Research and Communications in Mathematics and Mathematical Sciences
Vol. 13, Issue 2, 2021, Pages 129-151
ISSN 2319-6939
Published Online on January 29, 2022

http://jyotiacademicpress.org

POSITION VECTOR OF A DEVELOPABLE SPACE-LIKE *h*-SLANT RULED SURFACE IN MINKOWSKI 3-SPACE

ONUR KAYA and MUSTAFA KAZAZ

Department of Mathematics
Manisa Celal Bayar University
45140, Manisa
Turkey
e-mail: onur.kaya@cbu.edu.tr
mustafa.kazaz@cbu.edu.tr

Abstract

In this study, we investigate the position vector of a developable space-like h-slant ruled surface in Minkowski 3-space. First, we determine the conical curvature of a space-like h-slant ruled surface. Then, by achieving a differential equation with respect to the central normal vector of the ruled surface, we give the position vector of a developable space-like h-slant ruled surface with striction line as its base curve. Finally, we give some examples for the obtained results and construct new developable space-like h-slant ruled surfaces.

1. Introduction

In geometry, special curves and surfaces such as helices and ruled surfaces have a key place because of their significant importance to certain areas of science and technology. For instance, as helices find

2020 Mathematics Subject Classification: 53A25, 53A35, 14J26.
Keywords and phrases: Minkowski 3-space, ruled surface, developable surface, h-slant.
Communicated by Suayip Yuzbasi.
Received September 13, 2021; Revised December 17, 2021

applications to themselves in biology and genetics, ruled surfaces have many applications in architecture, engineering, mathematical physics, computer graphics and robotics [5, 19]. Therefore, many mathematicians were attracted to study these subjects of geometry in different spaces [11, 17, 18]. In 2004, an important development regarding such special curves were made and Izumiya and Takeuchi brought a new aspect to helices. They defined slant helices with the property which principal normal lines of the curve make a constant angle with a fixed direction [6]. Later, Ali studied on the position vector of general helices and slant helices in both Euclidean and Minkowski spaces [1, 2, 3]. Ali and Turgut also studied on the position vector of slant helices in Minkowski 3-space [4].

Önder applied the slant notion to ruled surfaces and defined slant ruled surfaces [13, 14]. Önder and Kaya studied different properties of slant ruled surfaces [9, 10, 15]. In this paper, we study such surfaces in Minkowski 3-space $\mathbb{E}_1^3$. We find the position vectors of developable space-like h-slant ruled surfaces and therefore make it easy to construct such surfaces by using their conical curvature.

2 Preliminaries

Let $\mathbb{E}_1^3$ be the Minkowski 3-space and the Lorentzian metric be $\langle , \rangle = dx^2 + dy^2 - dz^2$ for a standard rectangular coordinate system (x, y, z). A vector $\vec{u}$ is called to be space-like (respectively, time-like or light-like (null)) if $\langle \vec{u}, \vec{u} \rangle > 0$ or $\vec{u} = 0$ (respectively, $\langle \vec{u}, \vec{u} \rangle < 0$ or $\langle \vec{u}, \vec{u} \rangle = 0$). If we consider a space curve $\vec{\alpha} = \vec{\alpha}(t)$, then $\vec{\alpha}$ is also called a space-like (time-like) curve if the velocity vector $\frac{d\vec{\alpha}}{dt}$ is a space-like (time-like) vector. In the case of surfaces, a surface in Minkowski 3-space is called a space-like surface if the induced metric on the surface is

Riemannian, i.e., the normal vector on the surface is a time-like vector. Throughout this paper, we do not deal with light-like vectors since we investigate space-like surfaces.

Considering two vectors $\vec{u}_1 = (x_1, y_1, z_1)$ and $\vec{u}_2 = (x_2, y_2, z_2)$, the cross product of such vectors are given by

$$\vec{u}_1 \times \vec{u}_2 = (x_3 y_2 - x_2 y_3, x_1 y_3 - x_3 y_1, x_1 y_2 - x_2 y_1)$$

in Minkowski 3-space [7].

Definition 2.1. Let $\vec{x}$ and $\vec{y}$ be non-null vectors in $\mathbb{E}_1^3$. Then, the angle function between these vectors are given by the equations:

(i) $\langle \vec{x}, \vec{y} \rangle = -\|\vec{x}\|\|\vec{y}\| \cosh \theta$ when $\vec{x}$ and $\vec{y}$ are time-like vectors;

(ii) $\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\|\|\vec{y}\| \cos \theta$ when $\vec{x}, \vec{y}$ are space-like vectors and the plane they span is space-like;

(iii) $|\langle \vec{x}, \vec{y} \rangle| = \|\vec{x}\|\|\vec{y}\| \cosh \theta$ when $\vec{x}, \vec{y}$ are space-like vectors and the plane they span is time-like;

(iv) $|\langle \vec{x}, \vec{y} \rangle| = \|\vec{x}\|\|\vec{y}\| \sinh \theta$ when $\vec{x}$ is a space-like and $\vec{y}$ is a time-like vector,

where $\|\vec{x}\| = \sqrt{|\langle \vec{x}, \vec{x} \rangle|}$ is the norm of the vector $\vec{x}$ [12, 16].

Now, in order to define a ruled surface S in Minkowski 3-space $\mathbb{E}_1^3$, we consider an open interval I of $\mathbb{R}$ and a curve $\vec{k} = \vec{k}(u)$ in $\mathbb{E}_1^3$ which is defined on this interval. Let $\vec{q} = \vec{q}(u)$ be a transversal unit vector field along $\vec{k} = \vec{k}(u)$. We have the following parametrization:

$$\vec{r}(u, v) = \vec{k}(u) + v\vec{q}(u), \quad u \in I, v \in \mathbb{R}, \tag{1}$$

which is called a ruled surface. Here, the curve $\vec{k}$ is called the base curve, $\vec{q}$ is called the director curve and the straight lines $v \to \vec{k}(u) + v\vec{q}(u)$ are called the rulings. If $\vec{q}$ is constant, the surface is called cylindrical and non-cylindrical otherwise [8].

