Research and Communications in Mathematics and Mathematical Sciences
Vol. 13, Issue 1, 2021, Pages 13-29
ISSN 2319-6939
Published Online on February 12, 2021

http://jyotiacademicpress.org

ON MULTIPLICITY OF THE LAPLACIAN EIGENVALUE 2 IN BICYCLIC GRAPHS

M. FARKHONDEH[1], M. HABIBI[1] and D. A. MOJDEH[2]

[1]Department of Mathematics
Tafresh University
P. O. Box 39518-79611
Tafresh
Iran
e-mail: mfarkhondeh81@gmail.com
habibi.mohammad2@gmail.com

[2]Department of Mathematics
University of Mazandaran
P. O. Box 47416-95447
Babolsar
Iran
e-mail: damojdeh@umz.ac.ir

2020 Mathematics Subject Classification: 05C50, 11C08, 15A18.

Keywords and phrases: Laplacian eigenvalue, characteristic polynomial, multiplicity, unicyclic graph, bicyclic graph.

Communicated by Bao-Yuan Zhu.

Received November 5, 2020; Revised January 17, 2021

Abstract

The Laplacian matrix of a graph G is denoted by $L(G) = D(G) - A(G)$, where $D(G) = \text{diag}(d(v_1), \dots, d(v_n))$ and $A(G)$ is the adjacency matrix of G. A one-edge connection of two graphs G_1 and G_2 is a graph $G = G_1 \odot_{uv} G_2$ with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{e = uv\}$, where $u \in V(G_1)$ and $v \in V(G_2)$. In this article, we study the existence and multiplicity of Laplacian eigenvalue 2 in the bicyclic graph $G = G_1 \odot_{uv} G_2$ based on existence and multiplicity of this special Laplacian eigenvalue in G_1 and G_2. Examples to illustrate and delimit the results are provided.

1. Introduction

All graphs in this paper are finite and undirected with no loops or multiple edges. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Also, the degree of $v \in V(G)$ is denoted by $d(v)$. The *Laplacian matrix* of G is $L(G) = D(G) - A(G)$, where $D(G) = \text{diag}(d(v_1), \dots, d(v_n))$ is a diagonal matrix and $A(G)$ is the adjacency matrix of G. We shall use the notation $\lambda_k(G)$ to denote the k-th Laplacian eigenvalue of G and we assume that $\lambda_1(G) \geq \cdots \geq \lambda_n(G) = 0$. We also use the symbol $m_G(\lambda)$ to indicate the multiplication of the eigenvalue λ of $L(G)$. A vertex of degree one is called a *leaf* vertex and a vertex is said *quasi leaf* (support vertex) if it is incident to a leaf vertex. A *matching* of G is a set of pairwise disjoint edges of G. A perfect matching of a graph is a matching in which every vertex of the graph is incident to exactly one edge of the matching. A perfect matching is therefore a matching containing $\frac{n}{2}$ edges, meaning perfect matchings are only possible on graphs with an even number of vertices.

Connected graphs in which the number of edges equals the number of vertices are called *unicyclic graphs*. Therefore, a unicyclic graph is either a cycle or a cycle with some attached trees. Let $\mathfrak{U}_{n,g}$ be the set of all unicyclic graphs of order n with girth g. Throughout this paper, we suppose that the vertices of the cycle C_g are labelled by $v_1, \ldots, v_g$, ordered in a natural way around C_g, say in the clockwise direction. A *rooted tree* is a tree in which one vertex has been designated the root. Furthermore, assume that T_i is a rooted tree of order $n_i \geq 1$ attached to $v_i \in V(T_i) \cap V(C_g)$, where $\sum_{i=1}^{g} n_i = n$. This unicyclic graph is denoted by $C(T_1, \ldots, T_g)$. The *sun graph* of order $2n$ is a cycle C_n with an edge terminating in a leaf vertex attached to each vertex that is the corona of $C_n \circ K_1$. A *broken sun graph* is a unicyclic subgraph of a sun graph, so one can assume a sun graph is a broken sun graph too. A *one-edge* connection of two graphs G_1 and G_2 is a graph $G = G_1 \odot_{uv} G_2$ with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{e = uv\}$, where $u \in V(G_1)$ and $v \in V(G_2)$. We use the notation $u \sim v$ when $e = uv \in E(G)$.

