International Journal of Statistics: Advances in Theory and Applications
Vol. 1, Issue 1, 2020, Pages 45-74
Published Online on June 3, 2019

http://jyotiacademicpress.org

EFFICIENT ESTIMATION IN RESTRICTIVE PERIODIC EXPAR(1) MODELS

H. DRIDI, M. MERZOUGUI and A. CHADLI

LaPS Laboratory
University Badji Mokhtar
Annaba
Algeria
e-mail: merzouguimouna@yahoo.fr

Abstract

This paper is devoted to study the problem of estimation in the restricted periodic exponential autoregressive (PEXPAR(1)) model. The asymptotic optimality of the procedure is shown via local asymptotic normality (LAN). Once the LAN property is proved, we construct a parametric locally asymptotically minimax LAM estimator. Using these results, we construct the adaptive estimators for the parameters when the innovation density is unknown. The performance of the established estimators is shown via small simulation.

1. Introduction

Periodic time series models have been extensively used in the recent decades to describe many series with periodic dynamics. The inability of SARIMA models to adequately represent many seasonal time series exhibiting a periodic autocovariance structure has motivated the

---

2020 Mathematics Subject Classification: 62F12, 62M10.

Keywords and phrases: periodically correlated process, restricted periodic exponential autoregressive model, local asymptotic normality (LAN), local asymptotic minimax (LAM) estimator, adaptive estimation.

Communicated by Marcelo Bourguignon.

Received February 4, 2017; Revised May 10, 2017

research in the periodically correlated processes. This notion, introduced by Gladyshev [15], was exploited in a variety of new classes of time series models, among them, the periodic GARCH (Bollerslev and Ghysels [9]), the periodic bilinear (Bibi and Gautier [7]) and the mixture periodic autoregressive model (Shao [34]). Recently, Merzougui et al. [30] discussed a class of restricted periodic EXPAR(1) model.

The class of exponential autoregressive time series models introduced, by Ozaki [31] and Haggan and Ozaki [16], in order to describe nonlinear dynamics which are well known in random vibration theory such as amplitude-dependent frequency, jump phenomena and limit cycles. This class has been of interest for its potential applications in ecology, hydrology, speech signal, macroeconomic and others, see, for example, Haggan and Ozaki [16], Ozaki [32], Priestley [33], Terui and Van Dijk [39], Ishizuka et al. [20], Amiri [3]. Recently, Katsiampa [22] suggested the models EXPAR-GARCH which combine two forms of nonlinearity: conditional mean and conditional variance and have the potential of explaining financial time series. The theoretical properties of EXPAR models have been the subject of study by many authors: Chan and Tong [10] gave necessary and sufficient conditions of stationarity and geometric ergodicity for the EXPAR(1) model, a forecasting method is proposed by Al-Kassam and Lane [1], Koul and Schick [23] showed the LAN property and constructed asymptotically efficient estimates for the restricted EXPAR(1), Allal and El Melhaoui [2] constructed a parametric and nonparamtric test for the detection of exponential component in AR(1), Shi and Aoyama [35], Baragona et al. [4] used the genetic algorithm in order to estimate the parameters of EXPAR(p) models, Ismail [25] introduced the Bayesian analysis and Ghosh et al. [14] developed an estimation procedure using extended Kalman filter.

This paper is devoted to establish adaptive estimators of the unknown parameters of a restricted PEXPAR(1) model where the unknown innovation density is symmetric and satisfies only some mild regularity conditions. We recall that an adaptive estimator is efficient for a model where its distribution of the errors is only specified partially. Thus, the adaptive estimator based on nonparametric kernel density estimation is as efficient, asymptotically, as any optimal estimator. Consequently, adaptive estimation is recommended whenever information about the true density is not clear and/or complete. The asymptotic theory is based on the LAN property, due to Le Cam [25] and for the proof we use the version of Swensen [38]. In this work, we follow the procedure of Kreiss [24] who construct, firstly, a LAM estimator for an ARMA model when the innovation density was known, secondly, an adaptive estimator for the semiparametric model having a symmetric innovation density and uses the discretization technic for the proofs.

The problem of adaptive estimation has received considerable interest from many authors: Linton [29] for ARCH models, Drost et al. [12] for general time series models, Drost and Klaassen [11] for GARCH, Ling [27] for ARFIMA with GARCH errors, Ling and McAleer [28] for nonstationary ARMA-GARCH, Lee and Taniguchi [26] for the $\mathrm{ARCH}(\infty)-\mathrm{SM}$ model and Bentarzi et al. [6] for PAR model.

Our paper is organized as follows. The Section 2 introduces the notation and reviews the technical assumptions. In the third section, we establish, while adapting the Swensen's conditions [38] to our periodic model, the Local Asymptotic Normality (LAN) propriety and the local asymptotic linearity for the central sequence. In Section 4, in the case where the innovation density is specified and using a discrete and $\sqrt{n}$-consistent estimator, we obtain a parametric local asymptotic minimax estimator in the sense of Fabian and Hannan [13]. The fifth section, is devoted to the construction of an efficient estimator with unknown symmetric innovation density, based on the kernel estimator for the score function. Finally, in the sixth section, we present a small simulation study.

2. Notations and Assumptions

2.1. Main definitions and notations

The process $\{X_t;\, t \in \mathbb{Z}\}$ is said to follow a restricted periodic exponential autoregressive PEXPAR(1), with period $S\,(S \geq 2)$, if it is a solution of a nonlinear periodic stochastic difference equation of the form:

$$X_t = \left(\varphi_{t,1} + \varphi_{t,2} \exp\left(-\gamma X_{t-1}^2\right)\right) X_{t-1} + \varepsilon_t,\ t \in \mathbb{Z}, \tag{2.1}$$

where $\{\varepsilon_t;\, t \in \mathbb{Z}\}$ is a periodic white noise process, with mean 0 and finite variance σ_t^2, with probability density $f_{\sigma_t}(.)$, not necessarily Gaussian. The autoregressive parameters $\varphi_{t,1}, \varphi_{t,2}$ and the innovation variance σ_t^2 are periodic, in time, with period S, i.e.,

$$\varphi_{t+kS,\,j} = \varphi_{t,\,j} \text{ and } \sigma_{t+kS}^2 = \sigma_t^2,\ \forall k,\, t \in \mathbb{Z} \text{ and } j = 1, 2.$$

The nonlinear parameter, $\gamma > 0$, is known. In fact, Shi et al. [36] proposed a heuristic determination of this coefficient from the data and defined

$$\hat{\gamma} = -\frac{\log\epsilon}{\max\limits_{1 \leq t \leq n} X_t^2},$$

where ϵ is a small number.

Putting $t = s + Sr$, $s = 1, 2, \ldots, S$ and $r \in \mathbb{Z}$, one can rewrite the last periodic nonlinear stochastic difference equation in the equivalent form:

$$X_{s+Sr} = \left(\varphi_{s,1} + \varphi_{s,2} \exp\left(-\gamma X_{s+Sr-1}^2\right)\right) X_{s+Sr-1} + \varepsilon_{s+Sr},\ s = 1, 2, \ldots, S,\ r \in \mathbb{Z}. \tag{2.2}$$

Figure 1 shows a simulated series, monthplot and scatterplot of the $\text{PEXPAR}_4(1)$ model with $\underline{\varphi} = (-0.8, 2; 0.5, -1.5; 0.9, 1.1; -0.7, 0.6)'$, $\gamma = 1$ for all seasons and $n = 500$. The lag plot clearly indicate nonlinear behaviour.