From (1), the unit surface normal $\vec{n}$ of S is defined by $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$, where $\vec{r}_u = \dfrac{\partial \vec{r}}{\partial u}$ and $\vec{r}_v = \dfrac{\partial \vec{r}}{\partial v}$. When the parameter v increases infinitely along a ruling $u = u_1$ the unit normal $\vec{n}$ approaches a limiting direction which is called the asymptotic normal (central tangent) direction and it is defined by

$$\vec{a} = \lim_{v \to \pm\infty} \vec{n}(u_1, v) = \pm \frac{\vec{q} \times \dot{\vec{q}}}{\left\| \dot{\vec{q}} \right\|},$$

where $\vec{a}$ is unit, $\dot{\vec{q}} = d\vec{q}/du$ which are not light-like. The points where the unit surface normal and the asymptotic normal are perpendicular are called striction points and the set of all striction points is called the striction curve. Parametric representation of the striction curve is given by

$$\vec{c} = \vec{k} - \frac{\left\langle \dot{\vec{k}}, \dot{\vec{q}} \right\rangle}{\left\langle \dot{\vec{q}}, \dot{\vec{q}} \right\rangle} \vec{q}.$$

Hence, it is possible to represent a ruled surface by its striction curve as

$$\vec{r}(s, v) = \vec{c}(s) + v\vec{q}(s), \tag{2}$$

where s is the arc-length of $\vec{c}$ [8]. Henceforth, all given ruled surfaces will be in the form of (2).

The unit vector $\vec{h}$ which is called the central normal vector is defined by $\vec{h} = \pm \vec{a} \times \vec{q}$ (the sign differs due to the type of ruled surface) and thus the orthonormal frame $\left\{C, \vec{q}, \vec{h}, \vec{a}\right\}$ can be constructed at the striction point C and is called the Frenet frame of the surface S.

The set of unit space-like or time-like vectors $\vec{q}$ at origin constitutes a cone that is called the directing cone of the ruled surface S. The end points of those vectors drive a spherical curve $\vec{k}_1$ whose arc-length parameter is denoted by s_1 [8].

A ruled surface is called to be developable if the tangent plane contacts the surface at any point along a ruling. The Gaussian curvature vanishes for developable surfaces. Another characterization for developable surfaces is $\det(\vec{k}', \vec{q}, \vec{q}') = 0$ [8].

Lemma 2.1. *A ruled surface S is called space-like if the vector $\vec{q}$ is space-like, $\vec{h}$ is time-like and $\vec{a}$ is space-like* [17].

Let S be a space-like ruled surface in the Minkowski 3-space with the Frenet frame $\{\vec{q}, \vec{h}, \vec{a}\}$. From Lemma 2.1, we know that $\vec{q}$ is a space-like, $\vec{h}$ is a time-like and $\vec{a}$ is a space-like vector. Then, we have $\langle \vec{q}, \vec{q} \rangle = 1, \langle \vec{h}, \vec{h} \rangle = -1$ and $\langle \vec{a}, \vec{a} \rangle = 1$ and the striction curve of S is given by $\vec{c} = \vec{k} + \left\langle \frac{d\vec{k}}{ds}, \frac{d\vec{q}}{ds} \right\rangle \vec{q}$.

The Frenet formulae of space-like ruled surfaces are given by

$$\begin{bmatrix} \vec{q}' \\ \vec{h}' \\ \vec{a}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \kappa \\ 0 & \kappa & 0 \end{bmatrix} \begin{bmatrix} \vec{q} \\ \vec{h} \\ \vec{a} \end{bmatrix}, \tag{3}$$

where the sign "$'$" denotes derivation with respect to s_1, i.e., $\vec{q}' = dq/ds_1$ and $\kappa = \kappa(s_1)$ is the conical curvature of the directing cone which is defined by $\kappa = -\langle \vec{a}', \vec{h} \rangle$ [8, 14, 17].

Definition 2.2. Let S be a space-like ruled surface in Minkowski 3-space with Frenet frame $\{\vec{q}, \vec{h}, \vec{a}\}$ and conical curvature $\kappa \neq 0$. Then, S is a space-like h-slant slant ruled surface if the vector $\vec{h}$ of S makes a constant angle with a fixed non-null and non-zero direction $\vec{u}$, i.e.,

$$\langle \vec{h}, \vec{u} \rangle = \text{constant} \neq 0. \tag{4}$$

Here, the vector $\vec{u}$ is called the axis of the surface [14].

3. Position Vector of a Developable Space-Like *h*-Slant Ruled Surface

In this section, first, we calculate the conical curvature of space-like h-slant ruled surfaces. Then, we obtain a differential equation of third order for the central normal vector of a space-like ruled surface. By using these results, we give the position vector of developable space-like h-slant ruled surfaces.

Theorem 3.1. *Let S be a space-like ruled surface in Minkowski 3-space with Frenet frame $\{\vec{q}, \vec{h}, \vec{a}\}$ and conical curvature function $\kappa \neq 0$.*

(i) *Let $\vec{u}$ be a space-like vector. Then, the ruled surface S is a space-like h-slant ruled surface if and only if*

$$\kappa = \mp \frac{\tanh \theta s_1}{\sqrt{1 - \tanh^2 \theta s_1^2}}.$$

(ii) *Let $\vec{u}$ be a time-like vector. Then, the ruled surface S is a space-like h-slant ruled surface if and only if*

$$\kappa = \mp \frac{\coth \theta s_1}{\sqrt{1 - \coth^2 \theta s_1^2}},$$

where $\vec{u}$ is the of S and θ is the constant angle between the vectors $\vec{h}$ and $\vec{u}$.

