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VISCOSITY SOLUTIONS FOR THE INHOMOGENEOUS RELATIVISTIC VLASOV EQUATION ON A SPHERICALLY SYMMETRIC GRAVITATIONAL FIELD SPACE-TIME

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Abstract

The viscosity solution for the inhomogeneous relativistic Vlasov equation is investigated. We consider this equation on a spherically symmetric gravitational field space-time. We rigorously prove the existence result of the viscosity solution using method of vanishing viscosity and uniqueness result of this type of solution using comparison method in details.

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1. Introduction

The Vlasov equation is one of the basic equation of the relativistic kinetic theory, where particles are fast moving without collisions. It is distinguished from other equations of kinetic theory by the fact that there is no direct interaction between particles. Solely by the field which is collectively generated by the motion of particles. The field which are taken into account depend on the physical situation being modelled. In gravitational physics, which is considered here, the field are described by the Einstein's equations. The best known applications of the Vlasov equation to self-gravitating systems are to stellar dynamics (see [4]). It can also be applied in Cosmology. In the first case, considered systems are galaxies or parts of galaxies where there is not too much dust or gas which would require a hydrodynamical treatment. Possible applications are to globular clurters, elliptical galaxies and the central bulge of spiral galaxies. In all these cases, particles are stars. In the cosmological situation they might be galaxies or even clusters of galaxies. The fact that they are modelled as particles reflects the irrelevance of their internal structure to the dynamics of the system as a whole. The Vlasov equation or the Einstein-Vlasov system has been widely studied in literature, see [12, 17, 18, 19, 20, 23]. One of the simplest ways to describe the universe is to consider the spherically symmetric gravitational field space-time with suitable matter models. Considering this kind of space-time supposes that, the metric admits an action of the group $SO(3)$ by isometries and takes the form (3) below. The spherically symmetric gravitational field is one of the most important models used in relativistic kinetic theory. For this reason, a lot of works have been done on both the Vlasov equation and on the Einstein-Vlasov system, in spherically symmetric background, see [6, 14, 16, 21, 22]. First, we study the existence and uniqueness theorem for the viscosity solutions of the relativistic Vlasov equation on an inhomogeneous spherically symmetric gravitational field space-time. Note that, the primary objectives of the theory of viscosity solutions applied to certain partial differential equations are that it allows merely continuous functions to be solutions of these equations. This method has already been used to solve the relativistic Vlasov equation, but taking as background homogeneous Bianchi models [2, 11]. For more details on the notion of viscosity solutions, see [3, 5, 7, 8, 9, 10, 13, 15]. We investigate the inhomogeneous case and we rigorously prove the existence and uniqueness of the viscosity solution in details using the method of "vanishing viscosity" described in [13]. The method used here is completely different from the method used in literature to solve the relativistic Vlasov equation in the spherically symmetric gravitational field space-time.

The paper is organized as follows:

In Section 2, we provide the formulation of the relativistic Vlasov equation in Cartesian coordinates, taking as background a spherically symmetric gravitational field space-time.

In Section 3, we give the formulation of the relativistic Vlasov equation, in our geometrical background, as a Hamilton-Jacobi equation and we establish energy estimates.

In Section 4, we prove existence and uniqueness theorems for the viscosity solution of the relativistic Vlasov equation.

2. The Equation and the Space-Time

In this paper, greek indices $(\beta, \gamma, \lambda, \ldots)$ vary from 0 to 3 and latin ows $(i, j, l, \ldots)$ from 1 to 3. Unless otherwise specified, we adopt the Einstein summation convention $a_\alpha b^\alpha = \sum_\alpha a_\alpha b^\alpha$.

We consider the inhomogeneous Vlasov equation of the form

$$p^\alpha \frac{\partial f}{\partial x^\alpha} + P^\alpha \frac{\partial f}{\partial p^\alpha} = 0, \tag{1}$$

where

$$P^{\alpha} = -\Gamma^{\alpha}_{\lambda\gamma} p^{\lambda} p^{\gamma}. \tag{2}$$

The Equation (1) which governs the evolution without collisions of particles, with a non-zero rest mass m, is considered here on a spherically symmetric gravitational field. The 4-momentum of particles is denoted by $p = (p^{\alpha}) = (p^0, p^i)$, informs on their velocity. Their distribution function f is then a function of (x^{α}, p^{α}), where (x^{α}, p^{α}) denotes the usual coordinates of the tangent bundle $T(\mathbb{R}^4)$ of $\mathbb{R}^4$. The collisionless particles then evolve in the space time $(\mathbb{R}^4, g)$, under the action of their own gravitational field represented by the given metric tensor $g = (g_{\alpha\beta})$ that informs about gravitational effects. We consider that $(\mathbb{R}^4, g)$ is a spherical symmetric space-time whose symmetric metric tensor $g = (g_{\alpha\beta})$ is of Lorentzian signature $(-, +, +, +)$ and writes

$$g = -e^{\Phi(t,r)} dt^2 + e^{\Lambda(t,r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2. \tag{3}$$

where Φ and Λ are given class one differentiable functions which depend on the $t \in \mathbb{R}$ and $r \in [0; +\infty[$ (where r is given by $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} = \sqrt{\delta_{ij} x^i x^j}$).

Such a metric is said to be spherically symmetric since it admits an action of the group $SO(3)$ by isometries.

Trajectories $s \mapsto (x^{\alpha}(s), p^{\alpha}(s))$ of such particles are no longer geodesics, but are solutions of the differential system

$$\frac{dp^{\alpha}}{ds} = P^{\alpha}; \quad \frac{dx^{\alpha}}{ds} = p^{\alpha}. \tag{4}$$

In Equation (1), all variables are expressed in Cartesian coordinates while the Christoffel symbols are given with spherical coordinates using the metric tensor (3). Then we are going to write the Vlasov equation in Cartesian coordinates.