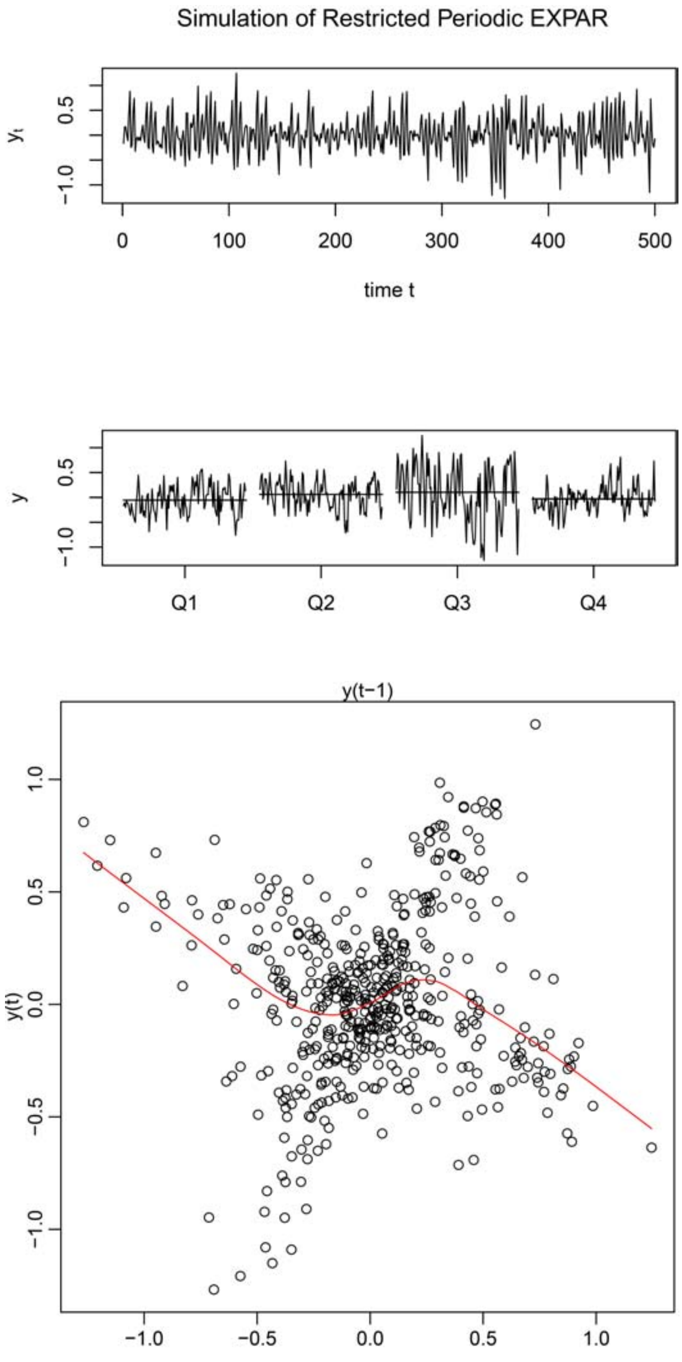

Figure 1. Simulated series, monthplot and scatterplot of the $\text{PEXPAR}_4(1)$.

Denoting $H_f^{(n)}(\underline{\varphi})$ a sequence of null hypotheses under which $\{X_t^{(n)}, t \in \mathbb{Z}\}$ satisfies the PEXPAR(1) model (2.2), where $\underline{\varphi} = (\underline{\varphi}'_1, \underline{\varphi}'_2, \dots, \underline{\varphi}'_S)' \in \mathbb{R}^{2S}$, where $\underline{\varphi}_s = (\varphi_{s,1}, \varphi_{s,2})' \in \mathbb{R}^2$, $s = 1, \dots, S$ and $H_f^{(n)}(\underline{\varphi}^{(n)})$ the sequence of local alternative hypotheses which are contiguous to $H_f^{(n)}(\underline{\varphi})$, under which $\{X_t^{(n)}, t \in \mathbb{Z}\}$ satisfies the PEXPAR(1) model (2.2), where

$$\underline{\varphi}^{(n)} = \left(\underline{\varphi}_1^{(n)'}, \underline{\varphi}_2^{(n)'}, \dots, \underline{\varphi}_S^{(n)'}\right)' \in \mathbb{R}^{2S},$$

$$\underline{\varphi}_s^{(n)} = \left(\varphi_{s,1} + \frac{1}{\sqrt{n}} h_{s,1}^{(n)}, \varphi_{s,2} + \frac{1}{\sqrt{n}} h_{s,2}^{(n)}\right)' \in \mathbb{R}^2, \; s = 1, \dots, S,$$

such that $\sup_n \left(h_{s,1}^{(n)2} + h_{s,2}^{(n)2}\right) < \infty$.

The terms $h_{s,1}^{(n)}$ and $h_{s,2}^{(n)}$, $s = 1, \dots, S$ can be interpreted as local perturbations of the parameters $\varphi_{s,1}$ and $\varphi_{s,2}$, respectively. Let $\underline{\tau}^{(n)} = (\underline{\tau}_1^{(n)}, \underline{\tau}_2^{(n)}, \dots, \underline{\tau}_S^{(n)})'$, where $\underline{\tau}_s^{(n)} = (h_{s,1}^{(n)}, h_{s,2}^{(n)})'$, $s = 1, \dots, S$. We can easily rewrite the sequence $\{\underline{\varphi}^{(n)}, n \in \mathbb{N}\}$ in the following form:

$$\underline{\varphi}^{(n)} = \underline{\varphi} + \frac{1}{\sqrt{n}} \underline{\tau}^{(n)}, \; \underline{\tau}^{(n)} \in \mathbb{R}^{2S} \text{ such that } \underline{\tau}^{(n)'} \underline{\tau}^{(n)} < \infty.$$

2.2. Technical regularity assumptions

Throughout this paper, we make the following assumptions:

Assumption (A1): The exponential autoregressive parameters $\underline{\varphi}$ satisfy the sufficient periodically stationary condition of (2.2), i.e., $|\varphi_{s,1}| < 1$, $\varphi_{s,2} \in \mathbb{R}$, $s = 1, \dots, S$.

Assumption (A2): The innovation density $f_{\sigma_t}(.)$ is supposed to satisfy the following conditions:

(a) $f_{\sigma_t}(x) > 0, \forall x \in \mathbb{R}$;

(b) $f_{\sigma_t}(.)$ is absolutely continuous with respect to the Lebesgue measure μ: that is there exists a function. $\dot{f}_{\sigma_t}(.)$ such that, for all $-\infty < a < b < \infty$, we have $f_{\sigma_t}(b) - f_{\sigma_t}(a) = \int_a^b \dot{f}_{\sigma_t}(x) d\mu(x)$;

(c) The Fisher information $I(f_{\sigma_t}) = \int (\phi_f(x))^2 f_{\sigma_t}(x) dx$ is finite, where $\phi_f = -\frac{\dot{f}_{\sigma_t}(.)}{f_{\sigma_t}(.)}$;

(d) $\int x f_{\sigma_t}(x) dx = 0$ and the variance is finite, i.e., $\sigma_t^2 = E(\varepsilon_t^2) < \infty$.

We note that the two conditions (b) and (c) imply the quadratic differentiability of the function $f_\sigma(.)^{1/2}$, i.e.,

$$\lim_{\lambda \to 0} \lambda^{-2} \int \left(f_{\sigma_t}^{1/2}(x+\lambda) - f_{\sigma_t}^{1/2}(x) - \lambda \frac{\dot{f}_{\sigma_t}(x)}{f_{\sigma_t}^{1/2}(x)} \right)^2 dx = 0.$$

(see, for instance Lemma 3, page 191, Hájek [17] or Hájek and Šidák [18]).

3. Local Asymptotic Normality for Restricted PEXPAR(1)

3.1. Sequence of likelihood ratios

Denote by $\underline{X}^{(n)} = \left(X_1^{(n)}, \ldots, X_n^{(n)}\right)$ a realization of a finite size n of a periodically correlated autoregressive process $\left\{X_t^{(n)}; t \in \mathbb{Z}\right\}$ satisfying the causal periodic restricted EXPAR(1) model (2.2) and let $X_0^{(n)}$ be the initial value whose densities are $g_0^{(n)}\left(X_0^{(n)};\ \underline{\varphi},\ \underline{\sigma}\right)$ and $g_0^{(n)}\left(X_0^{(n)};\ \underline{\varphi}^{(n)},\ \underline{\sigma}\right)$ under $H_f^{(n)}(\underline{\varphi})$ and $H_f^{(n)}\left(\underline{\varphi}^{(n)}\right)$, respectively, where $\underline{\sigma} = (\sigma_1, \ldots, \sigma_S)'$. Furthermore, we suppose that $g_0^{(n)}\left(X_0^{(n)};\ \underline{\varphi}^{(n)},\ \underline{\sigma}\right) - g_0^{(n)}\left(X_0^{(n)};\ \underline{\varphi},\ \underline{\sigma}\right)$ converges in probability to 0, when $\underline{\varphi}^{(n)} \to \underline{\varphi}$ with $n \to \infty$. Suppose, for simplicity of notation, that the size of observed time series n is a multiple of S, i.e., $n = mS$, $m \in \mathbb{N}^*$ and let $t = s + rS$, $s = 1, \ldots, S$ and $r = 0, 1, \ldots, m-1$. Denote by $Z_t^{(n)}(\underline{\varphi})$ and $Z_t^{(n)}\left(\underline{\varphi}^{(n)}\right)$, $t \in \mathbb{Z}$, the calculated residuals under $H_f^{(n)}(\underline{\varphi})$ and $H_f^{(n)}\left(\underline{\varphi}^{(n)}\right)$, respectively. Then, we have