Proof. (i) Let $\vec{u}$ be a space-like vector and S be a space-like h-slant ruled surface. Since θ is the angle between the vectors $\vec{h}$ and $\vec{u}$, from Definition 2.1 we have

$$\left|\left\langle \vec{h}, \vec{u} \right\rangle\right| = \sinh\theta \Rightarrow \left\langle \vec{h}, \vec{u} \right\rangle = \mp \sinh\theta.$$

By derivation of the above equation, we get

$$\langle \vec{q} + \kappa\vec{a}, \vec{u} \rangle = 0 \Rightarrow \langle \vec{q}, \vec{u} \rangle = -\kappa\langle \vec{a}, \vec{u} \rangle. \tag{5}$$

Substituting $\langle \vec{a}, \vec{u} \rangle = x$ yields $\langle \vec{q}, \vec{u} \rangle = -\kappa x$. Therefore, the vector $\vec{u}$ can be written as

$$\vec{u} = -\kappa x\vec{q} \mp \sinh\theta\vec{h} + x\vec{a}.$$

Since $\vec{u}$ is a unit vector, it follows

$$\begin{aligned} \langle \vec{u}, \vec{u} \rangle = 1 &\Rightarrow \kappa^2 x^2 - \sinh^2\theta + x^2 = 1 \\ &\Rightarrow x^2(\kappa^2 + 1) = 1 + \sinh^2\theta \\ &\Rightarrow x^2 = \frac{\cosh^2\theta}{\kappa^2 + 1} \\ &\Rightarrow x = \mp\frac{\cosh\theta}{\sqrt{\kappa^2 + 1}}. \end{aligned} \tag{6}$$

On the other hand, from the second derivative of Equation (5), we get

$$\begin{aligned} \langle \vec{h} + \kappa'\vec{a} + \kappa^2\vec{h}, \vec{u} \rangle = 0 &\Rightarrow \left\langle \vec{h}, \vec{u} \right\rangle + \kappa'\langle \vec{a}, \vec{u} \rangle + \kappa^2\left\langle \vec{h}, \vec{u} \right\rangle = 0 \\ &\Rightarrow \mp\sinh\theta + \kappa' x \mp \kappa^2 \sinh\theta = 0. \end{aligned} \tag{7}$$

Using the equalities (6) and (7), we have

$$\mp\frac{\kappa'}{(\kappa^2 + 1)^{3/2}} = \pm\tanh\theta,$$

and by integrating the last equation, it follows

$$\mp \frac{\kappa}{\sqrt{\kappa^2 + 1}} = \pm \tanh \theta(s_1 + c), \; c \in \mathbb{R}.$$

With the aid of a parameter change $s_1 \to s_1 - c$ we are able to eliminate the constant c. Therefore, from the last equation, we obtain

$$\mp \frac{\kappa}{\sqrt{\kappa^2 + 1}} = \pm \tanh \theta s_1 \Rightarrow \kappa = \mp \frac{\tanh \theta s_1}{\sqrt{1 - \tanh^2 \theta s_1^2}},$$

which is the desired result.

Conversely, let

$$\kappa = \mp \frac{\tanh \theta s_1}{\sqrt{1 - \tanh^2 \theta s_1^2}},$$

which can also be written as

$$\kappa = \mp \frac{s_1}{\sqrt{\coth^2 \theta - s_1^2}}.$$

Choosing x as

$$x = \pm \frac{\cosh \theta}{\sqrt{\kappa^2 + 1}} = \pm \sinh \theta \sqrt{\coth^2 \theta - s_1^2}, \tag{8}$$

yields

$$\kappa x = - \sinh \theta s_1. \tag{9}$$

Then, the vector $\vec{u}$ can be written as

$$\vec{u} = - \kappa x \vec{q} - \sinh \theta \vec{h} + x \vec{a}. \tag{10}$$

From (10), one can easily obtain that the vectors $\vec{h}$ and $\vec{u}$ make a constant angle, i.e., $\langle \vec{h}, \vec{u} \rangle = - \sinh \theta =$ constant. Finally, by substituting the Equations (8) and (9) in (10), we get

$$\vec{u} = \sinh\theta\left(s_1\vec{q} - \vec{h} \pm \sqrt{\coth^2\theta - s_1^2}\vec{a}\right),$$

which is a constant vector that can be checked by derivative $\vec{u}' = 0$. Therefore by Definition 2.2, S is a space-like h-slant ruled surface in Minkowski 3-space.

(ii) The proof of second case is similar to first case and left to the readers of the manuscript for the sake of avoiding recurrence. □

Theorem 3.2. *Let S be a space-like ruled surface in Minkowski 3-space with Frenet frame $\{\vec{q}, \vec{h}, \vec{a}\}$ and conical curvature function $\kappa \neq 0$. Then, the central normal vector $\vec{h}$ of S satisfy the differential equation given by*

$$\left[\frac{1}{\kappa'}\left(\vec{h}'' - (1+\kappa^2)\vec{h}\right)\right]' - \kappa\vec{h} = 0. \tag{11}$$

Proof. Let S be a space-like ruled surface. From (3) we have

$$\vec{h}' = \vec{q} + \kappa\vec{a}.$$

By differentiating the last equation, we get

$$\vec{a} = \frac{1}{\kappa'}\left(\vec{h}'' - (1+\kappa^2)\vec{h}\right),$$

and finally differentiating the last equation one more time, we obtain (11) which is the desired result. □

From now on, let us assume that S is always a developable space-like ruled surface in Minkowski 3-space. The tangent vector field of the striction line of a developable ruled surface coincides with the rulings. Therefore, we can write

$$\vec{c}' = \frac{dc}{ds}\frac{ds}{ds_1} = \vec{d}f, \tag{12}$$

where $f = \dfrac{ds}{ds_1}$.

By using Theorem 3.1, Theorem 3.2 and (12), we introduce the position vector of a developable space-like h-slant ruled surface.