So, let $x^1 = r \sin\theta \cos\varphi$, $x^2 = r \sin\theta \sin\varphi$, and $x^3 = r\cos\theta$. Setting $x_i = \delta_{ij} x^j$, g can be written as

$$g = g_{00} dt^2 + g_{ij} dx^i dx^j, \tag{5}$$

where components g_{00} and g_{ij} as obtained in [6]:

$$g_{00} = -e^{\Phi}, \quad g_{ij} = \delta_{ij} + (e^{\Lambda} - 1)\frac{x_i x_j}{r^2}, \quad g_{0i} = 0. \tag{6}$$

Using relations (5) and (6), we have the determinant of the metric g given by $\det(g_{\alpha\beta}) = -e^{\Phi+\Lambda}$. After computation, we have the reverse of $g = (g_{\alpha\beta})$, given by

$$g^{-1} = g^{00} dt^2 + g^{ij} dx_i dx_j, \tag{7}$$

where $g^{00} = e^{-\Phi}$, $g^{ij} = \delta^{ij} + (e^{-\Lambda} - 1)\dfrac{x_i x_j}{r^2}$.

The rest mass of particles is normalized to the unity, that is $m = 1$. Actually, the collisionless uncharged particles move on the future sheet of the mass hyperboloid $P(\mathbb{R}^4) \subset T(\mathbb{R}^4)$, whose equation is $P_{t,x}(p) : g(p, p) = -1$ or equivalently, using expression (3) of g:

$$P_{t,x} : p^0 = e^{-\frac{\Phi}{2}}\sqrt{1 + g_{ij} p^i p^j}, \tag{8}$$

where the choice $p^0 > 0$ symbolizes the fact that particles eject towards the future. For the transformation of the Equation (1), we shall need expressions of the Christoffel symbols $\Gamma^i_{\alpha\beta}$ which are given by (see [6])

$$\Gamma^i_{00} = \frac{1}{2}\partial_r\Phi \frac{x^i}{r} e^{\Phi-\Lambda}, \qquad \Gamma^i_{0j} = -\frac{1}{2}\partial_t\Lambda \frac{x^i x_j}{r^2}, \tag{9}$$

$$\Gamma^i_{jk} = \frac{1 - e^{-\Lambda}}{r}\left[\delta_{jk} - \frac{x_j x_k}{r^2}\right]\frac{x^i}{r} + \frac{1}{2}\partial_r\Phi \frac{x^i x_k x_j}{r^3}. \tag{10}$$

3. Hamilton-Jacobi Equation and Energy Estimates

We see that, terms $\frac{x^i x_k x_j}{r^4}$ in expressions of Christoffel symbols (10), imply some singularities at $r = 0$. To avoid these singularities and for some other purposes, we need some change of variables to the effect of having another expression of the relativistic Vlasov equation. To this end, we follow [22] and [14]. We then define the frame on $\mathbb{R}^3$ characterized by the vectors:

$$e_d^i = \delta_d^i + (e^{-\frac{\Lambda}{2}} - 1)\delta_{dj}\frac{x^i x^j}{r^2} \text{ such that } g_{ij}e_d^i e_s^j = \delta_{sd}. \tag{11}$$

Which is orthonormal with respect to the scalar product induced by $g_{\alpha\beta}$ on $\mathbb{R}^3$. The index d refers to the frame while the index i is for the coordinates.

Further, a dimensionless vector v^i is introduced such that

$$p^i = v^d e_d^i, \quad i = 1, 2, 3 \text{ with } v^i = p^i + (e^{\frac{\Lambda}{2}} - 1)\frac{(x \cdot p)x^i}{r^2}. \tag{12}$$

A vector in $\mathbb{R}^3$ will be parametrized not by components p^i but by components v^i. In terms of the dimensionless vector v^i, components of the momentum 4-vector p^α read

$$p^i = v^i + (e^{-\frac{\Lambda}{2}} - 1)\frac{(x \cdot v)x^i}{r^2}. \tag{13}$$

News coordinates in this change of variables (12) have the advantage that by the mass-shell condition (8), we have

$$p^0 = e^{-\frac{\Phi}{2}}\sqrt{1 + |v|^2}. \tag{14}$$

In the following we will use notations:

$$v^0 = \sqrt{1 + |v|^2}, \quad p^0 = e^{-\frac{\Phi}{2}} v^0. \tag{15}$$

By the above change of variables, instead of (t, x, p), we will use (t, x, v) as new variables. We henceforth set

$$\bar{f}(t, x, v) = f(t, x, p(t, x, v)) = f(t, x, v + (e^{-\frac{\Lambda}{2}} - 1)\frac{(x \cdot v)x}{r^2}). \tag{16}$$

To write the equation in new variables, we need derivatives of the distribution function that appears in the relativistic Vlasov equation (1) expressed in terms of new variables, which are given in following equalities (see [14]):

$$p^0 \frac{\partial f}{\partial t} = e^{-\frac{\Phi}{2}} v^0 \left[\frac{\partial \bar{f}}{\partial t} + \frac{1}{2} \partial_t \Lambda \delta_{jk} \frac{x^j v^k x^i}{r^2} \frac{\partial \bar{f}}{\partial v^i} \right], \tag{17}$$

$$p^i \frac{\partial f}{\partial x^i} = p^i \left(\frac{\partial \bar{f}}{\partial x^i} + \gamma_i^l \frac{\partial \bar{f}}{\partial v^l} \right), \tag{18}$$

where

$$\gamma_i^l = \frac{1}{2} \partial_r \Lambda e^{\frac{\Lambda}{2}} x^l x^k \delta_{ik} \delta_{nj} \frac{x^j v^n}{r^3} + (1 - e^{-\frac{\Lambda}{2}}) \delta_i^l \delta_{jk} \frac{v^k x^j}{r^2} + (e^{\frac{\Lambda}{2}} - 1) \delta_{ik} \frac{v^k x^l}{r^2}$$

$$+ (e^{-\frac{\Lambda}{2}} - e^{\frac{\Lambda}{2}}) \delta_{ik} \frac{x^k x^l}{r^2} \delta_{mn} \frac{x^n v^m}{r^2}, \tag{19}$$

and

$$\left(\Gamma_{00}^i (p^0)^2 + 2\Gamma_{0j}^i p^0 p^j + 2\Gamma_{jk}^i p^j p^k \right) \frac{\partial f}{\partial p^i} = \left(\Gamma_{00}^i (p^0)^2 + 2\Gamma_{0j}^i p^0 p^j + 2\Gamma_{jk}^i p^j p^k \right)$$

$$\times \frac{\partial \bar{f}}{\partial v^j} \left[\delta_i^j + (e^{\frac{\Lambda}{2}} - 1) \delta_{ik} \frac{x^j x^k}{r^2} \right]. \tag{20}$$