$$
\begin{aligned}
Z_{s,r}^{(n)}\left(\underline{\varphi}^{(n)}\right) = {} & \\
& X_{s+rS}^{(n)} - \left[\left(\varphi_{s,1} + \frac{1}{\sqrt{n}} h_{s,1}^{(n)}\right) + \left(\varphi_{s,2} + \frac{1}{\sqrt{n}} h_{s,2}^{(n)}\right) \exp\left(-\gamma X_{s+rS-1}^2\right)\right] X_{s+rS-1}^{(n)} \\
& = Z_{s,r}^{(n)}(\underline{\varphi}) - \frac{1}{\sqrt{n}}\left[h_{s,1}^{(n)} + h_{s,2}^{(n)} \exp\left(-\gamma X_{s+rS-1}^2\right)\right] X_{s+rS-1}^{(n)} \\
& = Z_{s,\tau}^{(n)}(\underline{\varphi}) - \frac{1}{\sqrt{n}} \underline{\tau}_s^{(n)'} \underline{X}_{s+rS-1}^{(n)} \\
& = Z_{s,\tau}^{(n)}(\underline{\varphi}) - \gamma_{s,r}^{(n)},
\end{aligned}
$$

where $\underline{X}_{s+rS-1}^{(n)} = \left(X_{s+rS-1}^{(n)},\ X_{s+rS-1}^{(n)} \exp\left(-\gamma X_{s+rS-1}^2\right)\right)'$ and $\gamma_{s,r}^{(n)} = \frac{1}{\sqrt{n}} \underline{\tau}_s^{(n)'}$ $\underline{X}_{s+rS-1}^{(n)}$, $s = 1, \ldots, S$. The corresponding empirical variances are then given by $\hat{\sigma}_s^2 = \frac{1}{m}\sum_{r=0}^{m-1}\left(Z_{s+rS}^{(n)}(\underline{\varphi})\right)^2$, $s = 1, 2, \ldots, S$. Hence, the logarithm of the likelihood ratio, $\Lambda_f^{(n)}\left(\underline{\varphi}^{(n)}\right) = \Lambda_f^{(n)}\left(\underline{\varphi} + \frac{1}{\sqrt{n}} \underline{\tau}^{(n)}\right)$ for $H_f^{(n)}(\underline{\varphi})$ versus $H_f^{(n)}\left(\underline{\varphi}^{(n)}\right)$, is then given, for $n = mS$, by:

$$\Lambda_f^{(n)}\left(\underline{\varphi} + \frac{1}{\sqrt{n}}\underline{\tau}^{(n)}\right) = \sum_{s=1}^{S}\sum_{r=0}^{m-1} \log \frac{f_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s^{(n)}\right)\right)}{f_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right)} + \log \frac{g_0^{(n)}\left(X_0^{(n)};\ \underline{\varphi}^{(n)},\ \sigma\right)}{g_0^{(n)}\left(X_0^{(n)};\ \underline{\varphi},\ \sigma\right)}.$$

Hence, we have, under $H_f^{(n)}(\underline{\varphi})$, the local asymptotic approximation:

$$\Lambda_f^{(n)}\left(\underline{\varphi} + \frac{1}{\sqrt{n}}\underline{\tau}^{(n)}\right) = \sum_{s=1}^{S}\sum_{r=0}^{m-1} \log \frac{f_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s^{(n)}\right)\right)}{f_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right)} + 0_P(1),$$

$$= \sum_{s=1}^{S}\sum_{r=0}^{m-1}\left[\log\left(f_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right) - \gamma_{s,r}^{(n)}\right)\right) - \log\left(f_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right)\right)\right] + 0_P(1),$$

where the $0_P(1)$ term accounts for the unobserved value $X_0^{(n)}$.

3.2. Local asymptotic normality

In order to prove LAN, we shall use a modification of Swensen's conditions to deal with our restricted PEXPAR(1) model. Let, for $s = 1, \ldots, S$ and $r = 0, \ldots, m-1$, the following random variables:

$$\xi_{s+rS}^{(n)}\left(\underline{\varphi}_s\right) = \frac{f_{\sigma_s}^{1/2}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right) - \gamma_{s,r}^{(n)}\right)}{f_{\sigma_s}^{1/2}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right)} - 1,$$

$$\zeta_{s+rS}^{(n)}\left(\underline{\varphi}_s\right) = \frac{1}{2}\phi_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right)\gamma_{s,r}^{(n)} = \frac{1}{2}\Delta_{s,r}^{(n)'}\underline{\tau}_s^{(n)},\ \text{where } \phi_{\sigma_s}(.) = -\frac{\dot{f}_{\sigma_s}(.)}{f_{\sigma_s}(.)},$$

and $\Delta_{s,.,r}^{(n)} = \frac{1}{\sqrt{n}}\phi_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right)\underline{X}_{s+rS-1}^{(n)}$,

$$\Delta^{(n)}\left(\underline{\varphi}_s\right) = \left(\Delta_1^{(n)'}\left(\underline{\varphi}_1\right), \ldots, \Delta_S^{(n)'}\left(\underline{\varphi}_S\right)\right)' \in \mathbb{R}^{2S}, \tag{3.1}$$

where $\Delta_s^{(n)}\left(\underline{\varphi}_s\right) = \frac{1}{\sqrt{n}}\sum_{r=0}^{m-1}\phi_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}\right)\right)\underline{X}_{s+rS-1}^{(n)}$, $s = 1, \ldots, S$,

$$\Delta_{s,1}^{(n)}\left(\underline{\varphi}_s\right) = \frac{1}{\sqrt{n}} \sum_{r=0}^{m-1} \phi_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}\right)\right) X_{s+rS-1}^{(n)}, \text{ and}$$

$$\Delta_{s,2}^{(n)}\left(\underline{\varphi}_s\right) = \frac{1}{\sqrt{n}} \sum_{r=0}^{m-1} \phi_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}\right)\right) X_{s+rS-1}^{(n)} \exp\left(-\gamma X_{s+rS-1}^{(n)2}\right).$$

Consider the $2S \times 2S$ block diagonal matrix:

$$\Gamma\left(\underline{\varphi}, \underline{\sigma}\right) = \begin{pmatrix} \frac{\Gamma_1\left(\underline{\varphi}, \underline{\sigma}\right)}{\sigma_1^2} & 0 & \cdots & 0 \\ 0 & \frac{\Gamma_2\left(\underline{\varphi}, \underline{\sigma}\right)}{\sigma_2^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \frac{\Gamma_S\left(\underline{\varphi}, \underline{\sigma}\right)}{\sigma_S^2} \end{pmatrix}, \tag{3.2}$$

where $\Gamma_s\left(\underline{\varphi}, \underline{\sigma}\right)$ is the variance matrix of the vector $\underline{X}_{s-1+Sr}^{(n)}$, $s = 1, \ldots, S$ and $r \in \mathbb{Z}$

$$\Gamma_s\left(\underline{\varphi}, \underline{\sigma}\right) = \begin{pmatrix} E(X_{s-1}^2) & E\left(e^{-\gamma X_{s-1}^2} X_{s-1}^2\right) \\ E\left(e^{-\gamma X_{s-1}^2} X_{s-1}^2\right) & E\left(e^{-2\gamma X_{s-1}^2} X_{s-1}^2\right) \end{pmatrix}.$$

Using the precedent definitions and notations, we are able to state the adapted Swensen's conditions, which imply the LAN property.

Proposition 3.1. *The following adapted conditions are, under the Assumptions* (A1) *and* (A2), *satisfied*:

(1) $\lim_{m\to\infty} E \sum_{s=1}^{S} \sum_{r=0}^{m-1} \left(\xi_{s+rS}^{(n)}\left(\underline{\varphi}_s\right) - \zeta_{s+rS}^{(n)}\left(\underline{\varphi}_s\right)\right)^2 = 0;$

(2) $\sup_{m} E \sum_{s=1}^{S} \sum_{r=0}^{m-1} \left(\zeta_{s+rS}^{(n)2}\left(\underline{\varphi}_s\right)\right) < \infty;$

(3) $\max_s \max_r \left|\zeta^{(n)}_{s+rS}\left(\underline{\varphi}_s\right)\right| = 0_P(1);$

(4) $\sum_{s=1}^{S}\sum_{r=0}^{m-1}\left(\zeta^{(n)2}_{s+rS}\left(\underline{\varphi}_s\right)\right) - \frac{1}{4S} I(f_1)\underline{\tau}^{(n)'}\Gamma\left(\underline{\varphi}, \underline{\sigma}\right)\underline{\tau}^{(n)} = 0_P(1);$

(5) $\sum_{s=1}^{S}\sum_{r=0}^{m-1} E\left[\zeta^{(n)2}_{s+rS}\left(\underline{\varphi}_s\right) I_{\left[\left|\zeta^{(n)}_{s+rS}\left(\underline{\varphi}_s\right)\right| > \frac{1}{2}\right]} / \mathcal{B}_{n, s-1+rS}\right] = 0_P(1),$

$\mathcal{B}_{n, s-1+rS}$ *is the* σ *-algebra generated by the past of the process up to time* $s - 1 + rS$;

(6) $E\left(\zeta^{(n)}_{s+rS}\left(\underline{\varphi}_s\right) / \mathcal{B}_{n, s-1+rS}\right) = 0_P(1).$

Proof. See Appendix.

