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LEAST SQUARES ESTIMATION IN PERIODIC RESTRICTED *EXPAR*(*p*) MODELS

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Abstract

A generalisation of the periodic restricted exponential autoregressive model (*PEXPAR*(1)) to order p is introduced. The least squares method is used for estimating the parameters. The asymptotic properties of estimates for strictly stationary restricted PEXPAR are derived. A small simulation study is carried out to check the asymptotic properties.

1. Introduction

Periodic time series models have been extensively used in the recent decades to describe many series with periodic dynamics. The inability of SARIMA models to adequately represent many seasonal time series exhibiting a periodic autocovariance structure has motivated the research in the periodically correlated processes. This notion, introduced by Gladyshev [13], was exploited in a variety of new classes of time series models, among them, the periodic GARCH (Bollerslev and Ghysels [8]), the periodic bilinear (Bibi and Gautier [7]) and the mixture periodic

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autoregressive model (Shao [29]). In this paper, we extend the class of periodic restricted exponential autoregressive model (*PEXPAR*(1)) discussed in Merzougui et al. [23] to order p. PEXPAR series satisfy a nonlinear difference equation similar to that for EXPAR models with parameters and white noise variances which change periodically with the season.

The class of exponential autoregressive (EXPAR) models introduced by Ozaki [25] and Haggan and Ozaki [15] has shown their appropriateness in capturing certain well-known features of nonlinear vibration theory, such as amplitude dependent frequency, jump phenomena and limit cycle behaviour, these models are autoregressive in form with amplitude dependent exponential coefficients. Many results and methods, including the stationarity, geometrical ergodicity, estimation, forecasting and testing, have been studied as Ozaki [26, 27], Chan and Tong [10], Al-Qassam and Lane [1], Koul and Schick [19] and Allal and El Melhaoui [2], in addition, this class of nonlinear time series found successful applications in analyzing data from a wide range of fields, including ecology (Haggan and Ozaki [15], Priestley [28]), hydrology (Ozaki [27]; Gurung [14]), speech signal (Ishizuka et al. [17]) and macroeconomic (Terui and Van Dijk [32]; Amiri [3]; Katsiampa [18]).

This paper deals with the least squares estimation of the periodic restricted *EXPAR*(p) model. Several methods for parameter estimation have been explored for the non-periodic EXPAR models, Haggan and Ozaki [15] proposed the faster method, it consists to fix the nonlinear parameter in the exponential term at one of a grid of values and then estimate the other parameters by the linear least squares method and use the AIC criterion to select the final parameters. Tjøstheim [34] treated the problem of estimation of nonlinear time series in a general framework, he gave the conditional LS and the ML estimates, Amondela and Francq [4] gave the QML estimator for the *EXPAR*(1) model, Shi and Aoyama [30] and Baragona et al. [5] used a genetic algorithm to estimate the parameters, Shi et al. [31] remarked that the objective function for the nonlinear coefficient is not convex and multiple local minima may exist and such estimation is time consuming, they proposed, when real

time estimation is needed, a heuristic estimate from the data for the nonlinear parameter. We can find an application for this method in Messaoud et al. [24] for modelling the vibrations and disturbances during the drilling process. Ghosh et al. [12] used the extended Kalman filter (EKF) to estimate the EXPAR models.

The rest of this paper is organized as follows. In Section 2, we introduce the restricted *PEXPAR*(p) models and provide some basic notations and technical assumptions. In Section 3, we estimate the parameters by least squares method, consistency and asymptotic normality of the LS estimators are established there. The performance of the estimators is shown via small simulation in Section 4. Finally, some concluding remarks are made in Section 5.

2. Periodic Restricted EXPAR Model

The process $\{Y_t;\, t \in \mathbb{Z}\}$ is said to follow a Periodic restricted Exponential Autoregressive $PEXPAR_S(p_t)$, with period $S(S \geq 2)$, if it is a solution of a nonlinear periodic stochastic difference equation of the form

$$Y_t = \sum_{j=1}^{p_t} \left(\varphi_{t,j} + \pi_{t,j} \exp\left(-\gamma Y_{t-1}^2\right)\right) Y_{t-j} + \varepsilon_t,\ t \in \mathbb{Z}, \tag{1}$$

