).

3. Coincidence and Common Fixed Point Theorems

The first main result of this paper reads as follow.

Theorem 3.1. *Let* (X, d) *be a metric space. Let* $f, T : X \to X$ *be selfmaps satisfying the following conditions*:

(H1) $T(X) \subseteq f(X)$.

(H2) *Either* $f(X)$ *or* $T(X)$ *is a complete subspace of X.*

(H3) (T, f) *is of Ćirić-Berinde pair of type* (C-B) *with parameters* $(\delta, L) \in [01) \times [0, +\infty)$.

Then the selfmaps f and T have at least a coincidence point in X. That is the set $C_{oin}\{f, T\}$ *is not empty.*

Proof. Let x_0 be an arbitrary point in X and choose a point x_1 in X such that $fx_1 = Tx_0$. This can be done since, by (H1) we know that $T(X) \subset f(X)$. By continuing this process, we construct two sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ of points of X fulfilling the following properties:

$$y_0 := fx_0 \quad \text{and} \quad y_n := fx_n = Tx_{n-1}, \quad n = 1, 2, 3, \cdots.$$

For all integer $n \geq 1$, we have

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), \\ &\qquad \frac{d(fx_{n-1}, Tx_n) + d(fx_n, Tx_{n-1})}{2}\} \\ &= \max\{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), \\ &\qquad \frac{d(y_{n-1}, y_{n+1}) + d(y_n, y_n)}{2}\} \\ &= \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}. \end{aligned}$$

Because $d(y_{n-1}, y_{n+1}) \leq d(y_{n-1}, y_n) + d(y_n, y_{n+1})$.

Thus by taking x_{n-1} for x and x_n for y in the inequality (C-B), it follows that

$$d(y_n, y_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \delta \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}$$
$$+ L \min\{d(y_{n-1}, y_{n+1}), d(y_n, y_n)\},$$

which further implies that

$$d(y_n, y_{n+1}) \leq \delta \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}. \tag{3.1}$$

To get a contradiction, suppose that for some positive integer n, we have

$$d(y_{n-1}, y_n) < d(y_n, y_{n+1}),$$

then, according to (3.1), we infer that $0 < d(y_n, y_{n+1}) \leq \delta d(y_n, y_{n+1})$ which gives $1 \leq \delta.$ This is impossible.

Hence, from (3.1), we deduce that

$$d(y_n, y_{n+1}) \leq \delta d(y_{n-1}, y_n)$$
$$\leq \ldots \leq \delta^n d(y_0, y_1),$$

Now, for any positive integers m and n with $m > n,$ we have

$$d(y_m, y_n) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m)$$
$$\leq [\delta^n + \delta^{n+1} + \cdots + \delta^{m-1})]d(y_0, y_1)$$
$$\leq \frac{\delta^n}{1-\delta} d(y_0, y_1).$$

which implies that $\{y_n\}$ is a Cauchy sequence.

(i) If $f(X)$ is a complete subspace of X, there exists a $y \in f(X)$ such that $y_n := fx_n \to y$. Hence we can find u in X such that $fu = y$. Now,

$$\begin{aligned} d(y, Tu) &\leq d(y, y_{n+1}) + d(y_{n+1}, Tu) \\ &= d(y, y_{n+1}) + d(Tx_n, Tu) \\ &\leq d(y, y_{n+1}) + \delta \max\{d(fx_n, fu), d(fx_n, Tx_n), d(fu, Tu), \\ &\quad \times \frac{d(fx_n, Tu) + d(fu, Tx_n)}{2}\} + L \min\{d(fx_n, Tu), d(fu, fx_{n+1})\} \\ &= d(y, y_{n+1}) + \delta \max\{d(y_n, y), d(y_n, y_{n+1}), d(y, Tu) \\ &\quad \times \frac{d(y_n, Tu) + d(y, y_{n+1})}{2}\} + L \min\{d(y_n, Tu), d(y, y_{n+1})\}, \end{aligned}$$

which by taking the limit as $n \to \infty$ gives that

$$d(y, Tu) \leq \delta \max\{d(y, y), d(y, y), d(y, Tu), \frac{d(y, Tu) + d(y, y)}{2}\}$$

$$+ L \min\{d(y, Tu), d(y, y)\},$$

which further implies

$$d(y, Tu) \leq \delta d(y, Tu).$$

Hence $d(y, Tu) = 0$ and then $fu = y = Tu$. This shows that u is a coincidence point of f and T. That is $u \in C_{oin}(f, T)$. Hence, the set $C_{oin}(f, T)$ is not empty.