Proposition 3.2. *Assuming that Assumptions* (A1) *and* (A2) *hold*, *then we have, under* $H_f^{(n)}\left(\underline{\varphi}\right)$, *as* $n \to \infty$ *the following two results*:

(i) $\Lambda_f^{(n)}\left(\underline{\varphi} + \frac{1}{\sqrt{n}}\underline{\tau}^{(n)}\right) = \underline{\tau}^{(n)'}\Delta^{(n)}\left(\underline{\varphi}\right) - \frac{I(f_1)}{2S}\underline{\tau}^{(n)'}\Gamma\left(\underline{\varphi}, \underline{\sigma}\right)\underline{\tau}^{(n)} + 0_P(1);$

(ii) $\Delta^{(n)}\left(\underline{\varphi}\right) \to N_{2S}\left(\underline{0}, \frac{I(f_1)}{S}\Gamma\left(\underline{\varphi}, \underline{\sigma}\right)\right).$

Proof. Since the Swensen's sufficient conditions are verified, then taking account of the fact that

$$2\sum_{s=1}^{S}\sum_{r=0}^{m-1}\zeta^{(n)}_{s+rS}\left(\underline{\varphi}\right) = \underline{\tau}^{(n)'}\Delta^{(n)}\left(\underline{\varphi}\right).$$

Local asymptotic normality follows immediately from Theorem 1 (Le Cam) (cf., Swensen [38]). We will use the notation $LAN\left(\underline{\varphi}, \frac{I(f_1)}{S}\Gamma\left(\underline{\varphi}, \underline{\sigma}\right), \Delta^{(n)}\left(\underline{\varphi}\right)\right)$.

Lemma 3.1. (i) *As a consequence of the property LAN, we have*: $H_f^{(n)}(\underline{\varphi})$ *and* $H_f^{(n)}(\underline{\varphi}^{(n)})$ *are contiguous.*

(ii) *The central sequence* $\Delta^{(n)}(\underline{\varphi})$ *satisfies the following local asymptotic linearity*:

$$\Delta^{(n)}\left(\underline{\varphi}^{(n)}\right) - \Delta^{(n)}\left(\underline{\varphi}\right) = -\frac{I(f_1)}{S}\Gamma\left(\underline{\varphi}, \underline{\sigma}\right)\underline{\tau}^{(n)} + 0_P(1), \tag{3.3}$$

where $\underline{\varphi}^{(n)} = \underline{\varphi} + \frac{1}{\sqrt{n}}\underline{\tau}^{(n)}$, *as* $n \to \infty$ *under* $H_f^{(n)}(\underline{\varphi})$ *hence also under* $H_f^{(n)}(\underline{\varphi}^{(n)})$.

Proof. For (ii), the proof is similar of Lemma 6.4 of Kreiss [24].

4. Existence and Construction of LAM Estimators

From the Proposition 3.2, we can construct sequences of estimates which are locally asymptotically minimax (LAM) as defined in Fabian and Hannan [13].

4.1. Lower bound in LAN models

Let $l : \mathbb{R}^{2S} \underset{x \to l(x)}{\to} \mathbb{R}$ a lost function which satisfied the following conditions:

(i) $l(x) \geq 0$;

(ii) $l(x) = l(-x)$, $\forall x \in \mathbb{R}^{2S}$;

(iii) $\{x \,/\, l(x) \leq u\}$ is convexe $\forall u \in \mathbb{R}_+^*$.

The following theorem gives a lower bound of the risk when we take $\{Z_n\}$ as a sequence of estimators of parameter $\underline{\varphi}$.

Theorem 4.1 (Fabian and Hannan [13]). *Let* $\{Z_n\}$ *any sequence of estimator of* $\underline{\varphi}$ *and suppose that the condition* $LAN\left(\underline{\varphi}, \frac{I(f_1)}{S}\Gamma(\underline{\varphi}, \underline{\sigma}), \Delta^{(n)}(\underline{\varphi})\right)$ *is verified, then*:

$$\lim_{K\to\infty}\liminf_{n\to\infty}\inf_{Z_n}\sup_{\|\sqrt{n}(\underline{\varphi}-\underline{\varphi}_0)\|\leq K} E_{n,\underline{\varphi}}l\{\sqrt{n}(Z_n-\underline{\varphi})\}$$

$$\geqslant \int l(x)dN\left(0, \left(\frac{I(f_1)}{S}\Gamma(\underline{\varphi}, \underline{\sigma})\right)^{-1}\right). \tag{4.1}$$

Definition 4.1. If the $LAN\left(\underline{\varphi}, \frac{I(f_1)}{S}\Gamma(\underline{\varphi}, \underline{\sigma}), \Delta^{(n)}(\underline{\varphi})\right)$ condition is satisfied, a sequence of estimator $\{Z_n\}$ is called locally and asymptotically minimax $LAM(\underline{\varphi})$, if the equality in (4.1) holds.

Definition 4.2. The sequence of estimators $\{Z_n\}$ is called $\underline{\varphi}$-regular, under $LAN\left(\underline{\varphi}, \frac{I(f_1)}{S}\Gamma(\underline{\varphi}, \underline{\sigma}), \Delta^{(n)}(\underline{\varphi})\right)$ if

$$\sqrt{n}(Z_n-\underline{\varphi}) - \frac{S}{I(f_1)}\left(\Gamma(\underline{\varphi}, \underline{\sigma})\right)^{-1}\Delta^{(n)}(\underline{\varphi}) = 0_P(1). \tag{4.2}$$

The following lemma is a result from Fabian and Hannan [13], Theorem 6.3, p.467.

Lemma 4.1. *Under the LAN condition, for any sequence of estimators* (Z_n), *if* (Z_n) *is* $\underline{\varphi}$*-regular, then* (Z_n) *is LAM.*

4.2. Construction of LAM estimators

In order to construct regular estimates the existence, of $\sqrt{n}$-consistent estimators of $\underline{\varphi}$, is essential. We assume that there exists a sequence of initial estimators $\widehat{\underline{\varphi}}^{(n)}$, $\sqrt{n}$-consistent, hence the following hypothesis:

Assumption (A3): A sequence of estimators $\overline{\varphi}_n$ exists, such that

(i) $\sqrt{n}\,(\hat{\underline{\varphi}}^{(n)} - \underline{\varphi}) = O_P(1)$, under $H_f^{(n)}(\underline{\varphi})$.

For technical reason, we also use discretized version $\overline{\varphi}_n$ of the $\hat{\underline{\varphi}}^{(n)}$ which is defined as follows:

(ii) $\overline{\varphi}_n$ is locally asymptotically discrete; that is, there exists $K \in N$ such that in dependently of $n \in N$, $\overline{\varphi}_n$ takes at most K different values in the set $Q_n = \left\{\theta \in \mathbb{R}^{2S} : \left(\underline{\nu}^{(n)}\right)^{-1}\left|\theta - \underline{\varphi}\right| \leq c\right\}$, $c > 0$ fixed.

The great advantage of discrete estimates is that we can replace a deterministic sequence $\underline{\varphi}^{(n)}$ by a sequence of discrete estimators, this result is formulated in Lemma 4.4 in Kreiss [24], so we can restrict the proofs to nonstochastic sequences.

The estimator given in the next proposition is valid and optimal when the density f is known.

Proposition 4.1. *Assume* $\{\overline{\varphi}_n\} \in \Theta$ *is discrete and* $\sqrt{n}$*-consistent sequences of estimators for* $\underline{\varphi} \in \Theta$. *Then* $\hat{\underline{\varphi}}_n$ *defined by*

$$\hat{\underline{\varphi}}_n = \overline{\varphi}_n + \frac{1}{\sqrt{n}}\left(\frac{I(f_1)}{S}\hat{\Gamma}(\overline{\varphi}_n, \underline{\sigma})\right)^{-1} \Delta_f^{(n)}(\overline{\varphi}_n), \tag{4.3}$$

is regular, with

$$\left(\hat{\Gamma}_n(\overline{\varphi}_n, \underline{\sigma})\right)^{-1} = \begin{pmatrix} \dfrac{\hat{\Gamma}_{n,1}(\overline{\varphi}_n)}{\hat{\sigma}_1^2} & \cdots & \cdots & 0 \\ 0 & \dfrac{\hat{\Gamma}_{n,2}(\overline{\varphi}_n)}{\hat{\sigma}_2^2} & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dfrac{\hat{\Gamma}_{n,S}(\overline{\varphi}_n)}{\hat{\sigma}_S^2} \end{pmatrix}^{-1},$$

where $\widehat{\Gamma}_{n,S}\,(\,\overline{\varphi}_n) = \frac{1}{m}\sum_{\tau=0}^{m-1} X\,(S\tau + s - 1)\,X\,(S\tau + s - 1)'$ *and* $\widehat{\Gamma}_n$ *is a consistent estimator of* $\Gamma(\underline{\varphi})$.