where $\{\varepsilon_t;\, t \in \mathbb{Z}\}$ is periodic i.i.d. process with continuous density $f_{\sigma_t}(.)$, not necessarily Gaussian, with mean 0 and finite variance σ_t^2. The autoregressive parameters $\varphi_{t,j}$, $\pi_{t,j}$, $\forall t \in \mathbb{Z}$ and $j = 1, \ldots, p$, the order p_t and the innovation variance σ_t^2 are periodic, in time, with period S, i.e.,

$$\varphi_{t+kS,j} = \varphi_{t,j},\ \pi_{t+kS,j} = \pi_{t,j},\ p_{t+kS} = p_t, \text{ and } \sigma_{t+kS}^2 = \sigma_t^2,\ \forall k,\ t \in \mathbb{Z} \text{ and } j = 1, \ldots, p_t.$$

The nonlinear parameter, $\gamma > 0$, is known. A heuristic determination of γ from data is $\hat{\gamma} = -\frac{\log \epsilon}{\max_{1 \le t \le n} Y_t^2}$, where ϵ is a small number (cf. Shi et al. [31]).

Putting $t = i + S\tau$, $i = 1, 2, \ldots, S$ and $\tau \in \mathbb{Z}$ and taking $p = \max_{i \in \{1, 2, \ldots, S\}} p_i$, where $\varphi_{i,j} = 0$, $\pi_{i,j} = 0$, for each $j > p_i$, one can rewrite Equation (1) in the equivalent form:

$$Y_{i+S\tau} = \sum_{j=1}^{p} \left(\varphi_{i,j} + \pi_{i,j} \exp\left(-\gamma Y_{i+S\tau-1}^2\right)\right) Y_{i+S\tau-j} + \varepsilon_{i+S\tau}, \; i = 1, \ldots, S, \; \tau \in \mathbb{Z}. \tag{2}$$

Let

$$\underline{\varphi}_i = \left(\varphi_{i,1}, \pi_{i,1}, \ldots, \varphi_{i,p}, \pi_{i,p}\right)', \; i = 1, \ldots, S, \text{ and } \underline{\varphi} = \left(\underline{\varphi}'_1, \ldots, \underline{\varphi}'_S\right)' \in \mathbb{R}^{2pS}.$$

We make the following assumptions:

A1: The periodic exponential autoregressive parameters $\underline{\varphi}$ satisfy the strict stationarity periodically condition of (2). A sufficient condition is: All the roots of the associated characteristic equation

$$z^p - c_{i,1} z^{p-1} \cdots - c_{i,p} = 0$$

are inside the unit circle, where $c_{i,j} = \max\left\{\left|\varphi_{i,j}\right|, \left|\varphi_{i,j} + \pi_{i,j}\right|\right\}$, $j = 1, \ldots, p$; $i = 1, \ldots, S$. In the nonperiodic case, see, for example, De Gooijer [11], page 37.

A2: The periodically ergodic process $\{Y_t; t \in \mathbb{Z}\}$ is such that $E(Y_t^4) < \infty$, for any $t \in \mathbb{Z}$.

3. Parameter Estimation

We consider the problem of estimating the parameters $\underline{\varphi}$ of the model (2), which is a linear optimisation problem, we can solve it using the least squares procedure.

Suppose that we have observations $\{Y_1, \ldots, Y_N\}$ from (2), $N = mS$, and define the conditional sum of squares

$$L_N(\underline{\varphi}) = \sum_{i=1}^{S} L_{i,m}(\underline{\varphi})$$

$$= \sum_{i=1}^{S}\left(\sum_{\tau=r+1}^{m-1}\left(Y_{S\tau+i} - E_{\underline{\varphi}}(Y_{S\tau+i}|B_{S\tau+i-1})\right)^2\right)$$

$$= \sum_{i=1}^{S}\left(\sum_{\tau=r+1}^{m-1}\left(Y_{S\tau+i} - \sum_{j=1}^{p}\left(\varphi_{i,j} + \pi_{i,j}\exp\left(-\gamma Y_{S\tau+i-1}^2\right)\right)Y_{S\tau+i-j}\right)^2\right),$$

where $r = \left[\frac{p}{S}\right]$, with $[x]$ denotes the integer part of x, $B_{S\tau+i-1}$ is the σ-algebra generated by the past of the process up to time $S\tau + i - 1$ and $E_{\underline{\varphi}}(.|.)$ is the conditional expectation assuming that $\underline{\varphi}$ is the true parameter.