(ii) If $T(X)$ is complete, then there exists a $z \in T(X)$ such that $Tx_n \to z$ as $n \to \infty$. Since $T(X) \subset f(X)$, then there exists a point $v \in X$ such that $z = f(v)$. Thus, we have $z \in f(X)$ and $fx_n \to z$ as $n \to \infty$. Now from the discussion made in the case (i), we infer that v is a coincidence point of f and T.

In all cases, the set $C_{oin}(f, T)$ is not empty and this ends the proof. □

By choosing $f = I_X$ in Theorem 3.1, we have the following corollary dealing with the existence of fixed points for Berinde maps of type (C-B).

Corollary 3.1. *Let* (X, d) *be a metric space. Let* $T : X \to X$ *satisfies the following condition*:

There exists $\delta \in\ ]0, 1[$ *and* $L \geq 0$ *such that*

$$d(Tx, Ty) \leq \delta M_1(x, y) + L \min\{d(x, Ty), d(y, Tx)\}, \tag{3.2}$$

for all $x, y \in X$, *where*

$$M_1(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}.$$

If $T(X)$ *is a complete subspace of X, then T has at least a fixed point.*

This corollary is Theorem 2.2 of [8].

Before stating the second main result of this section, we need to introduce the following class of functions.

Let $\mathcal{P}_6$ be the set of applications $F : \mathbb{R}^6 \to \mathbb{R}$ satisfying the following property:

(P): $F(t, t, 0, 0, t, t) > 0$, for every $t > 0$.

The following functions are examples of functions belonging to the class $\mathcal{P}_6$.

(i) $F(t_1, \ldots, t_6) = t_1 - c \max\{t_2, \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2}\}$, where $0 < c < 1$.

(ii) $F(t_1, \ldots, t_6) = t_1^4 - at_1^2t_2^2 - bt_1t_2t_3t_4 - ct_5^3t_6 - dt_5t_6^3$, where $a, b, c, d \geq 0$ and $0 < a + c + d < 1$.

(iii) $F(t_1, \ldots, t_6) = t_1^4 - c\frac{t_3^2t_4^2 + t_5^2t_6^2}{1 + t_2 + t_3t_4}$, where $0 < c < 1$.

Next we state the second main result of this paper.

Theorem 3.2. *Let* (X, d) *be a metric space. Let* $f, T : X \to X$ *be selfmaps satisfying the following conditions*:

(H1) $T(X) \subseteq f(X)$.

(H2) *Either* $f(X)$ *or* $T(X)$ *is a complete subspace of X.*

(H3) (T, f) *is of Berinde pair of type* (C-B) *with parameters* $(\delta, L) \in [01) \times [0, +\infty)$.

(H4) *There exists a function* $F \in \mathcal{P}_6$, $F : [0, +\infty)^6 \to \mathbb{R}$ *such that*

$$F(d(Tx, Ty), d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)) \leq 0,$$

$$\textit{for all } x, y \in X.$$

(H5) *The pair* (f, T) *is weakly compatible.*

Then the selfmaps f and T have a unique common fixed point in X.

Proof. By virtue of Theorem 3.1, the assumptions (H1), (H2) and (H3) ensure the existence of at least a coincidence point u in X. Therefore $y := f(u)$ is point of coincidence of the pair (f, T). It is easy to see that the condition (H4) ensures that this point of coincidence is unique. By using (H5) and Lemma 1.1, we infer that f and T have a unique common fixed point in X. This ends the proof. □

Corollary 3.2. *Let* (X, d) *be a metric space. Let* $f, T : X \to X$ *be selfmaps satisfying the following conditions*:

(A1) $T(X) \subseteq f(X)$.

(A2) *Either* $f(X)$ *or* $T(X)$ *is a complete subspace of X.*

(A3) *T satisfies generalized condition* (G.B) *associated with f.*

(A4) *The pair* (f, T) *is weakly compatible.*

Then the selfmaps f and T have a unique common fixed point in X.