Theorem 3.3. *Let S be a space-like ruled surface in Minkowski 3-space with Frenet frame $\{\vec{q}, \vec{h}, \vec{a}\}$, conical curvature function $\kappa \neq 0$ and striction line $\vec{c}$.*

(i) *Let $\vec{u}$ be space-like. Then, the position vector of the striction line of a developable space-like h-slant ruled surface is given by*

$$\begin{cases} c_1 = \cosh\theta \int f\left[\int \sinh[\operatorname{csch}\theta \arcsin(\tanh\theta s_1)]ds_1\right]ds_1, \\ c_2 = \pm\int f(\sinh\theta s_1 + n)ds_1, \\ c_3 = \cosh\theta \int f\left[\int \cosh[\operatorname{csch}\theta \arcsin(\tanh\theta s_1)]ds_1\right]ds_1, \end{cases}$$

or in parametric form

$$\begin{cases} c_1 = \cosh^2\theta \int \gamma\left[\int \sinh t \cos(\sinh\theta t)dt\right]dt, \\ c_2 = \pm\int \gamma[\cosh\theta \sin(\sinh\theta t) + n]\,dt, \\ c_3 = \cosh^2\theta \int \gamma\left[\int \sinh t \sin(\sinh\theta t)dt\right]dt. \end{cases}$$

(ii) *Let $\vec{u}$ be time-like. Then, the position vector of a developable space-like h-slant ruled surface is given by*

$$\begin{cases} c_1 = \sinh\theta \int f\left[\int \cos[\operatorname{sech}\theta \arcsin(\coth\theta s_1)]ds_1\right]ds_1, \\ c_2 = \sinh\theta \int f\left[\int \sin[\operatorname{sech}\theta \arcsin(\coth\theta s_1)]ds_1\right]ds_1, \\ c_3 = \int f(\cosh\theta s_1 + n)ds_1, \end{cases}$$

or in parametric form

$$\begin{cases} c_1 = \sinh^2\theta\int\gamma\left[\int\cos t\cos(\cosh\theta t)dt\right]dt, \\ c_2 = \sinh^2\theta\int\gamma\left[\int\sin t\cos(\cosh\theta t)dt\right]dt, \\ c_3 = \int\gamma[\sinh\theta\sin(\cosh\theta t)+n]\,dt, \end{cases}$$

where $\vec{c} = (c_1, c_2, c_3)$, θ *is the constant angle between the vectors* $\vec{h}$ *and* $\vec{u}$, $f = ds/ds_1$ *as defined in* (12), $n \in \mathbb{R}$ *and* $\gamma = ds/dt$.

Proof. Since S is a space-like ruled surface, the central normal vector $\vec{h}$ of S satisfy the differential equation

$$\left[\frac{1}{\kappa'}\left(\vec{h}'' - (1+\kappa^2)\vec{h}\right)\right]' - \kappa\vec{h} = 0,$$

that is given by (11).

(i) Let $\vec{u}$ be a space-like vector. In this case, from Theorem 3.1, we have

$$\kappa = \mp\frac{\tanh\theta s_1}{\sqrt{1-\tanh^2\theta s_1^2}}.$$

Substituting the last equation in (11), we get

$$(1-\tanh^2\theta s_1^2)\vec{h}''' - 3\tanh^2\theta s_1\vec{h}'' - \vec{h}' = 0. \tag{13}$$

Now, let the vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ be the basis vectors of E_1^3 and $h_1 = h_1(s_1)$, $h_2 = h_2(s_1)$, $h_3 = h_3(s_1)$ be smooth functions of the parameter s_1. Then the vector $\vec{h}$ can be written as

$$\vec{h} = h_1\vec{e}_1 + h_2\vec{e}_2 + h_3\vec{e}_3.$$

Since S is an h-slant ruled surface, without loss of generality, assume that the constant unit vector $\vec{u}$ be $\vec{e}_2$. Thus, we get $|h_2| = \sinh\theta$. Since the vector $\vec{h}$ is unit and time-like, we obtain

$$\left\langle \vec{h}, \vec{h} \right\rangle = -1 \Rightarrow h_1^2 + h_2^2 - h_3^2 = -1$$

$$\Rightarrow h_1^2 - h_3^2 = -1 - \sinh^2\theta$$

$$\Rightarrow h_3^2 - h_1^2 = \cosh^2\theta.$$

Considering $t = t(s_1)$ to be a smooth function, the solution of the last equation is

$$\begin{cases} h_1 = \pm\cosh\theta\sinh t, \\ h_3 = \pm\cosh\theta\cosh t. \end{cases}$$

From now on, the outcome of the theorem might vary due to the signs of h_1 and h_3. Let us choose the positive signs and take $h_1 = \cosh\theta\sinh t$ and $h_3 = \cosh\theta\cosh t$. Therefore, the vector $\vec{h}$ becomes

$$\vec{h} = (\cosh\theta\sinh t, \pm\sinh\theta, \cosh\theta\cosh t).$$

Since $\vec{h}$ satisfies (13), we obtain the system

$$\begin{cases} (1 - \tanh^2\theta s_1^2)[t''' + (t')^3] - 3\tanh^2\theta s_1 t'' - t' = 0, \\ (1 - \tanh^2\theta s_1^2)t'' - \tanh^2\theta s_1 t' = 0. \end{cases} \tag{14}$$

From the second equation of system (14), it follows:

$$t = n_1 \arccos(\tanh\theta s_1) + n_2 \text{ or } t = n_1 \arcsin(\tanh\theta s_1) + n_2,$$

where $n_1, n_2 \in \mathbb{R}$. Choosing $t = n_1 \arcsin(\tanh\theta s_1) + n_2$ and using the parameter change $t \to t + n_2$ yields $t = n_1 \arcsin(\tanh\theta s_1)$. Substituting $t = n_1 \arcsin(\tanh\theta s_1)$ in the first equation of the system (14), we get

$n_1 = \operatorname{csch}\theta$. Thus, we obtain $t = \operatorname{csch}\theta \arcsin(\tanh\theta s_1)$ and the components of the vector $\vec{h}$ become

$$\begin{cases} h_1 = \cosh\theta \sinh[\operatorname{csch}\theta \arcsin(\tanh\theta s_1)], \\ h_2 = \pm\sinh\theta, \\ h_3 = \cosh\theta \cosh[\operatorname{csch}\theta \arcsin(\tanh\theta s_1)]. \end{cases}$$