By [7, Theorem 13], due to Kelmans and Chelnokov, the Laplacian coefficient, ξ_{n-k}, can be expressed in terms of subtree structures of G, for $0 \leq k \leq n$. Suppose that F is a spanning forest of G with components T_i of order n_i, and $\gamma(F) = \prod_{i=1}^{k} n_i$. The *Laplacian characteristic polynomial* of G is denoted by $L_G(\lambda) = \det(\lambda I - L(G)) = \sum_{i=0}^{n} (-1)^i \xi_i \lambda^{n-i}$. If M is a square matrix, then the determinant of M is denoted by $|M|$ and the *minor* of the entry in the i-th row and j-th column is the determinant of the submatrix formed by deleting the i-th row and j-th column. This number is often denoted by $M_{i,j}$. A square matrix is non-singular if its determinant is non-zero or it has an inverse.

Let G be a graph with n vertices. It is convenient to adopt the following terminology from [4]: for a vector $X = (x_1, \ldots, x_n)^t \in \mathbb{R}^n$, we say X gives a valuation of the vertex of V, and with each vertex v_i of V, we associate the number x_i, which is the value of the vertex v_i, that is $x(v_i) = x_i$. Then λ is an eigenvalue of $L(G)$ with the corresponding eigenvector $X = (x_1, \ldots, x_n)$ if and only if $X \neq 0$ and

$$(d(v_i) - \lambda)x_i = \sum_{v_j \in N(v_i)} x_j, \quad \forall i = 1, \ldots, n. \tag{1.1}$$

In [3], authors have considered several conditions in the bicyclic graph $G = G_1 \odot_{uv} G_2$ for having the Laplacian eigenvalue 2. In this article, we would like to continue studying on the existence and multiplicity of Laplacian eigenvalue 2 in bicyclic graphs $G = G_1 \odot_{uv} G_2$ based on existence and multiplicity of this special Laplacian eigenvalue in G_1 and G_2.

2. Main Results

It is well-known that the multiplicity of Laplacian eigenvalue 2 of unicyclic graphs is less than 3. In [3, Lemma 2], it was shown that for a bicyclic graph G, $m_G(2) < 4$. Now, we try to examine the multiplication of the Laplacian eigenvalue 2 of $G = G_1 \odot_{uv} G_2$, where G_1 and G_2 are unicyclic graphs, in different following cases:

(1) G_1 and G_2 do not have 2 among their Laplacian eigenvalues.

(2) Just one of G_1 or G_2 has 2 among its Laplacian eigenvalues. Without loss of generality, let $m_{G_1}(2) = 1$ or $m_{G_1}(2) = 2$.

(3) Both G_1 and G_2 have 2 among their Laplacian eigenvalues with multiplicity 1 or 2.

We start this article by the following example which shows that $G = G_1 \odot_{uv} G_2$ may have 2 among its Laplacian eigenvalues, even if both G_1 and G_2 do not have 2 among their Laplacian eigenvalues.

Example 2.1. Let $G_i \in \mathfrak{U}_{n_i, g_i}$ be two broken sun graphs with $n_i = 2g_i - 1$, for $i = 1, 2$. We name the unique vertices of degree 2 of G_1 and G_2, by u and v, respectively. Then $L(G)$ has 2 among its Laplacian eigenvalues, Although both G_1 and G_2 do not have 2 among their Laplacian eigenvalues, by [1, Theorems 5 and 10]. In fact, we may assign 1 and −1 to the vertices of C_{g_1} and C_{g_2}, respectively; and assign −1 and 1 to leaves of G_1 and G_2, respectively. By this assigning, we obtain a vector in $R^{n_1 + n_2}$ that satisfies in Equation (1.1) for $\lambda = 2$. In the following (see Figure 1), we have given this assigning for two specific graphs with $g_1 = 4$ and $g_2 = 3$.