Proof. The assumptions (A1), (A2) and (A4) ensure the assumptions (H1), (H2) and (H5) of Theorem 3.2. Obviously, the assumption (A3) ensures the condition (H3).

The assumption (A3) ensures also the condition (H4) with the particular function F defined for all $(t_1, t_2, t_3, t_4, t_5, t_6) \in [0, +\infty)^6$ by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) := t_1 - \delta \max\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\} - L \min\{t_3, t_4, t_5, t_6\}.$$

Therefore, by Theorem 3.2, f and T have a unique common fixed point. This ends the proof. □

Corollary 3.3. *Let* (X, d) *be a metric space. Let* $f, T : X \to X$ *be selfmaps satisfying the following conditions*:

(A1) $T(X) \subseteq f(X)$.

(A2) *Either* $f(X)$ *or* $T(X)$ *is a complete subspace of X.*

(A′3) *There exist* $\delta \in]0, 1[$ *and* $L \geq 0$ *such that*

$$d(Tx, Ty) \leq \delta m(x, y) + L \min\{d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\} \tag{3.3}$$

for all $x, y \in X$, *where*

$$m(x, y) = \max\{d(fx, fy), \frac{1}{2}[d(fx, Tx) + d(fy, Ty)], \frac{1}{2}[d(fy, Tx) + d(fx, Ty)]\}.$$

(A4) *The pair* (f, T) *is weakly compatible.*

Then the selfmaps f and T have a unique common fixed point in X.

Proof. The assumptions (A1), (A2) and (A4) ensure the assumptions (H1), (H2) and (H5) of Theorem 3.2. Obviously, the assumption (A′3) ensures the condition (H3) of Theorem 3.2.

The assumption (A′3) ensures also the condition (H4) of Theorem 3.2 with the particular function F defined for all $(t_1, t_2, t_3, t_4, t_5, t_6) \in [0, +\infty)^6$, by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) := t_1 - \delta \max\{t_2, \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2}\} - L \min\{t_3, t_4, t_5, t_6\}.$$

Therefore, by Theorem 3.2, f and T have a unique common fixed point. This ends the proof. □

The following example is in support of Theorem 3.2.

Example 3.1. Let $X = \{0, 1\}$ with usual metric. Define $T, f : X \to X$ by the following:

$$T(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x = 1, \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x = 1. \end{cases}$$

We observe that $T(X) = f(X)$ and the pair (f, T) is weakly compatible on X. Also, f and T satisfy the inequality (C-B) with $\delta = \frac{1}{2}$ and $L = 1$. Hence f and T satisfy all hypotheses of Theorem 3.2. So coincidence points of f and T exist.

Indeed, here, the set of coincidence points of f and T is the whole set X.

We observe that $d(T0, T1) = 1$, and $d(f0, f1) = 1$ so that for any $\alpha \in [0, 1)$, (f, T) fails to satisfy the generalized condition (G.B). Hence Theorem 1.5 is not applicable. Indeed, here the maps f and T have no common fixed points in X.

This example shows that Theorem 3.1 is a generalization of Theorem 1.5.

We end this section by the following result where we study the continuity of T in the set of coincidence points of a Berinde pair of maps of type (C).

Theorem 3.3. *Let* (X, d) *be a metric space. Let* $f, T : X \to X$ *be selfmaps satisfying the following conditions*:

(H1) $T(X) \subseteq f(X)$.

(H2) *Either* $f(X)$ *or* $T(X)$ *is a complete subspace of X.*

(H3) (T, f) *is of Berinde pair of type* (C-B) *with parameters* $(\delta, L) \in [01) \times [0, +\infty)$.

Let $u \in C_{oin}(f, T)$ *and suppose that f is continuous at u.*

Then T is continuous at u.

Proof. According to Theorem 3.1, we know that the set $C_{oin}(f, T)$ is not empty. Let $u \in F(f, T)$ and suppose that f is continuous at u.