The estimate $\underline{\hat{\varphi}}_i = (\hat{\varphi}_{i,1}, \hat{\pi}_{i,1}, \ldots, \hat{\varphi}_{i,p}, \hat{\pi}_{i,p})'$, for a fixed season i, is a solution to the estimating equations

$$\frac{\partial L_{i,m}(\underline{\varphi})}{\partial \varphi_{i,j}} = 0 \text{ and } \frac{\partial L_{i,m}(\underline{\varphi})}{\partial \pi_{i,j}} = 0, \quad j = 1, \ldots, p.$$

The solution for a fixed season i is

$$\underline{\widehat{\varphi}}_i = \begin{pmatrix} M_{i,1,1} & \cdots & M_{i,1,p} \\ \vdots & \ddots & \vdots \\ M_{i,p,1} & \cdots & M_{i,p,p} \end{pmatrix}^{-1} \times \begin{bmatrix} \sum_{\tau=r+1}^{m-1} Y_{S\tau+i-1} Y_{S\tau+i} \\ \sum_{\tau=r+1}^{m-1} Y_{S\tau+i-1} Y_{S\tau+i} \exp\left(-\gamma Y_{S\tau+i-1}^2\right) \\ \vdots \\ \sum_{\tau=r+1}^{m-1} Y_{S\tau+i-p} Y_{S\tau+i} \\ \sum_{\tau=r+1}^{m-1} Y_{S\tau+i-p} Y_{S\tau+i} \exp\left(-\gamma Y_{S\tau+i-1}^2\right) \end{bmatrix}, \tag{3}$$

$$\widehat{\sigma}_i^2 = \frac{1}{m-r-1} \sum_{\tau=r+1}^{m-1} \left(Y_{S\tau+i} - \sum_{j=1}^{p} \left(\widehat{\varphi}_{i,j} + \widehat{\pi}_{i,j} \exp\left(-\gamma Y_{S\tau+i-1}^2\right) \right) Y_{S\tau+i-j} \right)^2, \tag{4}$$

where for $j, k = 1, \ldots, p$,

$$M_{i,j,k} =$$

$$\begin{pmatrix} \sum_{\tau=r+1}^{m-1} Y_{S\tau+i-j} Y_{S\tau+i-k} & \sum_{\tau=r+1}^{m-1} Y_{S\tau+i-j} Y_{S\tau+i-k} \exp\left(-\gamma Y_{S\tau+i-1}^2\right) \\ \sum_{\tau=r+1}^{m-1} Y_{S\tau+i-j} Y_{S\tau+i-k} \exp\left(-\gamma Y_{S\tau+i-1}^2\right) & \sum_{\tau=r+1}^{m-1} Y_{S\tau+i-j} Y_{S\tau+i-k} \exp\left(-2\gamma Y_{S\tau+i-1}^2\right) \end{pmatrix}.$$

Remark. For $p = 1$, we obtain the estimates of the periodic restricted *EXPAR*(1) model (cf. Merzougui [22]).

Theorem. *Suppose that* $\{Y_t\}$, *satisfying* (2), *is strictly stationary, then the least squares estimators* (3) *and* (4) *are strongly consistent as* $m \to \infty$. *That is,*

$$\underline{\widehat{\varphi}}_i \overset{a.s}{\to} \underline{\varphi}_i \text{ and } \widehat{\sigma}_i^2 \overset{a.s}{\to} \sigma_i^2,$$

and we have

$$\sqrt{m}\left(\underline{\widehat{\varphi}}_i - \underline{\varphi}_i\right) \underset{m\to\infty}{\overset{D}{\to}} N\left(\underline{0}_{2p},\ \sigma_i^2 \Gamma_i^{-1}\right),$$

where

$$\Gamma_i = \begin{pmatrix} \Gamma_{i,1,1} & \cdots & \Gamma_{i,1,\ p} \\ \vdots & \ddots & \vdots \\ \Gamma_{i,\ p,1} & \cdots & \Gamma_{i,\ p,\ p} \end{pmatrix},$$

and

$$\Gamma_{i,\ j,\ k} = \begin{pmatrix} E(Y_{i-j}Y_{i-k}) & E(Y_{i-j}Y_{i-k}\exp\left(-\gamma Y_{i-1}^2\right)) \\ E(Y_{i-j}Y_{i-k}\exp\left(-\gamma Y_{i-1}^2\right)) & E(Y_{i-j}Y_{i-k}\exp\left(-2\gamma Y_{i-1}^2\right)) \end{pmatrix},$$