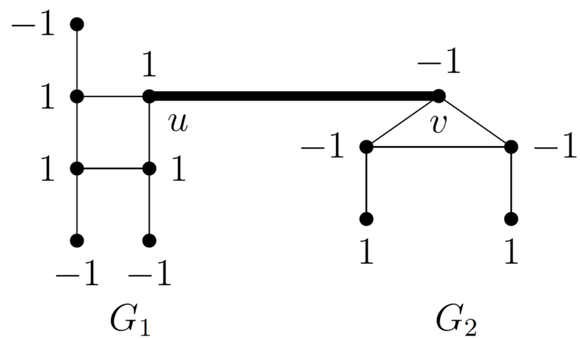

Figure 1. $G = G_1 \odot_{uv} G_2, \lambda = 2$ is an eigenvalue of $L(G)$.

Before proceeding, it is important to note that the selection of two vertices that connecting graphs G_1 and G_2 is very important and by changing them, different eigenvalues are obtained for the graph G, as in Figure 2 one can see.

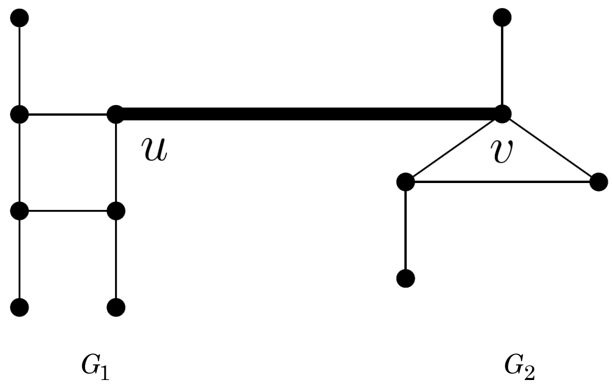

Figure 2. $L(G) = G_1 \odot_{uv} G_2$ and $m_G(2) = 0$.

Now, we give an example which shows that if $G_i \in \mathfrak{U}_{n_i, g_i}$ are two unicyclic graphs, for $i = 1, 2,$ in which G_1 has a perfect matching, $m_{G_1}(2) = 1$ and $m_{G_2}(2) = 0$, then it is possible $m_{G_1 \odot_{uv} G_2}(2) = 0$.

Example 2.2. It is well-known that each sun graph has 2 among its Laplacian eigenvalues. Also, it is easy to check that the broken sun graph G_2 and $G = G_1 \odot_{uv} G_2$ in Figure 3 do not have 2 among its Laplacian eigenvalues.

Next, we give a sufficient condition that $G = G_1 \odot_{uv} G_2$ has 2 among its Laplacian eigenvalues, for $G_i \in \mathfrak{U}_{n_i, g_i}$ with $m_{G_1}(2) = 1$ and $m_{G_2}(2) = 0$.

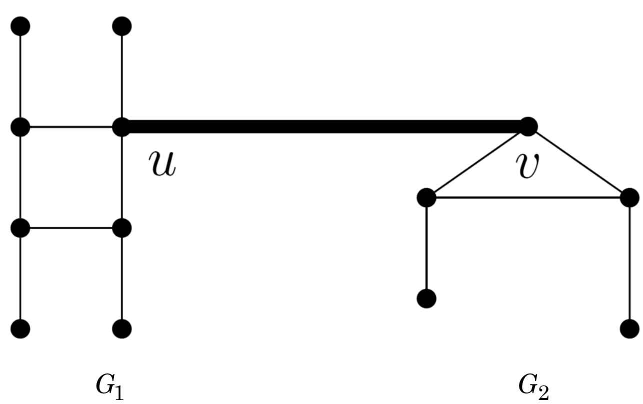

Figure 3. $L(G) = G_1 \odot_{uv} G_2$ and $m_G(2) = 0$.

Theorem 2.3. *Let* G_1 *be a broken sun graph of order* n_1 *which has no perfect matching and has* 2 *among its Laplacian eigenvalues. Also,* G_2 *is a unicyclic graph of order* n_2 *that does not have* 2 *among its Laplacian eigenvalues. Then* $G = G_1 \odot_{uv} G_2$ *has* 2 *among its Laplacian eigenvalues if one of the following conditions occurs*:

(1) $d(u) = 1$,

(2) $d(u) = 3$,

(3) $d(u) = 2$ *and* $d(u, w)$ *is even, where* w *has the shortest distance to* u *among all vertices of* G_1 *with degree* 3.