When we insert (17), (18) and (20) into the relativistic Vlasov equation (1), after some rearrangements and dividing the resulting equation by p^0 given by (15), we obtain

$$\frac{\partial f}{\partial t} + \mathcal{A}^i \frac{\partial f}{\partial x^i} + \mathcal{B}^i \frac{\partial f}{\partial v^i} = 0, \tag{21}$$

where

$$\mathcal{A}^i = \frac{e^{\frac{\Phi}{2}}}{v^0}\left[v^i + (e^{-\frac{\Lambda}{2}} - 1)\delta_{lk}\frac{x^i x^l v^k}{r^2}\right], \tag{22}$$

$$\mathcal{B}^i = \frac{e^{\frac{\Phi}{2}}}{v^0}\frac{e^{-\frac{\Lambda}{2}} - 1}{r^2}\delta_{lk}v^l(x^i v^k - x^k v^i) - \frac{\partial_r \Phi}{2}e^{\frac{(\Phi-\Lambda)}{2}}\frac{x^i}{r}v^0 - \frac{\partial_t \Lambda}{2}\delta_{lk}\frac{x^i x^l v^k}{r^2}. \tag{23}$$

One observe that the unknown function is still denoted by f instead of $\bar{f}$.

Assumption 1. The following assumptions used in [22] have been adopted:

(A1) We suppose that the function Λ is positive and independent of t, the function Φ is independent of r and that there exists a constant $C > 0$ such that

$$|\partial_r \Lambda| \leq C, \; |\partial_t \Phi| \leq C, \quad \forall\ r \in [0, +\infty[\text{ and } t \in [0; T[. \tag{24}$$

(A2) The distribution function is spherically symmetric, i.e., f is assumed to be invariant under simultaneous rotations of x and v. As a consequence, a direct calculation leads to the following condition:

$$\delta_{lk}v^l(v^k x^i - v^i x^k)\frac{\partial f}{\partial v^i} = \delta_{lk}x^l(x^k v^i - x^i v^k)\frac{\partial f}{\partial x^i}. \tag{25}$$

Now, using Assumption (A2), the relativistic Vlasov equation (21) reduces to

$$\frac{\partial f}{\partial t} + \frac{e^{\frac{\Phi}{2}}}{v^0} e^{-\frac{\Lambda}{2}} v^i \frac{\partial f}{\partial x^i} - \left[\frac{\partial_r \Phi}{2} e^{\frac{(\Phi - \Lambda)}{2}} \frac{x^i}{r} v^0 + \frac{\partial_t \Lambda}{2} \delta_{lk} \frac{x^i x^l v^k}{r^2} \right] \frac{\partial f}{\partial v^i} = 0. \tag{26}$$

Afterwards, using Assumption (A1), we have

$$\frac{\partial f}{\partial t} + \frac{e^{\frac{\Phi}{2}}}{v^0} e^{-\frac{\Lambda}{2}} v^i \frac{\partial f}{\partial x^i} = 0. \tag{27}$$

Now we consider the Hamiltonian, that is the function H defined by the second term of the relativistic Vlasov equation (27) as follows:

$$H = \frac{e^{\frac{\Phi}{2}}}{v^0} e^{-\frac{\Lambda}{2}} v^i \frac{\partial f}{\partial x^i}. \tag{28}$$

Setting $u_i = \frac{\partial f}{\partial x^i}$ and $w_i = \frac{\partial f}{\partial v^i}$ in the relation (28), the Hamiltonian H can be rewritten as a function

$$H : [0;\, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R};$$

$$(t,\, x,\, v,\, u,\, w) \mapsto H(t,\, x,\, v,\, u,\, w),$$

where $u = (u_i)$, $w = (w_i)$.

The relativistic Vlasov equation (1), in the spherical symmetric space-time has been transformed into the following Hamilton-Jacobi equation:

$$f_t(t,\, x,\, v) + H(t,\, x,\, v,\, u,\, w) = 0, \tag{29}$$

where H is given by (28).

Let us assume that a Lipschitz continuous and bounded function $f_0 : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ and a real number $T > 0$ are given and consider the following Cauchy problem:

$$\begin{cases} f_t(t, x, v) + H(t, x, v, u, w) = 0 \quad \text{in} \quad [0; T] \times \mathbb{R}^3 \times \mathbb{R}^3, \\ f(0, x, v) = f_0(x, v) \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}^3. \end{cases} \tag{30}$$

Our goal in the sequel, will be to prove that the Cauchy problem (30) has a unique viscosity solution $f \in C([0; T] \times \mathbb{R}^3 \times \mathbb{R}^3)$.

Firstly, let us state the following propositions.

Proposition 1. *The following functions* $\frac{v^i}{v^0}$, $\frac{1}{v^0}$, $e^{\frac{\Phi}{2}}$ *are bounded.*

Proof. Firstly, the expression (15) of v^0 shows the boundeness of $\frac{v^i}{v^0}$ and $\frac{1}{v^0}$. More precisely, we have

$$\left|\frac{v^i}{v^0}\right| \leq 1 \text{ and } \left|\frac{1}{v^0}\right| \leq 1. \tag{31}$$

• Secondly, the integration of relation (24) yields

$$e^{\frac{\Phi}{2}} \leq C_1 e^{\frac{CT}{2}}, \tag{32}$$

which shows the boundness of $e^{\frac{\Phi}{2}}$. □

Proposition 2. *Let* $T > 0$ *and* $R > 0$ *two given reals numbers, the Hamiltonian* H

$$H : [0; T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R};$$

$$(t, x, v, u, w) \mapsto H(t, x, v, u, w)$$

defined by (28) *satisfies following properties*:

(i) H *is continuous on* $[0; T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ *and uniformly continuous on* $[0; T] \times \overline{\mathbb{B}}_R \times \overline{\mathbb{B}}_R \times \overline{\mathbb{B}}_R \times \overline{\mathbb{B}}_R$, *where* $\overline{\mathbb{B}}_R$ *is the sphere of center* $0_{\mathbb{R}^3}$ *and radius* R;

(ii) H *is Lipschitz continuous with respect to* (t, x, v);

(iii) H *is Lipschitz continuous with respect to* (u, w);

(iv) *There exists one modulus* M_R *such that*

$$\left|H(t, x, v, u, w) - H(t', x', v', u', w')\right|$$

$$\leq M_R\left(\left|t - t'\right| + \left|x - x'\right| + \left|v - v'\right| + \left|u - u'\right| + \left|w - w'\right|\right),$$

$\forall\ (t, x, v, u, w), (t', x', v', u', w') \in [0; T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$, *where* $|u|, |u'| \leq R$.