$$j,\ k = 1,\ \ldots,\ p.$$

Proof. By replacing the lower limits of the sums in (3) and (4) by zero, we obtain the comparable estimators $\underline{\widetilde{\varphi}}_i = \left(\widetilde{\varphi}_{i,1},\ \widetilde{\pi}_{i,1}, \ldots, \widetilde{\varphi}_{i,\ p}, \widetilde{\pi}_{i,\ p}\right)'$:

$$\underline{\widetilde{\varphi}}_i = \begin{pmatrix} M_{i,1,1} & \cdots & M_{i,1,\ p} \\ \vdots & \ddots & \vdots \\ M_{i,\ p,1} & \cdots & M_{i,\ p,\ p} \end{pmatrix}^{-1} \times \begin{bmatrix} \sum_{\tau=0}^{m-1} Y_{S\tau+i-1}Y_{S\tau+i} \\ \sum_{\tau=0}^{m-1} Y_{S\tau+i-1}Y_{S\tau+i}\exp\left(-\gamma Y_{S\tau+i-1}^2\right) \\ \vdots \\ \sum_{\tau=0}^{m-1} Y_{S\tau+i-p}Y_{S\tau+i} \\ \sum_{\tau=0}^{m-1} Y_{S\tau+i-p}Y_{S\tau+i}\exp\left(-\gamma Y_{S\tau+i-1}^2\right) \end{bmatrix}, \tag{5}$$

$$\widetilde{\sigma}_i^2 = \frac{1}{m}\sum_{\tau=0}^{m-1}\left(Y_{S\tau+i} - \sum_{j=1}^{p}\left(\widetilde{\varphi}_{i,\ j} + \widetilde{\pi}_{i,\ j}\exp\left(-\gamma Y_{S\tau+i-1}^2\right)\right)Y_{S\tau+i-j}\right)^2, \tag{6}$$

where for $j, k = 1, \ldots, p$,

$$M_{i,j,k} =$$

$$\begin{pmatrix} \sum_{\tau=0}^{m-1} Y_{S_{\tau+i-j}} Y_{S_{\tau+i-k}} & \sum_{\tau=0}^{m-1} Y_{S_{\tau+i-j}} Y_{S_{\tau+i-k}} \exp\left(-\gamma Y_{S_{\tau+i-1}}^2\right) \\ \sum_{\tau=0}^{m-1} Y_{S_{\tau+i-j}} Y_{S_{\tau+i-k}} \exp\left(-\gamma Y_{S_{\tau+i-1}}^2\right) & \sum_{\tau=0}^{m-1} Y_{S_{\tau+i-j}} Y_{S_{\tau+i-k}} \exp\left(-2\gamma Y_{S_{\tau+i-1}}^2\right) \end{pmatrix},$$

as an approximation. $\underline{\tilde{\varphi}}_i$ and $\underline{\hat{\varphi}}_i$ have the same limiting distribution, there is no difference whether or not we observe the initial values in addition to the data for m large. (cf. Brockwell and Davis [9]; Chapter 8 and used in Basawa and Lund [6] for the PARMA case). Therefore, we study the asymptotic properties of $\underline{\tilde{\varphi}}_i$.

By replacing $Y_{S_{\tau+i}}$ in the formula (5), we obtain

$$\underline{\tilde{\varphi}}_i = \underline{\varphi}_i + \begin{pmatrix} M_{i,1,1} & \cdots & M_{i,1,p} \\ \vdots & \ddots & \vdots \\ M_{i,p,1} & \cdots & M_{i,p,p} \end{pmatrix}^{-1} \times \begin{bmatrix} \sum_{\tau=0}^{m-1} Y_{S_{\tau+i-1}} \varepsilon_{S_{\tau+i}} \\ \sum_{\tau=0}^{m-1} Y_{S_{\tau+i-1}} \exp\left(-\gamma Y_{S_{\tau+i-1}}^2\right) \varepsilon_{S_{\tau+i}} \\ \vdots \\ \sum_{\tau=0}^{m-1} Y_{S_{\tau+i-p}} \varepsilon_{S_{\tau+i}} \\ \sum_{\tau=0}^{m-1} Y_{S_{\tau+i-p}} \exp\left(-\gamma Y_{S_{\tau+i-1}}^2\right) \varepsilon_{S_{\tau+i}} \end{bmatrix}. \tag{7}$$