Proof. Let G_1 be a broken sun graph which has no perfect matching and has 2 among its Laplacian eigenvalues. So, $g_1 \equiv 0 \pmod 4$ and there exist odd number of vertices of degree 2 between any pair of consecutive vertices of degree 3, by [1, Theorem 10]. We may assign $\{-1, 0, 1\}$ to the vertices of C_{g_1}, by the pattern $0, 1, 0, -1$ consecutively starting with a vertex of degree 3, and assign to each leaf vertex the negative value of its neighbour to obtain an eigenvector, like X, of $L(G)$ corresponding to the eigenvalue 2. Note that by assigning vertices of G_1 in this method certainly $x(u) = 0$. So, by assigning 0 to all vertices of G_2, we can easily check that the eigenvector X satisfy in Equation (1.1) for $\lambda = 2$. This completes the proof. □

Example 2.4. (i) In Figure 4, G_1 and G_2 are broken sun graphs such that just one of them has 2 among its Laplacian eigenvalues. By Theorem 2.3 and by Equation (1.1),

$$X = (-1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)^t$$

is an eigenvector of $L(G)$ corresponding to the eigenvalue 2.

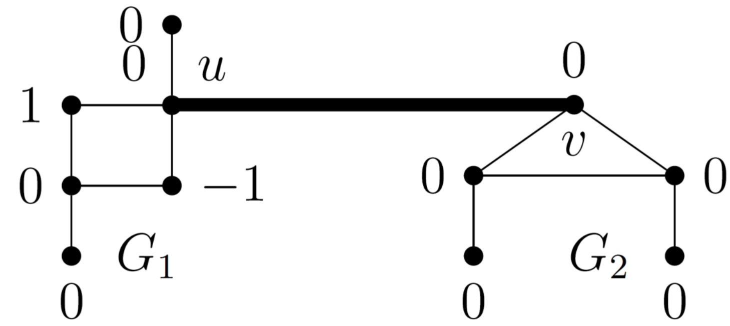

Figure 4. $G = G_1 \odot G_2$ has 2 as its Laplacian eigenvalue.

(ii) In Figure 5, we show that the condition for u in case $d(u) = 2$ in Theorem 2.3 is not superfluous.

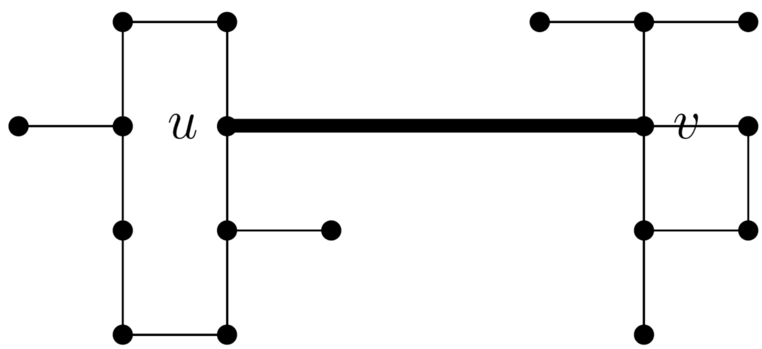

Figure 5. $G = G_1 \odot_{uv} G_2$ with $m_G(2) = 0$.

Now it's time to study the Laplacian eigenvalue 2 in $G = G_1 \odot_{uv} G_2$ for unicyclic graphs G_1 and G_2 with $m_{G_1}(2) = 2$ and $m_{G_2}(2) = 0$.

Theorem 2.5. *Let* $G_1 = C(T_1^{(1)}, \dots, T_{g_1}^{(1)})$ *be a unicyclic graph with a perfect matching and* $m_{G_1}(2) = 2$. *Also,* $G_2 = C(T_1^{(2)}, \dots, T_{g_2}^{(2)})$ *is a unicyclic graph such that* $m_{G_2}(2) = 0$. *It holds that* $G = G_1 \odot_{uv} G_2$ *has* 2 *among its Laplacian eigenvalues.*

Proof. Let $G_1 = C(T_1^{(1)}, \ldots, T_{g_1}^{(1)})$ be a unicyclic graph with a perfect matching and $m_G(2) = 2$. So, $g_1 \equiv 0 \pmod 4$ and $s_1 = g_1$, where s_1 is the number of trees of odd orders in G_1, by [1, Theorem 13]. So, to obtain an eigenvector like X of $L(G)$ corresponding to the eigenvalue 2, we have two situations:

(1) If $u \in C_{g_1}$, then we may assign $\{-1, 0, 1\}$ to the vertices of C_{g_1}, by the pattern $0, 1, 0, -1$ consecutively starting with the vertex u. Now, for u_i, the root of $T_i^{(1)}$, $i = 1, \ldots, C_{g_1}$ if $x(u_i) = 0$, then we assign 0 to each vertex of $T_i^{(1)}$, and if $x(u_i) \neq 0$, then we assign all vertices of the tree with $1, -1$ such that Equation (1.1) is satisfied for $\lambda = 2$. Furthermore, we assign 0 to each vertex of G_2 (note that Equation (1.1) is satisfied).

(2) If $u \notin C_{g_1}$, without loss of generality we can assume that $u \in V(T_1^{(1)})$. We assign 0 to each vertex of the tree $T_1^{(1)}$ and assign $\{-1, 0, 1\}$ to the vertices of C_{g_1}, by the pattern $0, 1, 0, -1$ consecutively starting with the root of $T_1^{(1)}$. The other vertices of G are assigned like case (1). So, G has 2 among its Laplacian eigenvalues and the proof is complete. □

Example 2.6. In Figure 6, G_1 and G_2 be two unicyclic graphs with $m_{G_1}(2) = 2$ and $m_{G_2}(2) = 0$. The bicyclic graph $G_1 \odot_{uv} G_2$ has 2 among its Laplacian eigenvalues with multiplicity 1.

We continue this article by studying the multiplicity of the Laplacian eigenvalue 2 of $G_1 \odot_{uv} G_2$, where the unicyclic graphs G_1 and G_2 have 2 among their Laplacian eigenvalues. For this, we need the following results.

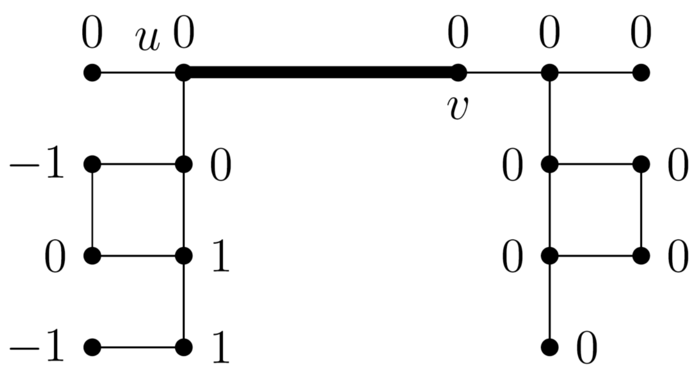

Figure 6. $G = G_1 \odot_{uv} G_2$ with $m_G(2) = 1$.

Lemma 2.7 ([2, Lemma 2.2]). *If* M *is a non-singular square matrix, then*

$$\det\begin{pmatrix} M & N \\ P & Q \end{pmatrix} = \begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M| \cdot |Q - PM^{-1}N|. \tag{2.1}$$

We use the notation M_u when in $M_{i,i}$, i is the row and column corresponding to the vertex u in a graph.

Lemma 2.8 ([6, Lemma 8]). *Let* G_1 *and* G_2 *be two graphs of order* n *and* m, *respectively. Then the Laplacian characteristic polynomial of* $G = G_1 \odot_{uv} G_2$ *is*

$$L_G(\lambda) = M.Q - M.Q_v - Q.M_u, \tag{2.2}$$

where $M = \lambda I - L(G_1)$ *and* $Q = \lambda I - L(G_2)$.

Corollary 2.9. *Let* G *be a unicyclic graph on* n *vertices. If* $\lambda_1, \ldots, \lambda_n$ *are the Laplacian eigenvalues of* G, *then* $\lambda_1, \ldots, \lambda_n$ *are the Laplacian eigenvalues of* $G' = G \odot_{uv} G$.