Proof. See Appendix.

5. Construction of Adaptive Estimators

The estimator given in the preceding proposition is valid and optimal when the density *f* is specified. However, in practice, *f* remains unknown. In this section, we consider the semi-parametric model whose parameter is $(\underline{\varphi}, f)$, where $\underline{\varphi}$ is the parameter of interest and *f* is the nuisance parameter belonging to the class of the symmetrical density functions F^+. An estimator is known as adaptive if it has for the unknown density function *f* the same efficacity as the optimal estimator in the model where *f* is supposed to be known. The requirement of adaptability given by Stein [37] is satisfied as soon as the density *f* is symmetrical, so we add the following hypothesis.

Assumption (A4): The innovation density function *f* is symmetric with finite fourth moment. This density intervenes in the estimate through the score function ϕ. One can estimate *f* by the kernel method, we envisage, in this work, the estimators proposed by Kreiss [24]. To this end, the following notations are introduced:

(i) $g(x;\ \eta) = \frac{1}{\sqrt{2\pi\eta^2}} \exp\left(-\frac{x^2}{2\eta^2}\right),\ x \in \mathbb{R};$

(ii) $f_\eta(x) = \int_{-\infty}^{+\infty} g(x - y;\ \eta) f(y) dy;$

(iii) $\widehat{f}_{\eta,\,r}\,(x;\ \varphi) = \frac{1}{2\,(m-1)} \sum_{\substack{r_0=0 \\ r_0 \neq r}}^{m-1} \{\, g\,(\,x + z_{s,\,\tau_{r0}},\ \eta\,) + g\,(\,x - z_{s,\,r_0},\ \eta\,)\,\};$

$r = 0,\ \ldots,\ m-1.$

Let $\widehat{q}_{n,r}(x, \varphi)$ be an estimator of ϕ

$$\widehat{q}_{n,r}(x, \varphi) = \begin{cases} -\dfrac{1}{2}\dfrac{\widehat{f'}_{\eta(n),r}(x, \varphi)}{\widehat{f}_{\eta(n),r}(x, \varphi)}, \text{ if } & \begin{cases} \widehat{f}_{\eta(n),r}(x, \varphi) \geq d_n, |x| \leq g_n \\ \widehat{f'}_{\eta(n),r}(x, \varphi) \leq c_n \widehat{f}_{\eta(n),r}(x, \varphi) \end{cases}, \\ 0, & \text{otherwise} \end{cases}$$

with $c_n \to \infty$, $g_n \to \infty$, $\eta(n) \to 0$, $d_n \to 0$. Define $\widetilde{\Delta}_s(\underline{\varphi}_s) = \frac{1}{\sqrt{n}} \sum_{r=0}^{m-1} \widehat{q}_{n,r}(x, \underline{\varphi}_s) \underline{X}_{s-1+Sr}$ and the estimator $\widetilde{\Delta}^{(n)}(\underline{\varphi})$ of $\Delta^{(n)}(\underline{\varphi})$ by $\widetilde{\Delta}^{(n)}(\underline{\varphi}) = (\widetilde{\Delta}_1(\underline{\varphi}_1), \widetilde{\Delta}_2(\varphi_2), \ldots, \widetilde{\Delta}_S(\underline{\varphi}_S))'$, and the unbiased estimator $\widehat{I}_n(\underline{\varphi}) = \frac{1}{S} \sum_{s=1}^{S} \left(\frac{1}{m} \sum_{r=0}^{m-1} \widehat{q}^{\,2}_{n,r} \left(Z^{(n)}_{s,r}(\underline{\varphi}_s), \underline{\varphi}_s \right) \right)$ of $I(f_1)$.

Lemma 5.1. *Let* $(\overline{\varphi}_n)$ *be a sequence of discrete* $\sqrt{n}$*-consistent of* $\underline{\varphi}$. *Then, under the hypotheses* (A1)-(A4), *we have*: $\widetilde{\Delta}^{(n)}(\overline{\varphi}_n) - \Delta^{(n)}(\overline{\varphi}_n) = 0_P(1)$ *if* $c_n \to \infty$, $g_n \to \infty$, $\eta(n) \to 0$, $d_n \to 0$, $\eta(n)c_n \to 0$, $g_n \eta(n)^{-4}/n \to 0$ *and* $n\eta(n)^9$ *stays bounded.*

Proof. See Appendix.

Lemma 5.2 (Estimation of the Fisher information)**.**

$$\widehat{I}_n(\underline{\varphi}) = \frac{1}{S} \sum_{s=1}^{S} \left(\frac{1}{m} \sum_{r=0}^{m-1} \widehat{q}^{\,2}_{n,r} \left(Z^{(n)}_{s,r}(\underline{\varphi}_s), \underline{\varphi}_s \right) \right),$$

is a consistent estimator of $I(f_1)$, *i.e.*, $\widehat{I}_n(\underline{\varphi}) = I(f_1) + 0_P(1)$.

Proof. From the WLLN, we have for each $\underline{\varphi}^{(n)}$ satisfying $\underline{\varphi}^{(n)} = \underline{\varphi} + \frac{1}{\sqrt{n}} \underline{\tau}^{(n)}$,

$$\frac{1}{S}\sum_{s=1}^{S}\left(\frac{1}{m}\sum_{r=0}^{m-1}\phi_{f_1}^{2}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s^{(n)}\right)\right)\right) \to I(f_1), \text{ as } n \to \infty, \text{ under } H_f^{(n)}\left(\underline{\varphi}^{(n)}\right).$$

The assertion follows from Lemma 4.1 of Bickel [8], the contiguity of $H_f^{(n)}(\underline{\varphi})$ and $H_f^{(n)}\left(\underline{\varphi}^{(n)}\right)$ and from the Lemma 4.4 of Kreiss [24].

The following proposition establishes the adaptive estimators for the parameters of the periodic EXPAR(1) models.

Proposition 5.1. *Under the hypotheses* (A1)-(A4), *the estimator* $\underline{\widetilde{\varphi}}_n$ *defined as*:

$$\underline{\widetilde{\varphi}}_n = \overline{\varphi}_n + \frac{1}{\sqrt{n}} \frac{S\widehat{\Gamma}_n(\overline{\varphi}_n)^{-1}}{\widetilde{I}_n} \widetilde{\Delta}^{(n)}(\overline{\varphi}_n),$$

is an LAM estimator, *consequently*, *it is adaptive*.

Proof. The proof of this proposition, which rests on the two preceding lemmas, is similar to that of Kreiss [24].

6. Simulation Results

The performance of the adaptive estimator is shown by a small simulation study. Two periodic $\text{PEXPAR}_S(1)$ models, with period $S = 2, 4$, are used to simulate the time series. For each data-generating process, we consider 1000 Monte Carlo replications and report the adaptive estimator (AE) and the LSE, which is used as initial estimator, and we compare them by the root mean square error (RMSE) criterion: RMSE $= \sqrt{(\text{Bias})^2 + \text{Variance}}$. The models are:

(1) $\text{PEXPAR}_2(1)$ model:

with $\underline{\varphi} = (-0.8, 1.5, 0.5, -2)$, $\sigma^2 = (0.4, 0.8)$ and $n = 200, 400$.

(2) $\text{PEXPAR}_4(1)$ model:

with $\varphi(0.4, 1.2; -0.5, 3; 0.9, 1.7; 0.7, 2)'$, $\sigma^2 = (0.2, 0.5, 0.8, 0.4)$ and $n = 1000$.

The innovation densities are taken to be the standard normal and the famous density of Kreiss [24] which is the case where the density is far away from normality:

$$f_1(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right),$$

$$f_2(x) = \frac{0.5\sqrt{10}}{\sqrt{2\pi}} \exp\left(-\frac{(\sqrt{10}x - 3)^2}{2}\right) + \frac{0.5\sqrt{10}}{\sqrt{2\pi}} \exp\left(-\frac{(\sqrt{10}x - 3)^2}{2}\right).$$

The second density is rescaled such that it has the required unit variance. The parameters of the score function was taken to be:

$$\eta(n) = m^{-\frac{1}{3}}, \ g_n = 4m^{\frac{1}{6}}, \ c_n = g_n, \ d_n = \frac{0.1}{m}.$$

The estimation results are reported in Tables 1 and 2, respectively.