From periodic ergodicity of $Y_{S\tau+i}$, we have

$$\frac{1}{m}\sum_{\tau=0}^{m-1}\varepsilon_{S\tau+i}Y_{S\tau+i-j} \overset{a.s}{\to} E(\varepsilon_i)E(Y_{i-j}) = 0, \quad j = 1, \ldots, p,$$

$$\frac{1}{m}\sum_{\tau=0}^{m-1}\varepsilon_{S\tau+i}Y_{S\tau+i-j}\exp\left(-\gamma Y_{S\tau+i-1}^2\right) \overset{a.s}{\to} E(\varepsilon_i)E\left(Y_{i-j}\exp\left(-\gamma Y_{i-1}^2\right)\right) = 0,$$

$$j = 1, \ldots, p.$$

Then $\underline{\tilde{\varphi}}_i \overset{a.s}{\to} \underline{\varphi}_i$ when $m \to \infty$.

From (6), we have

$$\tilde{\sigma}_i^2 = \frac{1}{m}\sum_{\tau=0}^{m-1}\left(\sum_{j=1}^{p}\left(\varphi_{i,j} + \pi_{i,j}\exp\left(-\gamma Y_{S\tau+i-1}^2\right)\right)Y_{S\tau+i-j} + \varepsilon_{S\tau+i}\right.$$

$$\left. - \sum_{j=1}^{p}\left(\tilde{\varphi}_{i,j} + \tilde{\pi}_{i,j}\exp\left(-\gamma Y_{S\tau+i-1}^2\right)\right)Y_{S\tau+i-j}\right)^2$$

$$= \frac{1}{m}\sum_{\tau=0}^{m-1}$$

$$\left(\varepsilon_{S\tau+i} - \sum_{j=1}^{p}(\tilde{\varphi}_{i,j} - \varphi_{i,j})Y_{S\tau+i-j} - \sum_{j=1}^{p}(\tilde{\pi}_{i,j} - \pi_{i,j})Y_{S\tau+i-j}\exp\left(-\gamma Y_{S\tau+i-1}^2\right)\right)^2.$$

From the previous results we have: when $m \to \infty$, $\tilde{\sigma}_i^2 \overset{a.s}{\to} E(\varepsilon_i^2) = \sigma_i^2$.

From (7),

$$\sqrt{m}\left(\underline{\tilde{\varphi}}_i - \underline{\varphi}_i\right) = \left(\frac{1}{m}\begin{pmatrix} M_{i,1,1} & \cdots & M_{i,1,\,p} \\ \vdots & \ddots & \vdots \\ M_{i,\,p,1} & \cdots & M_{i,\,p,\,p} \end{pmatrix}\right)^{-1}$$

$$\times\left(\frac{1}{\sqrt{m}}\begin{bmatrix} \sum_{\tau=0}^{m-1} Y_{S\tau+i-1}\varepsilon_{S\tau+i} \\ \sum_{\tau=0}^{m-1} Y_{S\tau+i-1}\exp\left(-\gamma Y_{S\tau+i-1}^2\right)\varepsilon_{S\tau+i} \\ \vdots \\ \sum_{\tau=0}^{m-1} Y_{S\tau+i-p}\varepsilon_{S\tau+i} \\ \sum_{\tau=0}^{m-1} Y_{S\tau+i-p}\exp\left(-\gamma Y_{S\tau+i-1}^2\right)\varepsilon_{S\tau+i} \end{bmatrix}\right).$$

Since $\{Y_{S\tau+i-j}\varepsilon_{S\tau+i}\}_\tau$ and $\{Y_{S\tau+i-j}\exp\left(-\gamma Y_{S\tau+i-1}^2\right)\varepsilon_{S\tau+i}\}_\tau$, $j = 1, \ldots, p$ are sequences of martingale differences, then we apply the central limit version for martingale differences (cf., Ibragimov [16]):

$$\frac{1}{\sqrt{m}}\begin{bmatrix} \sum_{\tau=1}^{m-1} Y_{S\tau+i-1}\varepsilon_{S\tau+i} \\ \sum_{\tau=1}^{m-1} Y_{S\tau+i-1}\exp\left(-\gamma Y_{S\tau+i-1}^2\right)\varepsilon_{S\tau+i} \\ \vdots \\ \sum_{\tau=1}^{m-1} Y_{S\tau+i-p}\varepsilon_{S\tau+i} \\ \sum_{\tau=1}^{m-1} Y_{S\tau+i-p}\exp\left(-\gamma Y_{S\tau+i-1}^2\right)\varepsilon_{S\tau+i} \end{bmatrix} \underset{m\to\infty}{\overset{D}{\to}} N\left(\underline{0}_{2p}, \sigma_i^2\Gamma_i\right),$$

which completes the proof with Slutzky's theorem.