Proof. Let $T > 0$ given

(i) $H(t, x, v, u, w)$ given by (28) is evidently defined and continuous in (t, x, v, u, w) since $v^0 \geqslant 1$.

(ii) Let $(u, w) \in \mathbb{R}^3 \times \mathbb{R}^3$ be fixed and $(t, x, v); (t', x', v') \in [0; T] \times \mathbb{R}^3 \times \mathbb{R}^3$.

Using relation (28) of H, we obtain

$$H(t, x, v, u, w) - H(t', x', v', u, w) = \left[e^{\frac{\Phi}{2}} e^{-\frac{\Lambda'}{2}}\left(\frac{(v')^i}{(v')^0} - \frac{v^i}{v^0}\right)\right.$$

$$\left. + \frac{v^i}{v^0} e^{\frac{\Phi}{2}}\left(e^{-\frac{\Lambda(x)}{2}} - e^{-\frac{\Lambda(x')}{2}}\right) + e^{-\frac{\Lambda'}{2}} \frac{v^i}{v^0}\left(e^{\frac{\Phi}{2}} - e^{\frac{\Phi'}{2}}\right)\right] u_i. \quad (33)$$

Now we have

$$\left|\frac{(v')^i}{(v')^0}-\frac{v^i}{v^0}\right| = \left|\frac{(v')^i}{(v')^0 v^0}\right|\left|(v')^0-v^0\right| + \left|\frac{1}{v^0}\right|\left|(v')^i-v^i\right| \leq \left|\frac{(v')^i}{(v')^0 v^0}\right|(|v'-v|) + \left|\frac{1}{v^0}\right|\left|(v')^i-v^i\right|;$$

From the inequality of finite increments, we obtain following inequalities in which y is an element on the segment $[x, x']$ and t_1 is also an element between t and t'.

$$\left|e^{-\frac{\Lambda(x')}{2}} - e^{-\frac{\Lambda(x)}{2}}\right| \leq \frac{1}{2} e^{-\frac{\Lambda(y)}{2}}\left|-\frac{y}{r}\right|\left|\frac{\partial\Lambda(y)}{\partial r}\right|\left|x-x'\right|,$$

and

$$\left|e^{\frac{\Phi(t')}{2}} - e^{\frac{\Phi(t)}{2}}\right| \leq \frac{1}{2} e^{\frac{\Phi(t_1)}{2}}\left|\partial_t\Phi(t_1)\right|\left|t-t'\right|.$$

We then deduce, using Proposition 1 and Assumption (A1), that

$$\left|\frac{(v')^i}{(v')^0}-\frac{v^i}{v^0}\right| \leq 2|v'-v|, \tag{34}$$

$$\left|e^{-\frac{\Lambda'}{2}} - e^{-\frac{\Lambda}{2}}\right| \leq C|x-x'|. \tag{35}$$

$$\left|e^{\frac{\Phi'}{2}} - e^{\frac{\Phi}{2}}\right| \leq \frac{C}{2}|t'-t|. \tag{36}$$

Previous inequalities in (33) yield:

$$\left|H(t, x, v, u, w) - H(t', x', v', u, w)\right| \leq K\left(\left|t'-t\right| + \left|x-x'\right| + \left|v'-v\right|\right), \tag{37}$$

with $K = \left[C_1 e^{\frac{CT}{2}}(2+C) + \frac{C}{2}\right]|u|$.

(iii) Let $(t, x, v) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3$ be fixed and $(u, w), (u', w') \in \mathbb{R}^3 \times \mathbb{R}^3$. Using relation (28) of H, we have

$$H(t, x, v, u, w) - H(t, x, v, u', w') = \frac{v^i}{v^0} e^{-\frac{\Lambda}{2}} e^{\frac{\Phi}{2}} (u_i - u_i'). \tag{38}$$

Using Proposition 1, we obtain from the relation (38), the following inequality:

$$\left| H(t, x, v, u, w) - H(t, x, v, u', w') \right| \leq K' \left(\left| u - u' \right| + \left| w - w' \right| \right), \tag{39}$$

with $K' = C_1 e^{\frac{CT}{2}}$.

(iv) Let $(t, x, v, u, w), (t', x', v', u', w') \in [0; T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$, we obtain from relations (37) and (39) the following inequality:

$$\left| H(t, x, v, u, w) - H(t', x', v', u', w') \right|$$

$$\leq K'' \left(\left| t - t' \right| + \left| x - x' \right| + \left| v - v' \right| + \left| u - u' \right| + \left| w - w' \right| \right), \tag{40}$$

with $K'' = K + K'$. $M_R = K''$ is a modulus. This ends the proof of our proposition. □

4. Existence and Uniqueness Theorems

In this section, we will show the existence and uniqueness theorems of our problem (30). Thus, we use in the sequel the method of *vanishing viscosity* given in [13]. The approach is to consider first the following approximate problem of (30):

$$\begin{cases} (f_\varepsilon)_t + H(t, x, v, u_\varepsilon, w_\varepsilon) - \varepsilon \Delta f_\varepsilon = 0 \text{ in } [0; T] \times \mathbb{R}^3 \times \mathbb{R}^3, \\ f_\varepsilon|_{t=0} = (f_0)_\varepsilon \text{ in } \mathbb{R}^3 \times \mathbb{R}^3. \end{cases} \tag{41}$$

(41) is an initial-value problem for a quasi-linear parabolic PDE, which turns out to have smooth solutions (see [13], paragraph 10). $\varepsilon\Delta f_\varepsilon$ in (41) is the term which regularizes the Hamilton-Jacobi equation. Thus we hope that as $\varepsilon \longrightarrow 0$, the solution f_ε of (40) will converge to a weak solution of (30). Then we suppose that $f_0 : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a Lipschitz continuous and bounded function and $T > 0$ a real number are given. We have the following result:

Lemma 1. *Let* Ω *be a compact subset of* $[0; T] \times \mathbb{R}^3 \times \mathbb{R}^3$. *Let* $f : \Omega \longrightarrow \mathbb{R}$ *be a continuous function such that there exists* $\varphi \in C^1(\Omega, \mathbb{R})$, $f - \varphi$ *has a strict local maximum at* (t_0, x_0, v_0).