Table 1. Empirical means and RMSE of LSE and AE for $\text{PEXPAR}_2(1)$

		$n = 200$				$n = 400$			
$f / \underline{\phi}$		$\varphi_{1,1}$	$\varphi_{1,2}$	$\varphi_{2,1}$	$\varphi_{2,2}$	$\varphi_{1,1}$	$\varphi_{1,2}$	$\varphi_{2,1}$	$\varphi_{2,2}$
f_1	LSE Mean	– 0.7991	1.4928	0.4916	– 1.9840	– 0.7977	1.5005	0.5008	– 1.9997
	RMSE	0.0804	0.2845	0.1688	0.3660	0.0576	0.1995	0.1164	0.2436
	AE Mean	– 0.8007	1.4976	0.4939	– 1.9885	– 0.7980	1.5027	0.5007	– 2.0033
	RMSE	0.0859	0.3018	0.1764	0.3880	0.0620	0.2116	0.1254	0.2640
f_2	LSE Mean	– 0.8023	1.5159	0.4974	– 2.0044	– 0.8023	1.4941	0.5041	– 2.0138
	RMSE	0.0752	0.3153	0.1860	0.3716	0.0549	0.2231	0.1239	0.2515
	AE Mean	– 0.7993	1.4942	0.4971	– 1.9970	– 0.8014	1.5111	0.5003	– 2.0006
	RMSE	0.0518	0.2169	0.0872	0.1570	0.0347	0.1393	0.0496	0.0976

Table 2. Empirical means and RMSE of LSE and AE for $\text{PEXPAR}_4(1)$

$f / \underline{\phi}$		$\phi_{1,1}$	$\phi_{2,1}$	$\phi_{1,2}$	$\phi_{2,2}$	$\phi_{3,1}$	$\phi_{3,2}$	$\phi_{4,1}$	$\phi_{4,2}$
f_1	LSE Mean	0.4011	1.2047	– 0.4982	2.9979	0.8988	1.7150	0.7000	2.0022
	RMSE	0.0282	0.2649	0.0527	0.1823	0.0702	0.2739	0.0346	0.1260
	AE Mean	0.4010	1.2027	– 0.4984	2.9967	0.8988	1.7175	0.7002	2.0020
	RMSE	0.0291	0.2806	0.0547	0.1911	0.0747	0.2900	0.0354	0.1319
f_2	LSE Mean	0.4017	1.1974	– 0.4989	2.9936	0.9005	1.6978	0.6977	2.0034
	RMSE	0.0312	0.2459	0.0525	0.2367	0.0908	0.2978	0.0352	0.1360
	AE Mean	0.3999	1.1991	– 0.5004	2.9987	0.9011	1.6967	0.7005	2.0000
	RMSE	0.0109	0.0893	0.0227	0.0986	0.0328	0.1054	0.0191	0.0779

The two tables show that the adaptive estimator is always better compared to LSE when the innovation density is f_2. In the normal case, the performance of the two estimators is not much different.

Acknowledgements

The authors would like to express their deep gratitude to the referee. His important suggestions and remarks improved the quality of the paper.

References

[1] M. S. Al-Kassam and J. A. Lane, Forecasting exponential autoregressive models of order 1, Journal of Time Series Analysis 10(2) (1989), 95-113.

[2] J. Allal and S. El Melhaoui, Optimal detection of exponential component, Journal of Time Series Analysis 27(6) (2006), 793-810.

[3] E. Amiri, Forecasting GDP growth rate with nonlinear models, 1st International Conference on Econometrics Methods and Applications, ICEKU2012 (2012), 25-27.

[4] R. Baragona, F. Battaglia and D. Cuccina, A note on estimating autoregressive exponential models, Quaderni di Statistica 4 (2002), 2002.

[5] Y. Benghabrit and M. Hallin, Optimal rank based tests against first order super-diagonal bilinear dependence, J. Statist. Plann. Inference 32 (1992), 45-61.

[6] M. Bentarzi, H. Guerbyenne and M. Merzougui, Adaptive estimation of causal periodic autoregressive model, Communications in Statistics Simulation and Computation 38 (2009), 1592-1609.

[7] A. Bibi and A. Gautier, Stationnarité et inférence asymptotique de modèles bilinéaires périodiques, Comptes rendus de l' Académie des Sciences – Mathématique 341 (2005), 679-682.

[8] P. J. Bickel, On adaptive estimation, Ann. Statist. Math. 10 (1982), 647-671.

[9] T. Bollerslev and E. Ghysels, Periodic autoregressive conditional heteroskedasticity, Journal of Business and Economic Statistics 14 (1996), 139-152.

[10] K. S. Chan and H. Tong, On the use of the deterministic Lyapunov function for the ergodicity of stochastic difference equations, Advances in Applied Probability 17(3) (1985), 666-678.

[11] F. C. Drost and C. A. J. Klaassen, Efficient estimation in semiparametric GARCH models, Journal of Econometrics 81 (1997), 193-221.

[12] F. C. Drost, C. A. J. Klaassen and B. J. M. Werker, Adaptive estimation in time series models, Annals of Statistics 25 (1997), 786-818.

[13] V. Fabian and J. Hannan, On estimation and adaptive estimation for locally asymptotically normal families, Z. Wahrsh. Verw. Gebiete 59 (1982), 459-478.

[14] H. Ghosh, B. Gurung and P. Gupta, Fitting EXPAR models through the extended Kalman filter, Sankhya: The Indian Journal of Statistics 77(1) (2015), 27-44.

[15] E. G. Gladyshev, Periodically correlated random sequences, Soviet. Math. 2 (1961), 385-388.

[16] V. Haggan and T. Ozaki, Modeling nonlinear random vibrations using an amplitude-dependent autoregressive time series model, Biometrik 68(1) (1981), 96-189.

[17] J. Hájek, Local Asymptotic Minimax and Admissibility in Estimation, Proc. Sixth Berkeley Symp. Math. Stat. Proba. 1 175-194. Univ. California Press, 1972.

[18] Hájek and Šidák, Theory of Rank Tests, Academic Press, New York, 1967.

[19] M. Hallin and M. L. Puri, Aligned rank tests for linear models with autocorrelated error termes, Journal of Multivariaye Analysis 50 (1994), 175-237.

[20] K. Ishizuka, H. Kato and T. Nakatani, Speech signal analysis with exponential autoregressive model, Proc. the 30th International Conference of Acoustics, Speech and Signal Processing 1 (2005), 225-228.

[21] M. A. Ismail, Bayesian analysis of exponential AR models, The Far East Journal of Statistics 5 (2001), 1-15 (India).

[22] P. Katsiampa, A new approach to modelling nonlinear time series: Introducing the ExpAR-ARCH and ExpAR-GARCH models and applications, 4th Student Conference on Operational Research P. (2014), 34-51.

[23] H. L. Koul and A. Schick, Efficient estimation in nonlinear autoregressive time series models, Bernoulli 3 (1997), 247-277.

[24] J. P. Kreiss, On adaptive estimation in stationary ARMA processes, The Annals of Statistics 15(1) (1987), 112-133.

[25] L. Le Cam, Locally asymptotically normal families of distributions, Univ. California Publ. Statist. 3 (1960), 37-98.

[26] S. Lee and M. Taniguchi, Asymptotic theory for ARCH-SM models: LAN and residual empirical processes, Statistica Sinica 15 (2005), 215-234.

[27] S. Ling, Adaptive estimators and tests of stationary and nonstationary short and long-memory ARFIMA-GARCH models, Journal of the American Statistical Association 98(464) (2003), 955-967.

[28] S. Ling and M. McAleer, Adaptative estimation in nonstationary ARMA models with GARCH errors, Annals of Statistics 31 (2003), 642-674.

[29] O. Linton, Adaptive estimation in ARCH models, Econometric Theory 9 (1993), 539-569.

[30] M. Merzougui, H. Dridi and A. Chadli, Test for periodicity in restrictive EXPAR models, Communications in Statistics: Theory and Methods 45(9) (2016), 2770-2783.

[31] T. Ozaki, Non-linear time series models for non-linear random vibrations, Journal of Applied Probability 17 (1980), 84-93.

[32] T. Ozaki, Nonlinear time series models and dynamical system, Handbook of Statistics 5 (1985), 25-83.

[33] M. B. Priestley, Non-linear and Non-stationary Time Series Analysis, Academic Press, New York, 1988.

[34] Q. Shao, Mixture periodic autoregressive time series model, Statistics and Probability Letters 76(6) (2006), 609-618.

[35] Z. Shi and H. Aoyama, Estimation of exponential autoregressive time series model by using genetic algorithm, Journal of Sound and Vibration 205(3) (1997), 309-321.

[36] Z. Shi, Y. Tamura and T. Ozaki, Monitoring the stability of BWR oscillation by nonlinear time series modeling, Annals of Nuclear Energy 28 (2001), 953-966.

[37] C. Stein, Efficient nonparametric estimation and testing, In J. Neyman (Editor), Proceedings of the Third Berkely Symposium on Mathematical Statistics and Probability, Berkely, University of California Press 1 (1956), 187-195.

[38] A. R. Swensen, The asymptotic distribution of the likelihood ratio for autoregressive time series with a regression trend, Journal of Multivariate Analysis 16 (1985), 54-70.