Since $\vec{q}\,' = \vec{h}$, the components of the vector $\vec{q}$ as in $\vec{q} = (q_1, q_2, q_3)$ are

$$\begin{cases} q_1 = \cosh\theta \int \sinh[\operatorname{csch}\theta \arcsin(\tanh\theta s_1)]ds_1, \\ q_2 = \pm\sinh\theta s_1 + n, \\ q_3 = \cosh\theta \int \cosh[\operatorname{csch}\theta \arcsin(\tanh\theta s_1)]ds_1. \end{cases}$$

Finally, since S is developable, from (12) we get $\vec{c}\,' = f\vec{q}$ and the components of the striction line of S are obtained as follows

$$\begin{cases} c_1 = \cosh\theta \int f\left[\int \sinh[\operatorname{csch}\theta \arcsin(\tanh\theta s_1)]ds_1\right]ds_1, \\ c_2 = \pm\int f(\sinh\theta s_1 + n)ds_1, \\ c_3 = \cosh\theta \int f\left[\int \cosh[\operatorname{csch}\theta \arcsin(\tanh\theta s_1)]ds_1\right]ds_1. \end{cases}$$

By substituting $t(s_1) = \operatorname{csch}\theta \arcsin(\tanh\theta s_1)$ we obtain

$$\begin{cases} c_1 = \cosh^2\theta \int \gamma\left[\int \sinh t \cos(\sinh\theta t)dt\right]dt, \\ c_2 = \pm\int \gamma[\cosh\theta \sin(\sinh\theta t) + n]\,dt, \\ c_3 = \cosh^2\theta \int \gamma\left[\int \sinh t \sin(\sinh\theta t)dt\right]dt, \end{cases}$$

which is parametric form.

(ii) The proof of the second case of this theorem is skipped to avoid recurrence. □

From Theorem 3.3, we have the following corollaries:

Corollary 3.4. *Let* S *be a space-like ruled surface in Minkowski 3-space with Frenet frame* $\{\vec{q}, \vec{h}, \vec{a}\}$, *conical curvature function* $\kappa \neq 0$, *striction line* $\vec{c} = (c_1, c_2, c_3)$ *and ruling* $\vec{q} = (q_1, q_2, q_3)$. *If* S *is a developable space-like h-slant ruled surface and*

(i) $\vec{u}$ *is a space-like vector, then the position vector of* S *with respect to the parameter* s_1 *is given by*

$$\vec{r}(s_1, v) = (c_1(s_1) + vq_1(s_1),\ c_2(s_1) + vq_2(s_1),\ c_3(s_1) + vq_3(s_1)),$$

where

$$\begin{cases} c_1 = \cosh\theta \int f \left[\int \sinh[\operatorname{csch}\theta \arcsin(\tanh\theta s_1)] ds_1\right] ds_1, \\ c_2 = \pm \int f(\sinh\theta s_1 + n) ds_1, \\ c_3 = \cosh\theta \int f \left[\int \cosh[\operatorname{csch}\theta \arcsin(\tanh\theta s_1)] ds_1\right] ds_1, \end{cases}$$

$$\begin{cases} q_1 = \cosh\theta \int \sinh[\operatorname{csch}\theta \arcsin(\tanh\theta s_1)] ds_1, \\ q_2 = \pm \sinh\theta s_1 + n, \\ q_3 = \cosh\theta \int \cosh[\operatorname{csch}\theta \arcsin(\tanh\theta s_1)] ds_1, \end{cases}$$

$n \in \mathbb{R}$ *and* θ *is the angle between the vectors* $\vec{h}$ *and* $\vec{u}$.

(ii) $\vec{u}$ *is a time-like vector, then the position vector of* S *with respect to the parameter* s_1 *is given by*

$$\vec{r}(s_1, v) = (c_1(s_1) + vq_1(s_1),\ c_2(s_1) + vq_2(s_1),\ c_3(s_1) + vq_3(s_1)),$$

where

$$\begin{cases} c_1 = \sinh\theta \int f \left[\int \cos[\operatorname{sech}\theta \arcsin(\coth\theta s_1)] ds_1\right] ds_1, \\ c_2 = \sinh\theta \int f \left[\int \sin[\operatorname{sech}\theta \arcsin(\coth\theta s_1)] ds_1\right] ds_1, \\ c_3 = \int f(\cosh\theta s_1 + n) ds_1, \end{cases}$$

$$\begin{cases} q_1 = \sinh\theta \displaystyle\int \cos[\operatorname{sech}\theta \arcsin(\coth\theta s_1)]\, ds_1, \\ q_2 = \sinh\theta \displaystyle\int \sin[\operatorname{sech}\theta \arcsin(\coth\theta s_1)]\, ds_1, \\ q_3 = \cosh\theta s_1 + n, \end{cases}$$

$n \in \mathbb{R}$ and θ is the angle between the vectors $\vec{h}$ and $\vec{u}$.

Corollary 3.5. *Let S be a space-like ruled surface in Minkowski 3-space with Frenet frame $\{\vec{q}, \vec{h}, \vec{a}\}$, conical curvature function $\kappa \neq 0$, striction line $\vec{c} = (c_1, c_2, c_3)$ and ruling $\vec{q} = (q_1, q_2, q_3)$. If S is a developable space-like h-slant ruled surface and*

(i) *$\vec{u}$ is a space-like vector, then the position vector of S with respect to the parameter t is given by*

$$\vec{r}(t, v) = (c_1(t) + vq_1(t),\ c_2(t) + vq_2(t),\ c_3(t) + vq_3(t)),$$

where

$$\begin{cases} c_1 = \cosh^2\theta \displaystyle\int \gamma\left[\int \sinh t \cos(\sinh\theta t) dt\right] dt, \\ c_2 = \pm \displaystyle\int \gamma[\cosh\theta \sin(\sinh\theta t) + n]\, dt, \\ c_3 = \cosh^2\theta \displaystyle\int \gamma\left[\int \cosh t \cos(\sinh\theta t) dt\right] dt, \end{cases}$$

$$\begin{cases} q_1 = \cosh^2\theta \displaystyle\int \sinh t \cos(\sinh\theta t) dt, \\ q_2 = \pm \cosh\theta \sin(\sinh\theta t) + n, \\ q_3 = \cosh^2\theta \displaystyle\int \cosh t \cos(\sinh\theta t) dt, \end{cases}$$

where θ is the angle between the vectors $\vec{h}$ and $\vec{u}$, $n \in \mathbb{R}$ and $\gamma = \frac{ds}{dt}$.