4. Simulation Results

The performance of the estimation is shown via small simulation. The restricted $PEXPAR_2(2)$ model is used to generate time series for sizes $n = 200, 400, 800$. We consider 1000 Monte Carlo replications and report the LS estimations, their bias and their standard deviations. The Table 1 gives the estimation with the parameters $\underline{\varphi} = (0.6, -1, 0.3, -0.5; -0.5, 1, -0.4, 0.8)'$, $\gamma = 1$ and normal white noise with $(\sigma_1^2, \sigma_2^2)' = (0.6, 1)'$. The choice of the values of the parameters was taken such that the model fulfill the condition A1, see Figure 1. The box plots and the Q-Q plots of the errors are given in Figures 2 and 3, respectively. Table 1 show that the estimates are close to the true values and the standard deviation decreases when n becomes larger and we remark that the standard deviation of $\varphi_{i,j}$ are smaller than those of $\pi_{i,j}$. This is confirmed by the box plots where we observe that the errors are more consistent for $\varphi_{i,j}$ and the range is larger for $\pi_{i,j}$, but in all cases the errors are centered on 0. On the other hand, the Q-Q plots show that the errors are normal.

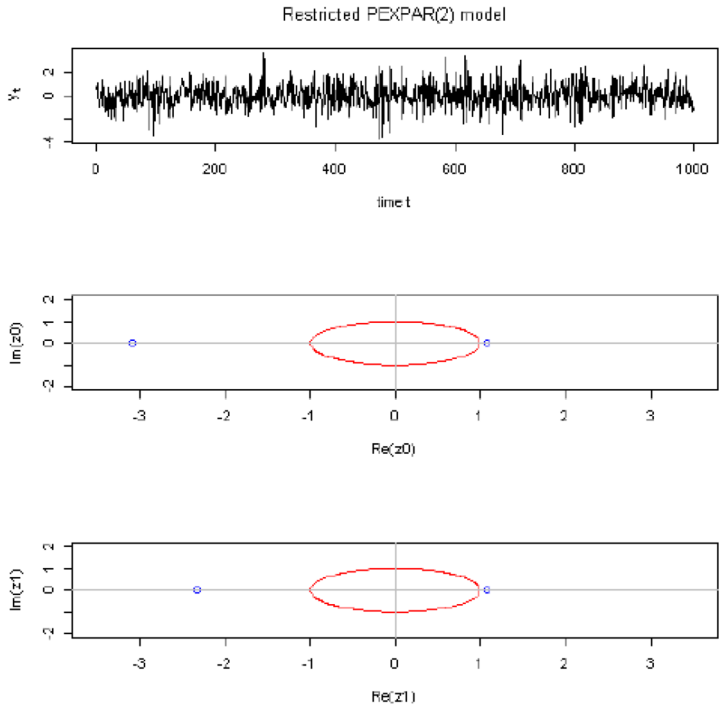

Figure 1. Simulated series and inverse roots of the characteristic equation of restricted $PEXPAR_2(2)$ and $n = 1000$.

Table 1. Estimation results for restricted $PEXPAR_2(2)$

	$\widehat{\varphi}_{1,1}$	$\widehat{\pi}_{1,1}$	$\widehat{\varphi}_{1,2}$	$\widehat{\pi}_{1,2}$	$\widehat{\varphi}_{2,1}$	$\widehat{\pi}_{2,1}$	$\widehat{\varphi}_{2,2}$	$\widehat{\pi}_{2,2}$
$n = 200$	0.5969	– 0.9943	0.3003	– 0.4951	– 0.5077	1.0083	– 0.3760	0.7644
bias	– 0.0030	0.0056	0.0003	0.0048	– 0.0077	0.0083	0.0239	– 0.0355
sd	0.0954	0.3473	0.1486	0.2513	0.2003	0.5182	0.1999	0.3008
$n = 400$	0.5996	– 0.9969	0.3055	– 0.5030	– 0.5013	1.0020	– 0.3901	0.7819
bias	– 0.0003	0.0030	0.0055	– 0.0030	– 0.0013	0.0020	0.0098	– 0.0180
sd	0.0654	0.2357	0.1028	0.1778	0.1381	0.3492	0.1354	0.2031
$n = 800$	0.5988	– 0.9893	0.3023	– 0.5049	– 0.5030	1.0045	– 0.3950	0.7894
bias	– 0.0011	0.0106	0.0023	– 0.0049	– 0.0030	0.0045	0.0049	– 0.0105
sd	0.0461	0.1643	0.0723	0.1226	0.0979	0.2463	0.0959	0.1443