If (f_n) *is a sequence of functions which uniformly converge to* f, *then there exists a sequence of points* $(t_n, x_n, v_n)_{n\in\mathbb{N}}$ *such that*

$$\begin{cases} (t_n, x_n, v_n) \longrightarrow (t_0, x_0, v_0) \\ f_n(t_n, x_n, v_n) \underset{n \longrightarrow +\infty}{\longrightarrow} f(t_0, x_0, v_0) \\ f_n - \varphi \textit{ has a local maximum at } (t_n, x_n, v_n), \textit{ for all } n \textit{ at some range.} \end{cases}$$

Proof. Let Ω be a compact subset of $[0; T] \times \mathbb{R}^3 \times \mathbb{R}^3$, f be a continuous function on Ω such that there exists $\varphi \in C^1(\Omega, \mathbb{R})$ such that $f - \varphi$ has a strict local maximum at (t_0, x_0, v_0). We assume that $(f_n)_n$ is a sequence of functions which uniformly converges to *f* and we show that there exists a sequence of points $(t_n, x_n, v_n)_{n\in\mathbb{N}}$ such that

$$\begin{cases} (t_n, x_n, v_n) \longrightarrow (t_0, x_0, v_0) \\ f_n(t_n, x_n, v_n) \underset{n \longrightarrow +\infty}{\longrightarrow} f(t_0, x_0, v_0) \\ f_n - \varphi \text{ has a local maximum at } (t_n, x_n, v_n), \text{ for all } n \text{ at some range.} \end{cases}$$

As $f - \varphi$ has a strict local maximum at (t_0, x_0, v_0), for $\rho = \frac{1}{n}$ enough small we can find ε_ρ such that if

$$|(t, x, v) - (t_0, x_0, v_0)| \leq \rho,$$

then

$$f(t, x, v) - \varphi(t, x, v) + \varepsilon_\rho < f(t_0, x_0, v_0) - \varphi(t_0, x_0, v_0). \tag{42}$$

Since $(f_n)_n$ is uniformly convergent to f, then there exists $N_\rho \in \mathbb{N}$ such that

$$\forall n > N_\rho, \forall z \in \Omega, |f_n(z) - f(z)| \leq \frac{\varepsilon_\rho}{4}. \tag{43}$$

Thus we can write

$$f(t_0, x_0, v_0) - \frac{\varepsilon_\rho}{4} \leq f_n(t_0, x_0, v_0), \quad f_n(t, x, v) \leq f(t, x, v) + \frac{\varepsilon_\rho}{4}. \tag{44}$$

Now relations (42) and (44), yield to the inequality:

$$f_n(t, x, v) - \varphi(t, x, v) + \frac{\varepsilon_\rho}{2} \leq f_n(t_0, x_0, v_0) - \varphi(t_0, x_0, v_0). \tag{45}$$

$f_n - \varphi$ is then bounded, hence there exists (t_n, x_n, v_n) such that $f_n - \varphi$ has a maximum at (t_n, x_n, v_n) on $\overline{B((t_0, x_0, v_0), \rho)}$ at some range $N > N_\rho$.

In the other hand, when $n \longrightarrow +\infty$, we have $\rho \longrightarrow 0$ and then we deduce that

$$(t_n, x_n, v_n) \longrightarrow (t_0, x_0, v_0). \tag{46}$$

Since $(f_n)_n$ uniformly converges to f, then we have

$$f_n(t_n, x_n, v_n) \longrightarrow f(t_0, x_0, v_0). \tag{47}$$

So, relations (45), (46) and (47) conclude the proof. $\square$

Theorem 1 (Existence)**.** *Let* $(u_n)_{n\in\mathbb{N}}$ *be a non negative sequence of reals which converges to* 0; $(f_{u_n})_{n\in\mathbb{N}}$ *a smooth sequence of functions, solutions of* (41) *for* $\varepsilon = u_n$.

(i) *There exists a sub-sequence of* $(f_{u_n})_{n\in\mathbb{N}}$, *denoted by* $(f_{u_n})_{n\in\mathbb{N}}$ *and a function* f *such that* $f_{u_n} \underset{n \to +\infty}{\longrightarrow} f$ *for the infinite norm on all compact*;

(ii) f *is a viscosity solution of problem* (30).

Proof. (i) Let $(f_{u_n})_n$ be a smooth sequence of real-value functions of (41), Ω be a compact subset of $[0; T[\times\mathbb{R}^3\times\mathbb{R}^3$. Thus $\forall n \in \mathbb{N}$, f_{u_n} be a uniformly continuous function on Ω, hence $(f_{u_n})_n$ is a uniformly equicontinuous family of functions and bounded. Then by Arzela-Ascoli theorem, there exists a subsequence $(f_{u_{n_j}})_j \subset (f_{u_n})_n$ and a continuous function f, such that

$$(f_{u_{n_j}})_j \longrightarrow f \text{ uniformly on } \Omega.$$

(ii) First of all, we show that f is a viscosity subsolution. We will then suppose that $f - \varphi$ has a strict local maximum at (t_0, x_0, v_0). Consider first that φ is a class two differentiable function.

Applying the previous lemma in $\overline{B((0_{\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3}, \varepsilon))}$ which is compact, there exists (t_n, x_n, v_n) such that

$$\begin{cases} (t_n, x_n, v_n) \longrightarrow (t_0, x_0, v_0), \quad f_{u_n}(t_n, x_n, v_n) \underset{n \to +\infty}{\longrightarrow} f(t_0, x_0, v_0) \\ f_{u_n} - \varphi \text{ has a local maximum at } (t_n, x_n, v_n), \text{ for all } n \text{ at some range.} \end{cases}$$

Thus

$$\partial_t f_{u_n}(t_n, x_n, v_n) = \partial_t \varphi(t_n, x_n, v_n), \quad \partial_x f_{u_n}(t_n, x_n, v_n)$$
$$= \partial_x \varphi(t_n, x_n, v_n), \, \partial_v f_{u_n}(t_n, v_n, v_n) = \partial_v \varphi(t_n, x_n, v_n), \tag{48}$$

and

$$\Delta f_{u_n}(t_n, x_n, v_n) \leq \Delta\varphi(t_n, x_n, v_n). \tag{49}$$

Since f_{u_n} is solution of (41) then (48) and (49) yield

$$\varphi_t(t_n, x_n, v_n) + H_\varphi(t_n, x_n, v_n, \bar{u}_x, \bar{w}_v) \leq u_n \Delta\varphi(t_n, x_n, v_n). \tag{50}$$

Now, passing to the limit as $n \longrightarrow +\infty$ in inequality (50), we have

$$\varphi_t(t_n, x_n, v_n) + H_\varphi(t_n, x_n, v_n, \bar{u}_x, \bar{w}_v) \leq 0, \tag{51}$$

since $u_n \longrightarrow 0$ as $n \longrightarrow +\infty$.