(ii) *$\vec{u}$ is a time-like vector, then the position vector of S with respect to the parameter t is given by*

$$\vec{r}(t, v) = (c_1(t) + vq_1(t),\ c_2(t) + vq_2(t),\ c_3(t) + vq_3(t)),$$

where

$$\begin{cases} c_1 = \sinh^2\theta \int \gamma \left[\int \cos t \cos(\cosh\theta t) dt \right] dt, \\ c_2 = \sinh^2\theta \int \gamma \left[\int \sin t \cos(\cosh\theta t) dt \right] dt, \\ c_3 = \int \gamma [\sinh\theta \sin(\cosh\theta t) + n] dt, \end{cases}$$

$$\begin{cases} q_1 = \sinh^2\theta \int \cos t \cos(\cosh\theta t) dt, \\ q_2 = \sinh^2\theta \int \sin t \cos(\cosh\theta t) dt, \\ q_3 = \sinh\theta \sin(\cosh\theta t) + n, \end{cases}$$

where θ *is the angle between the vectors* $\vec{h}$ *and* $\vec{u}$, $n \in \mathbb{R}$ *and* $\gamma = \dfrac{ds}{dt}$.

Similar to Theorem 3.3, we have the following theorem:

Theorem 3.6. *Let* S *be a space-like ruled surface in Minkowski 3-space with Frenet frame* $\{\vec{q}, \vec{h}, \vec{a}\}$, *conical curvature function* $\kappa \neq 0$, *striction line* $\vec{c} = (c_1, c_2, c_3)$ *and ruling* $\vec{q} = (q_1, q_2, q_3)$.

(i) *Let* $\vec{u}$ *be a space-like vector. Then, the position vector of the striction line of a developable space-like h-slant ruled surface is given by*

$$\begin{cases} c_1 = \cosh\theta \int f \left[\int \sinh[\operatorname{csch}\theta \arccos(\tanh\theta s_1)] ds_1 \right] ds_1, \\ c_2 = \pm \int f(\sinh\theta s_1 + n) ds_1, \\ c_3 = \cosh\theta \int f \left[\int \cosh[\operatorname{csch}\theta \arccos(\tanh\theta s_1)] ds_1 \right] ds_1, \end{cases}$$

or in the parametric form

$$\begin{cases} c_1 = -\cosh^2\theta \int \gamma \left[\int \sinh t \cos(\sinh\theta t)dt\right]dt, \\ c_2 = \pm \int \gamma[\cosh\theta \sin(\sinh\theta t) + n]dt, \\ c_3 = -\cosh^2\theta \int \gamma \left[\int \sinh t \cos(\sinh\theta t)dt\right]dt, \end{cases}$$

(ii) $\vec{u}$ *Let* $\vec{u}$ *be a time-like vector. Then, the position vector of the striction line of a developable space-like h-slant ruled surface is given by*

$$\begin{cases} c_1 = \sinh\theta \int f \left[\int \cos[\operatorname{sech}\theta \arccos(\coth\theta s_1)]ds_1\right]ds_1, \\ c_2 = \sinh\theta \int f \left[\int \sin[\operatorname{sech}\theta \arccos(\coth\theta s_1)]ds_1\right]ds_1, \\ c_3 = \int f\,(\cosh\theta s_1 + n)\, ds_1, \end{cases}$$

or in the parametric form

$$\begin{cases} c_1 = -\sinh^2\theta \int \gamma \left[\int \cos t \cos(\cosh\theta t)dt\right]dt, \\ c_2 = -\sinh^2\theta \int \gamma \left[\int \sin t \cos(\cosh\theta t)dt\right]dt, \\ c_3 = \int \gamma[\sinh\theta \cos(\cosh\theta t) + n]dt, \end{cases}$$

where θ *is the angle between the vectors* $\vec{h}$ *and* $\vec{u}$, $f = ds/ds_1$ *as defined in* (12), $n \in \mathbb{R}$ *and* $\gamma = ds/dt$.

By Theorem 3.6, following corollaries are achieved:

Corollary 3.7. *Let* S *be a space-like ruled surface in Minkowski* 3*-space with Frenet frame* $\{\vec{q}, \vec{h}, \vec{a}\}$, *conical curvature function* $\kappa \neq 0$, *striction line* $\vec{c} = (c_1, c_2, c_3)$ *and ruling* $\vec{q} = (q_1, q_2, q_3)$. *If* S *is a developable space-like h-slant ruled surface and*

(i) $\vec{u}$ *is a space-like vector, then the position vector of* S *with respect to the parameter* s_1 *is given by*

$$\vec{r}(s_1, v) = (c_1(s_1) + vq_1(s_1),\ c_2(s_1) + vq_2(s_1),\ c_3(s_1) + vq_3(s_1)),$$

where

$$\begin{cases} c_1 = \cosh\theta \int f \left[\int \sinh[\operatorname{csch}\theta \arccos(\tanh\theta s_1)] ds_1\right] ds_1, \\ c_2 = \pm \int f(\sinh\theta s_1 + n)\, ds_1, \\ c_3 = \cosh\theta \int f \left[\int \cosh[\operatorname{csch}\theta \arccos(\tanh\theta s_1)] ds_1\right] ds_1, \end{cases}$$

$$\begin{cases} q_1 = \cosh\theta \int \sinh[\operatorname{csch}\theta \arccos(\tanh\theta s_1)]\, ds_1, \\ q_2 = \pm \sinh\theta s_1 + n, \\ q_3 = \cosh\theta \int \cosh[\operatorname{csch}\theta \arccos(\tanh\theta s_1)]\, ds_1, \end{cases}$$

$n \in \mathbb{R}$ *and* θ *is the angle between the vectors* $\vec{h}$ *and* $\vec{u}$.