Proof. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are the vertices of G and the copy of G, respectively. Let $G' = G \odot_{u_i v_j} G$ and $M = L_G(\lambda)$. Therefore $L_{G'}(\lambda) = M^2 - M.M_{v_j} - M.M_{v_i}$, by Equation (2.2). So $L_{G'}(\lambda) = M(M - M_{u_i} - M_{v_j})$ and the result follows. □

Let G_1 and G_2 be two unicyclic graphs such that $m_{G_1}(2) = m_{G_2}(2) = 1$. In [3], it has been proven that $G = G_1 \odot_{uv} G_2$ has 2 as its Laplacian eigenvalue. In this case, the multiplicity of the eigenvalue 2 in bicyclic graph G may be 1, 2 or 3.

Example 2.10. Let G be a unicyclic graph as follows. It is not hard to check that $m_G(2) = 1$ and $m_{G \odot_{uv} G}(2) = 3$ (see Figure 7).

Figure 7. $G' = G \odot_{uv} G$ with $m_{G'}(2) = 3$.

Now, we identified some bicyclic graphs that having 2 among their Laplacian eigenvalues with multiplicity 3.

Theorem 2.11 ([Main Theorem])**.** *Let G_1 and G_2 be unicyclic graphs containing a perfect matching with $m_{G_1}(2) = m_{G_2}(2) = 2$. It holds that $m_{G_1 \odot_{uv} G_2}(2) = 3$, where $u \in C_{g_1}$ and $v \in C_{g_2}$.*

Proof. If $m_{G_1}(2) = m_{G_2}(2) = 2$, then there are two situations.

(1) If $G_1 = C_m$ and $G_2 = C_n$ then $m \equiv n \equiv 0 \pmod 4$, by [1, Theorem 12].

(2) If $G_1 = C(T_1^{(1)}, \ldots, T_{g_1}^{(1)})$ and $G_2 = C(T_1^{(2)}, \ldots, T_{g_2}^{(2)})$ such that $\sum_{k=1}^{k=g_1} |V(T_k^{(1)})| = m$, $\sum_{k=1}^{k=g_2} |V(T_k^{(2)})| = n$ and there exists at least one i (or j) so that $|V(T_i^{(1)})| \geq 3$ (or $|V(T_j^{(2)})| \geq 3$). Let

$$s_i = |\{T_k^{(i)} : |V(T_k^{(i)})| \text{ is odd}; 1 \leq k \leq g_i\}|; \quad i = 1, 2,$$

so $s_i = g_i$ and $g_i \equiv 0 \pmod 4$, for $i = 1, 2$, by [1, Theorem 13].

In addition, $G = G_1 \odot uvG_2$ is a bicyclic graph so $m_G(2) \leq 3$, by [3, Lemma 2]. On the other hand,

$$L_{G_1}(\lambda) = |M| = (\lambda - 2)^2 f(\lambda), \qquad L_{G_2}(\lambda) = |Q| = (\lambda - 2)^2 g(\lambda).$$

According to Equation (2.2), it is enough to show that M_u and Q_v have $\lambda - 2$ as a factor.

Case 1: Let $G_1 = C_m$, $G_2 = C_n$ and $m \equiv n \equiv 0 \pmod 4$.

We use induction on m. If $G_1 = C_4$, then the result follows and so the induction basis holds.

$$M_u = \begin{vmatrix} \lambda - 2 & 1 & 0 \\ 1 & \lambda - 2 & 1 \\ 0 & 1 & \lambda - 2 \end{vmatrix} = (\lambda - 2)(\lambda^2 - 4\lambda + 2).$$

Suppose in the cyclic graph C_{m-4}, $M_u(C_{m-4})$ with $m - 4 \equiv 0 \pmod 4$ has $\lambda - 2$ as a factor.

Let $G_1 = C_m$ and without loss of generality, assume that $u = u_m$. So

$$M_u = \begin{vmatrix} M' & N \\ N^T & Q' \end{vmatrix},$$

such that

$$M' = \begin{pmatrix} \lambda - 2 & 1 & 0 & 0 \\ 1 & \lambda - 2 & 1 & 0 \\ 0 & 1 & \lambda - 2 & 1 \\ 0 & 0 & 1 & \lambda - 2 \end{pmatrix},$$

$$Q' = (q'_{ij});\ q'_{ij} = \begin{cases} \lambda - 2, & i = j; \\ a_{ij}, & i \neq j; \end{cases} \quad 1 \leq i,\ j \leq m - 5,$$

and

$$N_{4\times(m-5)} = (n_{ij});\ n_{ij} = \begin{cases} 1, & i = 4,\ j = 1; \\ 0, & o.w; \end{cases} \quad 1 \leq i \leq 4 \text{ and } 1 \leq j \leq m - 5.$$

Therefore $M_u = |M'||Q' - N^T M'^{-1} N|$, by Lemma 2.1.