Now, by the continuity of H given by (i) of the Proposition 2, we obtain the fact that f is a viscosity subsolution of (30).

For φ a class one differentiable function, we rather consider a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of class two differentiable functions which uniformly converges to φ on all compact and the sequence of their derivatives also converges uniformly on all compact to the derivative of φ. Thus using the same previous restriction and the previous lemma, we have

$$(\varphi_n)_t(t'_n, x'_n, v'_n) + H(t'_n, x'_n, v'_n, \bar{u}_x, \bar{w}_v) \leq 0.$$

When $n \longrightarrow +\infty$, the uniform convergence and the property (i) of Proposition 2 give the result.

(ii) To show that f is a super-viscosity solution, we use the same previous argument with reverse inequalities (49) and (50).

(iii), (i) and (ii) show that f is the viscosity solution of (30). □

Theorem 2. *If* H *in* (28) *verifies estimations of the Proposition* 2, *then the problem* (30) *has one and only one uniformly continuous and bounded viscosity solution on* $[0; T] \times \mathbb{R}^3 \times \mathbb{R}^3$.

Proof. We suppose that there exists two viscosity solutions f and $\bar{f}$ of problem (30), uniformly continuous and bounded on $[0; T] \times \mathbb{R}^3 \times \mathbb{R}^3$, with the same initial condition f_0.

We take f as viscosity sub-solution and $\bar{f}$ as viscosity supersolution who have same initial data.

We are interested in the $\max_{[0;\, T] \times \mathbb{R}^3 \times \mathbb{R}^3} \left(f - \bar{f}\right) = M$. By absurd, we suppose that $M > 0$.

Let $(\varepsilon, \alpha) \in \,]0, 1[^2$, we set : $\forall (t, x, v), (t', x', v') \in [0; T] \times \mathbb{R}^3 \times \mathbb{R}^3$,

$$\Phi_\varepsilon(t, t', x, x', v, v') = f(t, x, v) - \bar{f}(t', x', v')$$
$$- \frac{1}{\varepsilon^2}\left(\left|v - v'\right|^2 + \left|x - x'\right|^2 + (t - t')^2\right)$$
$$- \varepsilon(|x|^2 + |x'|^2 + |v|^2 + |v'|^2) - \alpha(t + t'). \tag{52}$$

We choose in inequality (52)

$$(t, x, v) = (t', x', v') \in [0; T] \times \mathbb{R}^3 \times \mathbb{R}^3,$$

and we can fixed α and ε too small such that

$$\sup_{[0;\, T] \times \mathbb{R}^3 \times \mathbb{R}^3} \{f(t, x, v) - \bar{f}(t, x, v) - 2\alpha t - 2\varepsilon(|x|^2 + |v|^2)\} \geq \frac{M}{2}.$$

Since $\Phi_\varepsilon \to -\infty$ when $|x|, |x'|, |v|, |v'| \to +\infty$ then the supremum of Φ_ε is reached at a certain point $(t_\varepsilon, t'_\varepsilon, x_\varepsilon, x'_\varepsilon, v_\varepsilon, v'_\varepsilon)$.

In other side, we have

$$f(t_\varepsilon, x_\varepsilon, v_\varepsilon) - \bar{f}(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon) - \varepsilon(|x_\varepsilon|^2 + |x'_\varepsilon|^2 + |v_\varepsilon|^2 + |v'_\varepsilon|^2)$$

$$- \frac{1}{\varepsilon^2}\left(|v_\varepsilon - v'_\varepsilon|^2 + |x_\varepsilon - x'_\varepsilon|^2 + |t_\varepsilon - t'_\varepsilon|^2\right) - \alpha(t_\varepsilon + t_\varepsilon) \geq 0,$$

hence

$$f(t_\varepsilon, x_\varepsilon, v_\varepsilon) - \bar{f}(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon) \geq + \alpha(t_\varepsilon + t_\varepsilon) + \varepsilon(|x_\varepsilon|^2 + |x'_\varepsilon|^2 + |v_\varepsilon|^2 + |v'_\varepsilon|^2)$$

$$+ \frac{1}{\varepsilon^2}\left(|v_\varepsilon - v'_\varepsilon|^2 + |x_\varepsilon - x'_\varepsilon|^2 + |t_\varepsilon - t'_\varepsilon|^2\right).$$

Since f and $\bar{f}$ are bounded, the above inequality yield

$$\mathcal{R} + \bar{\mathcal{R}} \geq + \alpha\,(t_\varepsilon + t_\varepsilon) + \varepsilon(|x_\varepsilon|^2 + |x'_\varepsilon|^2 + |v_\varepsilon|^2 + |v'_\varepsilon|^2)$$

$$+ \frac{1}{\varepsilon^2}\left(|v_\varepsilon - v'_\varepsilon|^2 + |x_\varepsilon - x'_\varepsilon|^2 + |t_\varepsilon - t'_\varepsilon|^2\right), \tag{53}$$

with $\mathcal{R}$ and $\bar{\mathcal{R}}$ are respectively, bounded constants of f and $\bar{f}$.

We obtain

$$|t_\varepsilon - t'_\varepsilon| \leq \varepsilon\sqrt{2\mathbf{R}}, \quad |x_\varepsilon - x'_\varepsilon| \leq \varepsilon\sqrt{2\mathbf{R}}, \quad |v_\varepsilon - v'_\varepsilon| \leq \varepsilon\sqrt{2\mathbf{R}}, \tag{54}$$

with $\mathbf{R} = \max\{\mathcal{R};\ \bar{\mathcal{R}}\}$.