(ii) $\vec{u}$ *is a time-like vector, then the position vector of* S *with respect to the parameter* s_1 *is given by*

$$\vec{r}(s_1, v) = (c_1(s_1) + vq_1(s_1),\ c_2(s_1) + vq_2(s_1),\ c_3(s_1) + vq_3(s_1)),$$

where

$$\begin{cases} c_1 = \sinh\theta \int f \left[\int \cos[\operatorname{sech}\theta \arccos(\coth\theta s_1)] ds_1\right] ds_1, \\ c_2 = \sinh\theta \int f \left[\int \sin[\operatorname{sech}\theta \arccos(\coth\theta s_1)] ds_1\right] ds_1, \\ c_3 = \int f(\cosh\theta s_1 + n) ds_1, \end{cases}$$

$$\begin{cases} q_1 = \sinh\theta \int \cos[\operatorname{sech}\theta \arccos(\coth\theta s_1)] ds_1, \\ q_2 = \sinh\theta \int \sin[\operatorname{sech}\theta \arccos(\coth\theta s_1)] ds_1, \\ q_3 = \cosh\theta s_1 + n, \end{cases}$$

$n \in \mathbb{R}$ *and* θ *is the angle between the vectors* $\vec{h}$ *and* $\vec{u}$.

Corollary 3.8. *Let S be a space-like ruled surface in Minkowski 3-space with Frenet frame $\{\vec{q}, \vec{h}, \vec{a}\}$, conical curvature function $\kappa \neq 0$, striction line $\vec{c} = (c_1, c_2, c_3)$ and ruling $\vec{q} = (q_1, q_2, q_3)$. If S is a developable space-like h-slant ruled surface and*

(i) $\vec{u}$ *is a space-like vector, then the position vector of S with respect to the parameter t is given by*

$$\vec{r}(t, v) = (c_1(t) + vq_1(t),\ c_2(t) + vq_2(t),\ c_3(t) + vq_3(t)),$$

where

$$\begin{cases} c_1 = -\cosh^2\theta \int \gamma\left[\int \sinh t \cos(\sinh\theta t)dt\right]dt, \\ c_2 = \pm \int \gamma[\cosh\theta \sin(\sinh\theta t) + n]\,dt, \\ c_3 = -\cosh^2\theta \int \gamma\left[\int \sinh t \cos(\sinh\theta t)dt\right]dt, \end{cases}$$

$$\begin{cases} q_1 = -\cosh^2\theta \int \sinh t \cos(\sinh\theta t)dt, \\ q_2 = \pm\cosh\theta \sin(\sinh\theta t) + n, \\ q_3 = -\cosh^2\theta \int \sinh t \cos(\sinh\theta t)dt, \end{cases}$$

where θ is the angle between the vectors $\vec{h}$ and $\vec{u}$, $n \in \mathbb{R}$ and $\gamma = \frac{ds}{dt}$.

(ii) $\vec{u}$ *is a time-like vector, then the position vector of S with respect to the parameter t is given by*

$$\vec{r}(t, v) = (c_1(t) + vq_1(t),\ c_2(t) + vq_2(t),\ c_3(t) + vq_3(t)),$$

where

$$\begin{cases} c_1 = -\sinh^2\theta \int \gamma \left[\int \cos t \cos(\cosh \theta t) dt\right] dt, \\ c_2 = -\sinh^2\theta \int \gamma \left[\int \sin t \cos(\cosh \theta t) dt\right] dt, \\ c_3 = \int \gamma [\sinh\theta \cos(\cosh \theta t) + n] dt, \end{cases}$$

$$\begin{cases} q_1 = -\sinh^2\theta \int \cos t \cos(\cosh \theta t) dt, \\ q_2 = -\sinh^2\theta \int \sin t \cos(\cosh \theta t) dt, \\ q_3 = \sinh\theta \cos(\cosh \theta t) + n, \end{cases}$$

where θ *is the angle between the vectors* $\vec{h}$ *and* $\vec{u}$, $n \in \mathbb{R}$ *and* $\gamma = \dfrac{ds}{dt}$.

4. Examples

In this section, we will demonstrate some examples by using the results obtained in Section 3. One of the straight lines on the ruled surfaces are highlighted with red color.

Example 4.1. Let us consider the first case of Corollary 3.5. By taking $\theta = \pi/3$, $\gamma = t^2$ and $n = 1$, we obtain the position vector of a developable space-like h-slant ruled surface with a space-like axis $\vec{u}$. The graph of this surface is given by Figure 1 with the intervals $t \in [-1, 1]$ and $v \in [-1, 1]$.

Example 4.2. Let us consider the first case of Corollary 3.5. By taking $\theta = \pi/4$, $\gamma = e^t$ and $n = 2$, we obtain the position vector of a developable space-like h-slant ruled surface with a time-like axis $\vec{u}$. The graph of this surface is given by Figure 2 with the intervals $t \in [-1, 1]$ and $v \in [-2, 2]$.

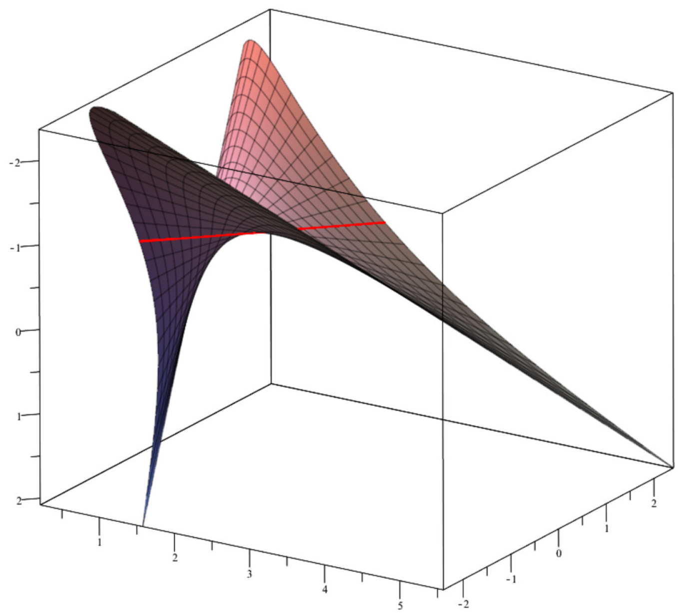

Figure 1. Developable space-like h-slant ruled surface with space-like axis.