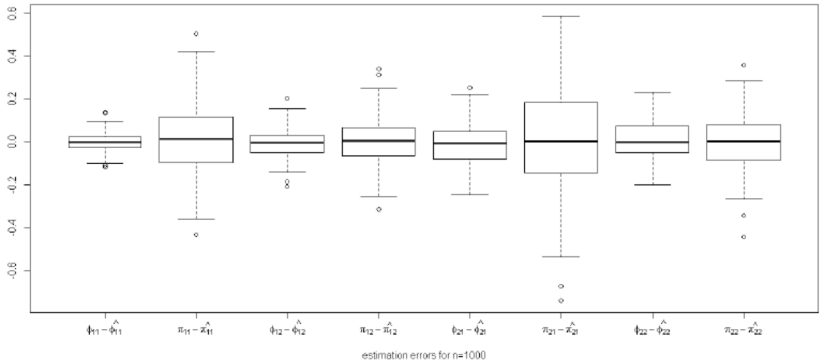

Figure 2. Box plots of the errors from estimates of 200 replications of restricted $PEXPAR_2(2)$ and $n = 1000$.

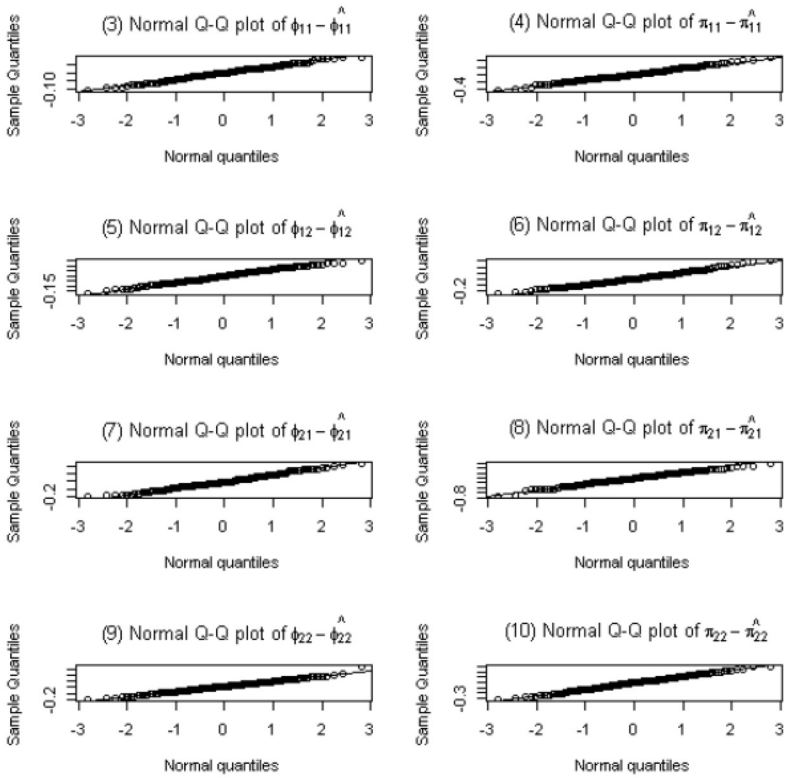

Figure 3. The Q-Q plots of the errors from estimates of 200 replications of restricted $PEXPAR_2(2)$ and $n = 1000$.

5. Conclusion

In this study, we have used the linear least squares method for the estimation of the periodic restricted *EXPAR*(p) model, consistency and asymptotic normality are derived and simulated series checked the asymptotic properties. This LS estimator can be used as an initial estimator in adaptive estimation. As a part of future research, the authors study the Nonlinear LS and Quasi ML estimation of the periodic (unrestricted) *EXPAR*(p) model. We have considered, here, a sufficient condition of strict stationarity but this subject merit further research.

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