Since $\bar{f}$ is uniform continuous on $[0;\ T[\times \mathbb{R}^3 \times \mathbb{R}^3$, then we introduce the modulus of continuity of $\bar{f}$ whose expression is given by

$$m_{\bar{f}}(l) = \sup_{|t-t'|\leq l,\ |x-x'|\leq l,\ |v-v'|\leq l} \left|\bar{f}(t, x, v) - \bar{f}(t', x', v')\right|,$$

which $m_{\bar{f}}(l) \longrightarrow 0$ when $l \longrightarrow 0$ or simply $m_{\bar{f}}(0) = 0$.

Furthermore, we obtain after substitutions

$$\frac{M}{2} \le \Phi(t_\epsilon, t'_\epsilon, x'_\epsilon, x_\epsilon, v'_\epsilon, v_\epsilon)$$

$$\le f(t_\varepsilon, x_\varepsilon, v_\varepsilon) - \bar{f}(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon)$$

$$\le f(t_\varepsilon, x_\varepsilon, v_\varepsilon) - f(0, x_\varepsilon, v_\varepsilon) + f(0, x_\varepsilon, v_\varepsilon) - \bar{f}(0, x_\varepsilon, v_\varepsilon)$$

$$+ \bar{f}(0, x_\varepsilon, v_\varepsilon) - \bar{f}(t_\varepsilon, x_\varepsilon, v_\varepsilon) + \bar{f}(t_\varepsilon, x_\varepsilon, v_\varepsilon) - \bar{f}(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon)$$

$$\le m_f(l) + 0 + m_{\bar{f}}(l) + m_{\bar{f}}(l).$$

Finally, we obtain

$$\frac{M}{2} \le m_f(l) + m_{\bar{f}}(l) + m_{\bar{f}}\left(\varepsilon\sqrt{2\mathbf{R}}\right).$$

In particular, if $l = t_\varepsilon$, we have

$$\frac{M}{2} \le m_f(t_\varepsilon) + m_{\bar{f}}(t_\varepsilon) + m_{\bar{f}}\left(\varepsilon\sqrt{2\mathbf{R}}\right).$$

Moreover, if $t_\varepsilon = 0$, we get $M = 0$ when ε tends to 0, what is absurd. So there exists $\nu > 0$ such that $t_\varepsilon \ge \nu$. Similarly, following the same method as before with the variable $(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon)$, it is shown that there exists $\nu > 0$ such that $t'_\varepsilon \ge \nu$.

Next, we see that $(t_\varepsilon, x_\varepsilon, v_\varepsilon)$ is one maximum point of

$$\Theta(t, x, v) = f(t, x, v) - \varphi_\varepsilon^1(t, x, v),$$

where

$$\varphi_\varepsilon^1(t, x, v) = \bar{f}(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon) + \frac{1}{\varepsilon^2}\left(|v - v'_\varepsilon|^2 + |x - x'_\varepsilon|^2 + (t - t'_\varepsilon)^2\right)$$

$$+ \varepsilon(|x|^2 + |x'_\varepsilon|^2 + |v|^2 + |v'_\varepsilon|^2) + \alpha(t + t'_\varepsilon). \tag{55}$$

$\Theta(t, x, v)$ is class $\mathcal{C}^1$ and thus $(t_\varepsilon, x_\varepsilon, v_\varepsilon)$ is one maximum point of $\varphi_\varepsilon^1(t, x, p) = f(t, x, v) - \Theta(t, x, v)$. f is a viscosity subsolution of (30) and $(t_\varepsilon, x_\varepsilon, v_\varepsilon) \in [0; T] \times \mathbb{R}^3 \times \mathbb{R}^3$, thus we obtain

$$(\varphi_\varepsilon^1)_t(t_\varepsilon, x_\varepsilon, v_\varepsilon) = \alpha + \frac{2}{\varepsilon^2}(t_\varepsilon - t'_\varepsilon),$$

$$(\nabla\varphi_\varepsilon^1)_x(t_\varepsilon, x_\varepsilon, v_\varepsilon) = \frac{2}{\varepsilon^2}(x_\varepsilon - x'_\varepsilon) + 2\varepsilon x_\varepsilon,$$

$$(\nabla\varphi_\varepsilon^1)_v(t_\varepsilon, x_\varepsilon, v_\varepsilon) = \frac{2}{\varepsilon^2}(v_\varepsilon - v'_\varepsilon) + 2\varepsilon v_\varepsilon,$$

thus

$$\frac{\partial\varphi_\varepsilon^1}{\partial t_\varepsilon} + H(t_\varepsilon, x_\varepsilon, v_\varepsilon, (\nabla\varphi_\varepsilon^1)_x, (\nabla\varphi_\varepsilon^1)_v) \le 0. \tag{56}$$

Finally,

$$\alpha + \frac{2}{\varepsilon^2}(t_\varepsilon - t'_\varepsilon) + H\bigg(t_\varepsilon, x_\varepsilon, v_\varepsilon, \frac{2}{\varepsilon^2}(x_\varepsilon - x'_\varepsilon) + 2\varepsilon x_\varepsilon, \frac{2}{\varepsilon^2}(v_\varepsilon - v'_\varepsilon) + 2\varepsilon v_\varepsilon\bigg) \le 0. \tag{57}$$

Using the same skills, $(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon)$ is one minimum point of

$$\sum(t', x', v') = \bar{f}(t', x', v') + \varphi_\varepsilon^2(t', x', v'),$$

with

$$\varphi_\varepsilon^2(t', x', v') = f(t_\varepsilon, x_\varepsilon, v_\varepsilon) - \frac{1}{\varepsilon^2}\left(|v_\varepsilon - v'|^2 + |x_\varepsilon - x'|^2 + (t_\varepsilon - t)^2\right) - \varepsilon(|x_\varepsilon|^2 + |x'|^2 + |v_\varepsilon|^2 + |v'|^2) - \alpha(t + t'_\varepsilon),$$

thus we see that $(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon)$ is a one minimum point of $\varphi_\varepsilon^2(t', x', v') = \sum(t', x', v') - \bar{f}(t', x', v')$. $\bar{f}$ is a viscosity super-solution of (30) and $(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon) \in [0; T] \times \mathbb{R}^3 \times \mathbb{R}^3$, we obtain

$$(\varphi_\varepsilon^2)'_t(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon) = -\alpha + \frac{2}{\varepsilon^2}(t'_\varepsilon - t),$$

$$(\nabla\varphi^2)_{x'}(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon) = \frac{2}{\varepsilon^2}(x_\varepsilon - x'_\varepsilon) - 2\varepsilon x'_\varepsilon,$$

$$(\nabla\varphi^2)_{v'}(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon) = \frac{2}{\varepsilon^2}(v_\varepsilon - v'_\varepsilon) - 2\varepsilon v'_\varepsilon,$$

thus

$$\frac{\partial\varphi_\varepsilon^2}{\partial t'_\varepsilon} + H(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon, (\nabla\varphi^2)_{x'}, (\nabla\varphi^2)_{v'}) \geq 0. \tag{58}$$