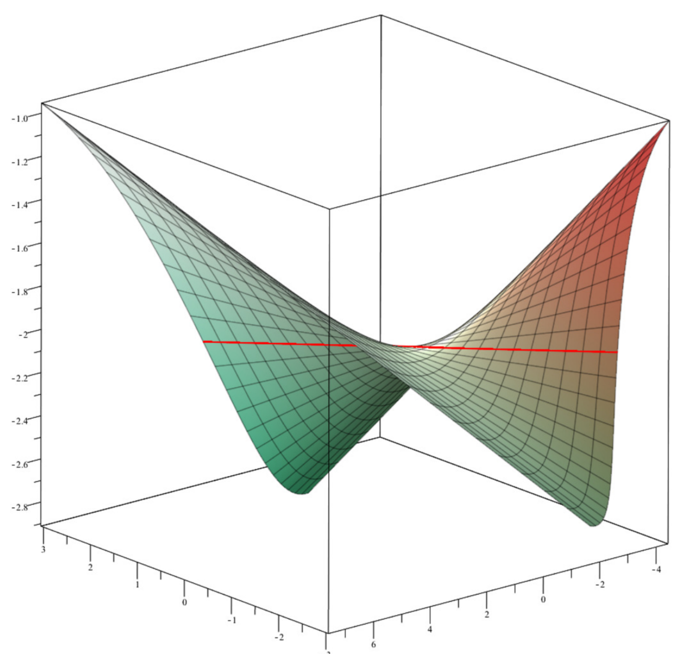

Figure 2. Developable space-like h-slant ruled surface with time-like axis.

References

[1] A. T. Ali, Position vectors of spacelike general helices in Minkowski 3-space, Nonlinear Analysis: Theory, Methods & Applications 73(4) (2010), 1118-1126.

DOI: https://doi.org/10.1016/j.na.2010.04.051

[2] A. T. Ali, Position vectors of general helices in Euclidean 3-space, Bulletin of Mathematical Analysis and Applications 3(2) (2010), 198-205.

[3] A. T. Ali, Position vectors of slant helices in Euclidean 3-space, Journal of the Egyptian Mathematical Society 20(1) (2012), 1-6.

DOI: https://doi.org/10.1016/j.joems.2011.12.005

[4] A. T. Ali and M. Turgut, Position vector of a time-like slant helix in Minkowski 3-space, Journal of Mathematical Analysis and Applications 365(2) (2010), 559-569.

DOI: https://doi.org/10.1016/j.jmaa.2009.11.026

[5] S. Flöry and H. Pottmann, Ruled Surfaces for Rationalization and Design in Architecture, LIFE in: Formation, On Responsive Information and Variations in Architecture (2010), 103-109.

[6] S. Izumiya and N. Takeuchi, New special curves and developable surfaces, Turkish Journal of Mathematics 28(2) (2004), 153-163.

[7] R. López, Differential geometry of curves and surfaces in Lorentz-Minkowski space, International Electronic Journal of Geometry 7(1) (2014), 44-107.

DOI: https://doi.org/10.36890/iejg.594497

[8] A. Karger and J. Novak, Space Kinematics and Lie Groups, STNL Publishers of Technical Lit., Prague, Czechoslovakia, 1978.

[9] O. Kaya and M. Önder, Position vector of a developable h-slant ruled surface, TWMS Journal of Applied and Engineering Mathematics 7(2) (2017), 322-331.

[10] O. Kaya and M. Önder, Position vector of a developable q-slant ruled surface, Korean Journal of Mathematics 26(4) (2018), 545-559.

DOI: https://doi.org/10.11568/kjm.2018.26.4.545

[11] Y. H. Kim and D. W. Yoon, Classification of ruled surfaces in Minkowski 3-spaces, Journal of Geometry and Physics 49(1) (2004), 89-100.

DOI: https://doi.org/10.1016/S0393-0440(03)00084-6

[12] E. Nešović, M. Petrović-Torgašev and L. Verstraelen, Curves in Lorentzian spaces, Bolletino dell'Unione Matematica Italiana 8(3) (2005), 685-696.

[13] M. Önder, Slant ruled surfaces, Transnational Journal of Pure and Applied Mathematics 1(1), (2018), 63-82.

[14] M. Önder, Non-null slant ruled surfaces, AIMS Mathematics 4(3) (2019), 384-396.

DOI: https://doi.org/10.3934/math.2019.3.384

[15] M. Önder and O. Kaya, Slant null scrolls in Minkowski 3-space E_1^3, Kuwait Journal of Science 43 (2016), 31-47.

[16] J. G. Ratcliffe, S. Axler and K. A. Ribet, Foundations of Hyperbolic Manifolds, Vol. 149. New York, Springer, 2006.

[17] A. Turgut and H. H. Hacısalihoğlu, Space-like ruled surfaces in the Minkowski 3-space, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 46 (1997), 83-91.

DOI: https://doi.org/10.1501/Commua1_0000000427

[18] A. Turgut and H. H. Hacısalihoğlu, Time-like ruled surfaces in the Minkowski 3-space-II, Turkish Journal of Mathematics 22(1) (1998), 33-46.

[19] B. van Sosin, M. Barton and G. Elber, Accessibility for line-cutting in free-form surfaces, Computer-Aided Design 114 (2019), 202-214.

DOI: https://doi.org/10.1016/j.cad.2019.05.014

■