Also

$$[N^T M'^{-1} N]_{m-5\,\times\,m-5} = \begin{pmatrix} \frac{M'_{4,4}}{|M'|} & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then

$Q' - N^T M'^{-1} N = (b'_{ij});$

$$b'_{ij} = \begin{cases} \lambda - 2 - \dfrac{M'_{4,4}}{|M'|}, & i = j = 1; \\ \lambda - 2, & i = j \neq 1; \quad 1 \leq i, j \leq m - 5. \\ a_{ij}, & i \neq j; \end{cases}$$

Furthermore, $|Q' - N^T M'^{-1} N| = |Q'| + |H|,$ such that

$$H = (h_{ij}); \quad h_{ij} = \begin{cases} -\dfrac{M'_{4,4}}{|M'|}, & i = j = 1; \\ 0, & i = 1 \neq j; \\ \lambda - 2, & i = j \neq 1; \\ a_{ij}, & i \neq 1, i \neq j; \end{cases} \quad 1 \leq i, j \leq m - 5.$$

Consequently,

$$|Q' - N^T M'^{-1} N| = |Q'| - \frac{M'_{4,4}}{|M'|} Q'_{1,1}$$

$$\Rightarrow M_u = |M'||Q'| - M'_{4,4} Q'_{1,1}.$$

On the other hand

$$M'_{4,4} = \begin{vmatrix} \lambda - 2 & 1 & 0 \\ 1 & \lambda - 2 & 1 \\ 0 & 1 & \lambda - 2 \end{vmatrix} = (\lambda - 2)(\lambda^2 - 4\lambda + 2).$$

Also, $|Q'_{(m-5)\times(m-5)}|$ according to the induction hypothesis has $\lambda - 2$ as a factor therefore, M_u has $\lambda - 2$ as a factor. With a similar method, one can check that Q_v has $\lambda - 2$ as a factor. Thus, $m_G(2) = 3$ and we are done.

Case 2: Let $G_1 = C(T_1^{(1)}, \ldots, T_{g_1}^{(1)})$ and $G_2 = C(T_1^{(2)}, \ldots, T_{g_2}^{(2)})$ so $s_i = g_i$ and $g_i \equiv 0 \pmod 4$, for $i = 1, 2$. In addition, there exists some $i(j)$, such that $|V(T_i^{(1)})| \geq 3$ $(|V(T_j^{(2)})| \geq 3)$. Let $u' \in V(T_i^{(1)})$ and $d(u', u_i) = \max_{x \in V(T_i^{(1)})} d(x, u_i)$, where u_i is the root of $T_i^{(1)}$. Since G_1 has a perfect matching, u' is a leaf vertex and its neighbour, say u'', has degree 2. Thus, $G = (G \setminus \{u', u''\}) \odot S_2$. Using [5, Theorem 2.5], we obtain $m_G(2) = m_{G \setminus \{u', u''\}}(2)$. So by repeating this method in the graph $G \setminus \{u', u''\}$ and omitting all leaf vertices and quasi leaf vertices of degree 2 and using [5, Theorem 2.5], the obtained graph is two cycles like graph in the case 1. So $m_G(2) = 3$ and the proof is complete. □

Example 2.12. G_1 and G_2 are two unicyclic graphs with $m_{G_1}(2) = m_{G_2}(2) = 2$. Also, there exists at least one k such that $|T_k^{(i)}| \geq 3$ for $i = 1$ or $i = 2$. In Figure 8, the bicyclic graph $G_1 \odot_{uv} G_2$ has 2 among its Laplacian eigenvalues with multiplicity 3.