Finally,

$$\begin{aligned}
&-\alpha + \frac{2}{\varepsilon^2}(t_\varepsilon - t'_\varepsilon) \\
&\quad + H\left(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon, \frac{2}{\varepsilon^2}(x_\varepsilon - x'_\varepsilon) - 2\varepsilon x'_\varepsilon, \frac{2}{\varepsilon^2}(v_\varepsilon - v'_\varepsilon) - 2\varepsilon v'_\varepsilon\right) \geq 0.
\end{aligned} \tag{59}$$

When we subtract inequality (59) of inequality (57), we have

$$\begin{aligned}
2\alpha \leq\ & H\left(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon, \frac{2}{\varepsilon^2}(x_\varepsilon - x'_\varepsilon) + 2\varepsilon x'_\varepsilon, \frac{2}{\varepsilon^2}(v_\varepsilon - v'_\varepsilon) + 2\varepsilon v_\varepsilon\right) \\
& - H\left(t_\varepsilon, x_\varepsilon, v_\varepsilon, \frac{2}{\varepsilon^2}(x_\varepsilon - x'_\varepsilon) - 2\varepsilon x'_\varepsilon, \frac{2}{\varepsilon^2}(v_\varepsilon - v'_\varepsilon) - 2\varepsilon v'_\varepsilon\right).
\end{aligned}$$

In addition,

$$\begin{aligned}
2\alpha \leq\ & H(t'_\varepsilon, x'_\varepsilon, v'_\varepsilon, \frac{2}{\varepsilon^2}(x_\varepsilon - x'_\varepsilon) + 2\varepsilon x_\varepsilon, \frac{2}{\varepsilon^2}(v_\varepsilon - v'_\varepsilon) + 2\varepsilon v_\varepsilon) \\
& - H(t_\varepsilon, x_\varepsilon, v_\varepsilon, \frac{2}{\varepsilon^2}(x_\varepsilon - x'_\varepsilon) + 2\varepsilon x_\varepsilon, \frac{2}{\varepsilon^2}(v_\varepsilon - v'_\varepsilon) + 2\varepsilon v_\varepsilon) \\
& + H(t_\varepsilon, x_\varepsilon, v_\varepsilon, \frac{2}{\varepsilon^2}(x_\varepsilon - x'_\varepsilon) + 2\varepsilon x_\varepsilon, \frac{2}{\varepsilon^2}(v_\varepsilon - v'_\varepsilon) + 2\varepsilon v_\varepsilon) \\
& - H(t_\varepsilon, x_\varepsilon, v_\varepsilon, \frac{2}{\varepsilon^2}(x_\varepsilon - x'_\varepsilon) - 2\varepsilon x'_\varepsilon, \frac{2}{\varepsilon^2}(v_\varepsilon - v'_\varepsilon) - 2\varepsilon v'_\varepsilon).
\end{aligned}$$

Using Proposition 2, we obtain

$$2\alpha \le C_1 e^{\frac{CT}{2}} 2\varepsilon(|x_\varepsilon + x'_\varepsilon| + |v_\varepsilon + v'_\varepsilon|) + \left[C_1 e^{\frac{CT}{2}}(2 + C) + \frac{C}{2}\right]$$

$$\times \left(\frac{2}{\varepsilon^2}|x_\varepsilon - x'_\varepsilon| + 2\varepsilon|x'_\varepsilon|\right)(|t_\varepsilon - t'_\varepsilon| + |x_\varepsilon - x'_\varepsilon| + |v'_\varepsilon - v_\varepsilon|), \tag{60}$$

so

$$\alpha \le C_1 e^{\frac{CT}{2}} 2\varepsilon\,(|x_\varepsilon + x'_\varepsilon| + |v_\varepsilon + v'_\varepsilon|) + \left[C_1 e^{\frac{CT}{2}}(2 + C) + \frac{C}{2}\right]$$

$$\times \left(\frac{1}{\varepsilon^2}|x_\varepsilon - x'_\varepsilon| + \varepsilon\,|x'_\varepsilon|\right)(|t_\varepsilon - t'_\varepsilon| + |x_\varepsilon - x'_\varepsilon| + |v'_\varepsilon - v_\varepsilon|), \tag{61}$$

and using (54), we easily obtain $\alpha = 0.$

This is absurd because $\alpha > 0$ hence $M \le 0,$ thus $f \le \bar{f}.$

Reverse inequality can be proof of the same way, just reverse the role in f and $\bar{f},$ so $f = \bar{f}.$ □

Theorem 3 (Uniqueness)**.** *Let* $T > 0,\ f_0 : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ *be given*

(1) *The Cauchy problem* (30)

$$\begin{cases} f_t(t, x, v) + H(t, x, v, u, w) = 0 \quad in \quad [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3, \\ f(0, x, v) = f_0(x, v) \quad in \quad \mathbb{R}^3 \times \mathbb{R}^3, \end{cases}$$

has one unique continuous viscosity solution.

(2) *The relativistic Vlasov equation* (1) *in spherical symmetric space-time, admits a unique continuous viscosity solution* $f = f(t, x, v)$ *in* $[0, +\infty[\times \mathbb{R}^3 \times \mathbb{R}^3$ *which satisfies the initial condition* $f(0, x, v) = f_0(x, v)$ *in* $\mathbb{R}^3 \times \mathbb{R}^3$.

Proof. (1) Let $R \geq 0$ and consider two continuous viscosity solutions f and g of problem (30) on $\overline{B((0_{\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3}), R)}_{R \geq 0}$, where they are uniformly continuous and bounded, next apply the previous theorem.

(2) The conclusion comes naturally from the equivalence between the relativistic Vlasov equation with the initial condition $f(0, x, v) = f_0(x, v)$ in $\mathbb{R}^3 \times \mathbb{R}^3$ and the Cauchy problem (30). □

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