[39] N. Terui and H. K. Van Dijk, Combined Forecasts from Linear and Nonlinear Time Series Models, Econometric Institute Report EI-9949/A. (1999).

■

Appendix

Proof of Proposition 3.1

Proof of condition (1)

Replacing $\xi^{(n)}_{s+Sr}\left(\underline{\varphi}_s\right)$ and $\zeta^{(n)}_{s+Sr}\left(\underline{\varphi}_s\right)$ by their values in the expectation expression, given in (3.1), we obtain

$$E\sum_{s=1}^{S}\sum_{r=0}^{m-1}\left(\xi^{(n)}_{s+Sr}\left(\underline{\varphi}_s\right)-\zeta^{(n)}_{s+Sr}\left(\underline{\varphi}_s\right)\right)^2 = \sum_{s=1}^{S}\sum_{r=0}^{m-1}E\left(f^{1/2}_{\sigma_s}\left(Z^{(n)}_{s,r}\left(\underline{\varphi}_s\right)-\gamma^{(n)}_{s,r}\right)-f^{1/2}_{\sigma_s}\left(Z^{(n)}_{s,r}\left(\underline{\varphi}_s\right)\right)\right.$$

$$\left.-\frac{1}{2}\left(-\gamma^{(n)}_{s,r}\right)\frac{\dot{f}_{\sigma_s}\left(Z^{(n)}_{s,r}\left(\underline{\varphi}_s\right)\right)}{f^{1/2}_{\sigma_s}\left(Z^{(n)}_{s,r}\left(\underline{\varphi}_s\right)\right)}\right)^2 f^{-1}_{\sigma_s}\left(Z^{(n)}_{s,r}\left(\underline{\varphi}_s\right)\right)\Bigg),$$

$$= E\left(\sum_{s=1}^{S}(B_{1,s,m}+B_{2,s,m})\right),$$

where

$$B_{1,s,m} = \sum_{r=0}^{m-1}E$$

$$\left\{I_{[|X_{s-1+Sr}|<K]}\left(f^{1/2}_{\sigma_s}\left(Z^{(n)}_{s,r}\left(\underline{\varphi}_s\right)-\gamma^{(n)}_{s,r}\right)-f^{1/2}_{\sigma_s}\left(Z^{(n)}_{s,r}\left(\underline{\varphi}_s\right)\right)\right.\right.$$

$$\left.\left.-\frac{1}{2}\left(-\gamma^{(n)}_{s,r}\right)\frac{\dot{f}_{\sigma_s}\left(Z^{(n)}_{s,r}\left(\underline{\varphi}_s\right)\right)}{f^{1/2}_{\sigma_s}\left(Z^{(n)}_{s,r}\left(\underline{\varphi}_s\right)\right)}\right)^2\times f^{-1}_{\sigma_s}\left(Z^{(n)}_{s,r}\left(\underline{\varphi}_s\right)\right)\right\},$$

and

$$B_{2,s,m} = \sum_{r=0}^{m-1} E$$

$$\left\{ I_{[|X_{s-1+Sr}| \geqslant K]} \left(f_{\sigma_s}^{1/2}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right) - \gamma_{s,r}^{(n)}\right) - f_{\sigma_s}^{1/2}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right) \right.\right.$$

$$\left.\left. - \frac{1}{2}\left(-\gamma_{s,r}^{(n)}\right) \frac{\dot{f}_{\sigma}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right)}{f_{\sigma}^{1/2}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right)} \right)^2 \times f_{\sigma_s}^{-1}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right) \right\},$$

it is enough to show that $B_{1,s,m}$ and $B_{2,s,m}$ converge to 0, for s fixed, $s = 1, \ldots, S$ and for any positive real number K.

$$B_{1,s,m} = \sum_{r=0}^{m-1} \int_{|X_{s-1+Sr}|<K} \gamma_{s,r}^{(n)2} \int_{|X_{s-1+Sr}|<K}$$

$$\left(\frac{f_{\sigma_s}^{1/2}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right) - \gamma_{s,r}^{(n)}\right) - f_{\sigma_s}^{1/2}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right)}{-\gamma_{s,r}^{(n)}} - \frac{1}{2} \frac{\dot{f}_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right)}{f_{\sigma_s}^{1/2}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right)} \right)^2 dZ_{s,r}^{(n)}\left(\underline{\varphi}_s\right) dG_X,$$

where G_X is the distribution of random variable X_{s-1+Sr}. In order to use the Lemma 2 of Swensen [38], let $u = -\underline{\tau}_s^{(n)'} \underline{X}_{s+rS-1}^{(n)}$ and $y = Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)$, we have

$$B_{1,s,m} \leq \sum_{r=0}^{m-1} \int_{\mathbb{R}} \gamma_{s,r}^{(n)2} \int_{|X_{s-1+Sr}|<K} \left(\frac{\sqrt{f_{\sigma_s}\left(y + \frac{u}{\sqrt{n}}\right)} - \sqrt{f_{\sigma_s}(y)}}{\frac{u}{\sqrt{n}}} - \frac{\dot{f}_{\sigma_s}(y)}{2\sqrt{f_{\sigma_s}(y)}} \right)^2 dy dG_X$$

$$\leq C_m(K) \sum_{r=0}^{m-1} E\left(\gamma_{s,r}^{(n)}\right)^2,$$

where

$$C_m(K) = \int_{|u|<b(K)} \left(\frac{\sqrt{f_{\sigma_s}\left(y + \frac{u}{\sqrt{n}}\right)} - \sqrt{f_{\sigma_s}(y)}}{\frac{u}{\sqrt{n}}} - \frac{\dot{f}_{\sigma_s}(y)}{2\sqrt{f_{\sigma_s}(y)}} \right)^2 dy,$$

and $b(K) = K\left|h_{s,1}^{(n)} + h_{s,2}^{(n)}\right|$.

Since $\sum_{r=0}^{m-1} E\left(\gamma_{s,r}^{(n)}\right)^2$ is uniformly bounded (proved in condition (2) below) and $\lim_{m\to\infty} C_m(K) = 0$ (Lemma 2 of Swensen [38]), then $B_{1,s,m}$ converge to 0 as $m \to \infty$.

For $B_{2,s,m}$ and using the second part of Lemma 2 of Swensen [38], we have

$$B_{2,s,m} \leq \sum_{r=0}^{m-1} \int_{|X_{s-1+Sr}|\geqslant K} \int_{\mathbb{R}} \left(\sqrt{f_{\sigma_s}\left(y - \gamma_{s,r}^{(n)}\right)} - \sqrt{f_{\sigma_s}(y)} - \frac{1}{2}\gamma_{s,r}^{(n)} \phi(y) \sqrt{f_{\sigma_s}(y)} \right)^2 dy dG_X$$

$$\leq \sum_{r=0}^{m-1} \int_{|X_{s-1+Sr}|\geqslant K} \gamma_{s,r}^{(n)2} I(f_{\sigma_s}) dG_X$$

$$= I(f_{\sigma_s}) \sum_{r=0}^{m-1} E\left(I_{[|X_{s-1+Sr}|\geqslant K]} \gamma_{s,r}^{(n)2} \right).$$

Since the processes $\{X_{s-1+Sr},\, r \in \mathbb{Z}\}$ are stationary $B_{2,s,m}$ can be made small uniformly in m by choosing K large enough.

Proof of condition (2)

It is sufficient to show that, for fixed s, $\sup_m E \sum_{r=0}^{m-1} \left(\zeta_{s+Sr}^{(n)2}\left(\underline{\varphi}_s\right) \right) < \infty.$

Using the definition of $\zeta_{s+Sr}^{(n)}\left(\underline{\varphi}_s\right)$, we obtain

$$E \sum_{r=0}^{m-1} \left(\zeta_{s+Sr}^{(n)2}\left(\underline{\varphi}_s\right) \right) = \frac{I(f_{\sigma_s})}{4} \sum_{r=0}^{m-1} E\left(\gamma_{s,r}^{(n)}\right)^2, \text{ for a fixed } s.$$

We must show, for s fixed, that the sum $\sum_{r=0}^{m-1} E\left(\gamma_{s,r}^{(n)}\right)^2$ is uniformly bounded

$$\sum_{r=0}^{m-1} E\left(\gamma_{s,r}^{(n)}\right)^2 = \frac{1}{n}\sum_{r=0}^{m-1} E\left(h_{s,1}^{(n)} X_{s+rS-1} + h_{s,2}^{(n)} \exp\left(-\gamma X_{s+rS-1}^2\right) X_{s+rS-1}\right)^2$$

$$\leq \frac{2}{n}\left(\sup_m \left(h_{s,1}^{(n)2}\right)\sum_{r=0}^{m-1} E\left(X_{s+rS-1}^2\right) + \sup_m \left(h_{s,2}^{(n)2}\right)\sum_{r=0}^{m-1} E\left(\left(e^{-\gamma X_{s+rS-1}^2} X_{s+rS-1}\right)^2\right)\right)$$

$$= \frac{2}{S}\left(\sup_m \left(h_{s,1}^{(n)2}\right) E\left(X_{s-1}^2\right) + \sup_m \left(h_{s,1}^{(n)2}\right) E\left(e^{-2\gamma X_{s-1}^2} X_{s-1}^2\right)\right).$$

It follows from the fact that $\sup_m \left(h_{s,1}^{(n)2} + h_{s,2}^{(n)2}\right) < \infty$ and the stationarity of X_{s-1+Sr} that $\sum_{r=0}^{m-1} E\left(\gamma_{s,r}^{(n)}\right)^2$ is uniformly bounded in m.