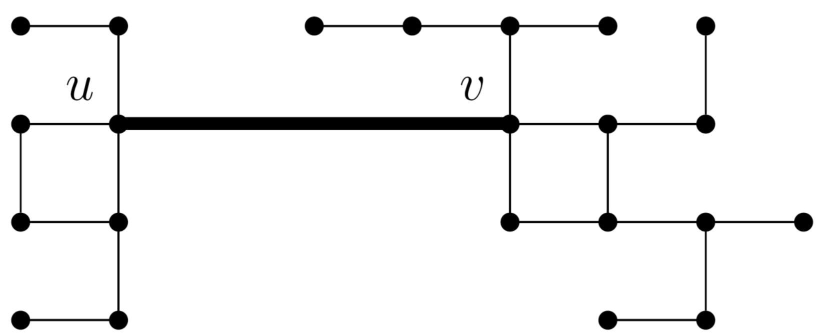

Figure 8. $G = G_1 \odot_{uv} G_2$ with $m_G(2) = 3$.

According to the Example 2.10, we can see that the converse of Theorem 2.11 does not necessarily true, in general.

Theorem 2.13. *Let* G_1 *be a unicyclic graph with a perfect matching and* $m_{G_1}(2) = 2$. *Also,* G_2 *is a unicyclic graph such that* $m_{G_2}(2) = 1$. *So,* $m_{G_1 \odot_{uv} G_2}(2) \geq 2$, *where* $u \in C_{g_1}$.

Proof. It is proven like Theorem 2.11, by a similar method. □

Example 2.14. G_1 and G_2 are two unicyclic graphs with $m_{G_1}(2) = 2$ and $m_{G_2}(2) = 1$. The bicyclic graph $G_1 \odot_{uv} G_2$ has 2 among its Laplacian eigenvalues with multiplicity 2 (see Figure 9).

Figure 9. $G = G_1 \odot_{uv} G_2$ with $m_G(2) = 2$.

In this paper, we studied the existence and multiplicity of Laplacian eigenvalue 2 in the bicyclic graph $G = G_1 \odot_{uv} G_2$ based on existence and multiplicity of this special Laplacian eigenvalue in G_1 and G_2.

Now, we pose the following problem.

Problem. In the definition of $G = G_1 \odot_{uv} G_2$, if we replace the edge uv with the path $p_n = u_1 - u_2 - \cdots - u_n$ for $n \geq 3$, then whether we can obtain similar results.

Acknowledgements

We would like to thank the anonymous referee for his/her valuable suggestions and comments.

References

[1] S. Akbari, D. Kiani and M. Mirzakhah, The multiplicity of Laplacian eigenvalue two in unicyclic graphs, Linear Algebra and its Applications 445 (2014), 18-28.

DOI: https://doi.org/10.1016/j.laa.2013.11.022

[2] D. Cvetkovic, M. Doob and H. Sachs, Spectra of Graphs Theory and Applications, 3rd Edition, Johann Ambrosius Barth Verlag, Heidelberg-Leipzig, 1995.

[3] M. Farkhondeh, M. Habibi, D. A. Mojdeh and Y. Rao, Some bicyclic graphs having 2 as their Laplacian eigenvalues, Mathematices 7(12) (2019); Article 1233.

DOI: https://doi.org/10.3390/math7121233

[4] M. Fiedler, A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory, Czechoslovak Mathematical Journal 25(4) (1975), 619-633.

DOI: https://doi.org/10.21136/CMJ.1975.101357

[5] R. Grone, R. Merris and V. S. Sunder, The Laplacian spectrum of a graph, SIAM Journal on Matrix Analysis and Applications 11(2) (1990), 218-238.

DOI: https://doi.org/10.1137/0611016

[6] J. M. Guo, On the second largest Laplacian eigenvalue of trees, Linear Algebra and its Applications 404 (2005), 251-261.

DOI: https://doi.org/10.1016/j.laa.2005.02.031

[7] P. V. Mieghem, Graph Spectra for Complex Networks, Cambridge University Press, Cambridge, 2011.

DOI: https://doi.org/10.1017/CBO9780511921681

[8] B. Mohar, The Laplacian Spectrum of Graphs, In: Y. Alavi, G. Chartrand, O. R. Oellermann and A. J. Schwenk (Editors), Graph Theory, Combinatorics, and Applications 2 (1991), 871-898.

■