Let $\{u_n\}$ be any sequence in X converging to u. Then by taking $y := u_n$ and $x := u$ in (C-B), we get

$$d(Tu, Tu_n) \leq \delta M(u, u_n) + L \min\{d(fu, Tu_n), d(fu_n, Tu)\}, \; n = 1, 2, \cdots,$$

where

$$M(u, u_n) = \max\{d(fu, fu_n), d(fu, Tu), d(fu_n, Tu_n), \frac{d(fu, Tu_n) + d(fu_n, Tu)}{2}\},$$

which, in view of $Tu = fu$, implies

$$d(Tu, Tu_n) \leq \delta \max\{d(fu, fu_n), d(fu_n, Tu_n), \frac{d(Tu, Tu_n) + d(fu_n, fu)}{2}\} + Ld(fu_n, fu)$$

$$\leq \delta(d(fu_n, fu) + d(Tu, Tu_n)) + Ld(fu_n, fu),$$

which further implies

$$d(Tu, Tu_n) \leq \frac{\delta + L}{1 - \delta} d(fu_n, fu), \quad n = 1, 2, \cdots.$$

Now, by letting $n \to \infty$ we get $Tu_n \to Tu$ as $n \to \infty$, because f is continuous at u. This shows that T is continuous at u. This ends the proof.

□

4. On Ćirić Almost Contractions

Berinde [7] introduced the concept of Ćirić almost contraction, that is, a mapping for which there exist a constant $\alpha \in [0, 1[$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \alpha M(x, y) + Ld(y, Tx), \text{ for all } x, y \in X, \tag{4.1}$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

By using symmetry, we see that (4.1) is equivalent to the following inequality:

$$d(Tx, Ty) \leq \alpha M(x, y) + L \min\{d(y, Tx), d(x, Ty)\}, \text{ for all } x, y \in X. \tag{4.2}$$

Berinde (see [8]) proved by an example that the condition (4.1) does not ensure the existence of fixed point in a complete metric space.

Following the similar arguments to those given in the proof of Theorem 3.1, we can prove the following theorem.

Theorem 4.1. *Let (X, d) be a complete metric space. Let $T : X \to X$ satisfying the following condition*:

There exist a constant $\delta \in \left]0, \frac{1}{2}\right[$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta M(x, y) + L \min\{d(x, Ty), d(y, Tx)\}, \tag{4.3}$$

for all $x, y \in X$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Then T has at lest a fixed point. That is Fix(T) is not empty.

Also by using the well known Theorem of Ćirić (see [9]), one can also deduce easily the following result.

Theorem 4.2. *Let* (X, d) *be a complete metric space. Let* $T : X \to X$ *satisfying the following condition*:

There exist a constant $\delta \in \,]0, 1[$ *and some* $L \geq 0$ *with* $L = 1 - \alpha$ *such that*

$$d(Tx, Ty) \leq \alpha M(x, y) + L \min\{d(x, Ty), d(y, Tx)\}, \tag{4.4}$$

for all $x, y \in X$, *where*

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Then T has at lest a fixed point. That is Fix(T) is not empty.

We define Δ a the set of all pairs of numbers $(\alpha, L) \in [0, 1) \times [0, +\infty)$ satisfying the following property:

(FPP): For all complete metric space (X, d) and for all strong Ćirić almost contraction with parameter (α, L), we have *Fix*(*T*) is not empty.

For every $(\alpha, L) \in [0, 1) \times [0, +\infty)$, let us denote $\mathcal{C}_{ac}(\alpha, L)$ the set of all Ćirić almost contractions with parameter (α, L). Then (FPP) means that all complete metric space (X, d) has the fixed point property for the class of mappings $\mathcal{C}_{ac}(\alpha, L)$.

Next we make a list of some observations:

(a) According to Theorem 4.1, we deduce that

$$[0, \frac{1}{2}) \times [0, +\infty) \subset \Delta.$$

(b) According to Theorem 4.2, we deduce that

$$\{(\alpha, L) \in [0, 1) \times [0, +\infty) : 0 \leq \alpha + L < 1\} \subset \Delta.$$

(c) A direct consequence of the classical theorem due to Ćirić (see [9]), we deduce that

$$[0, 1) \times \{0\} \subset \Delta.$$

Now the following question is natural:

Open problem: What is exactly the set Δ ?.

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