Proof of condition (3)

We show that $\max_r \left|\zeta_{s+Sr}^{(n)}\left(\underline{\varphi}_s\right)\right| = \max_r \left|\frac{1}{2}\phi_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right)\left(\underline{\tau}_s^{(n)'}\, \underline{X}_{s+rS-1}^{(n)}\right)\right| = 0_P(1),$

for any s. Thus, we need to prove that

$$\max_r \frac{1}{\sqrt{n}}\left|\phi_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right) h_{s,1}^{(n)} X_{s+rS-1}^{(n)}\right| = 0_P(1), \text{ under } H_f^{(n)}\left(\underline{\varphi}\right),$$

and

$$\max_r \frac{1}{\sqrt{n}}\left|\phi_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right) h_{s,2}^{(n)} X_{s+rS-1}^{(n)} \exp\left(-\gamma X_{s+rS-1}^2\right)\right| = 0_P(1), \text{ under } H_f^{(n)}\left(\underline{\varphi}\right).$$

Indeed, we have

$$p\left(\max_r \frac{1}{\sqrt{n}}\left|\phi_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right) h_{s,1}^{(n)} X_{s+rS-1}^{(n)}\right| > \varepsilon\right)$$

$$\leq \sum_{r=0}^{m-1} p\left(\frac{1}{\sqrt{n}}\left|\phi_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right) h_{s,1}^{(n)} X_{s+rS-1}^{(n)}\right| > \varepsilon\right),$$

$$\leq \frac{1}{n\varepsilon^2}\sum_{r=0}^{m-1} E\left\{\phi_{\sigma_s}^2\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right)\left(h_{s,1}^{(n)}X_{s+rS-1}^{(n)}\right)^2 I_{\left[\left|\frac{1}{\sqrt{n}}\phi_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right)h_{s,1}^{(n)}X_{s+rS-1}^{(n)}\right|>\varepsilon\right]}\right\},$$

$$\leq \frac{1}{S\varepsilon^2} E\left(\phi_{\sigma_s}^2\left(Z_{s,0}^{(n)}\left(\underline{\varphi}_s\right)\right)\left(h_{s,1}^{(n)}X_{s-1}^{(n)}\right)^2 I_{\left[\left|\phi_{\sigma_s}\left(Z_{s,0}^{(n)}\left(\underline{\varphi}_s\right)\right)h_{s,1}^{(n)}X_{s-1}^{(n)}\right|>\sqrt{n\varepsilon}\right]}\right),$$

thus the right hand side converges to 0, as $n \to \infty$. In a same manner, we can show that the second expression also converges to zero.

Proof of condition (4)

We have

$$\sum_{s=1}^{S}\sum_{r=0}^{m-1}\left(\zeta_{s+Sr}^{(n)2}\left(\underline{\varphi}_s\right)\right) = \frac{1}{4n}\sum_{s=1}^{S}\sum_{r=0}^{m-1}\phi_{\sigma_s}^2\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right)\underline{\tau}_s^{(n)'}\underline{X}_{Sr+s-1}\underline{X}'_{Sr+s-1}\underline{\tau}_s^{(n)},$$

using the fact that the process is ergodic, we have

$$\frac{1}{m}\sum_{r=0}^{m-1}\phi_{\sigma_s}^2\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s\right)\right)\underline{X}_{Sr+s-1}\underline{X}'_{Sr+s-1} \to E(\phi_{\sigma_s}^2)E\left(\underline{X}_{s-1}\underline{X}'_{s-1}\right),$$

where $E(\phi_{\sigma_s}^2) = I(f_{\sigma_s})$ with $I(f_{\sigma_s}) = \frac{1}{\sigma_s^2} I(f_1)$.

$$E\left(X_{s-1}X'_{s-1}\right) = \Gamma_s\left(\underline{\varphi}, \underline{\sigma}\right) = \begin{pmatrix} E\left(X_{s-1}^2\right) & E\left(e^{-\gamma X_{s-1}^2}X_{s-1}^2\right) \\ E\left(e^{-\gamma X_{s-1}^2}X_{s-1}^2\right) & E\left(e^{-2\gamma X_{s-1}^2}X_{s-1}^2\right)\end{pmatrix}.$$

Thus, we have

$$\sum_{s=1}^{S}\sum_{r=0}^{m-1}\left(\zeta_{s+Sr}^{(n)2}\left(\underline{\varphi}_s\right)\right) = \frac{1}{4S}\sum_{s=1}^{S}\underline{\tau}_s^{(n)'} I(f_1)\left(\frac{1}{\sigma_s^2}\Gamma_s\left(\underline{\varphi}, \underline{\sigma}\right)\right)\underline{\tau}_s^{(n)} + 0_P(1)$$

$$= \frac{I(f_1)}{4S}\underline{\tau}^{(n)'}\Gamma\left(\underline{\varphi}, \underline{\sigma}\right)\underline{\tau}^{(n)} + 0_P(1),$$

where $\Gamma\left(\underline{\varphi}, \underline{\sigma}\right)$ is given by (3.2).

Proofs of conditions (5) and (6) are similar to those of Swensen's [38], hence they are omitted.

Proof of Proposition 4.1

We will show that $\underline{\hat{\varphi}}_n$ is regular

$$\sqrt{n}\left(\underline{\hat{\varphi}}_n - \underline{\varphi}\right) - \frac{S}{I(f_1)}\left(\Gamma\left(\underline{\varphi}, \underline{\sigma}\right)\right)^{-1} \Delta^{(n)}\left(\underline{\varphi}\right)$$

$$= \sqrt{n}\left(\overline{\varphi}_n - \underline{\varphi}\right) + \left[\left(\frac{I(f_1)}{S}\hat{\Gamma}(\overline{\varphi}_n, \underline{\sigma})\right)^{-1} - \frac{S}{I(f_1)}\left(\Gamma\left(\underline{\varphi}, \underline{\sigma}\right)\right)^{-1}\right]\Delta_f^{(n)}(\overline{\varphi}_n)$$

$$+ \frac{S}{I(f_1)}\left(\Gamma\left(\underline{\varphi}, \underline{\sigma}\right)\right)^{-1}\left(\Delta_f^{(n)}(\overline{\varphi}_n) - \Delta^{(n)}\left(\underline{\varphi}\right)\right)$$

$$= \sqrt{n}\left(\overline{\varphi}_n - \underline{\varphi}\right) + \frac{S}{I(f_1)}\left(\Gamma\left(\underline{\varphi}, \underline{\sigma}\right)\right)^{-1}\left(\Delta_f^{(n)}(\overline{\varphi}_n) - \Delta^{(n)}\left(\underline{\varphi}\right)\right) + 0_P(1),$$

because $\Delta_f^{(n)}(\overline{\varphi}_n) = O_P(1)$ and $\hat{\Gamma}_n\left(\underline{\varphi}, \underline{\sigma}\right) = \Gamma\left(\underline{\varphi}, \underline{\sigma}\right) + 0_P(1)$. On the other side, from Proposition 3.2,

$$\sqrt{n}\left(\underline{\hat{\varphi}}_n - \underline{\varphi}\right) - \frac{S}{I(f_1)}\left(\Gamma\left(\underline{\varphi}, \underline{\sigma}\right)\right)^{-1} \Delta^{(n)}\left(\underline{\varphi}\right)$$

$$= \frac{S}{I(f_1)}\left(\Gamma\left(\underline{\varphi}, \underline{\sigma}\right)\right)^{-1}\left[\frac{I(f_1)}{S}\Gamma\left(\underline{\varphi}, \underline{\sigma}\right)\sqrt{n}\left(\overline{\varphi}_n - \underline{\varphi}\right) + \Delta_f^{(n)}(\overline{\varphi}_n) - \Delta^{(n)}\left(\underline{\varphi}\right)\right] + 0_P(1)$$

$$= 0_P(1).$$

Proof of Lemma 5.1

By Lemma 4.4 of Kreiss [24], it suffices to verify that $\tilde{\Delta}^{(n)}\left(\underline{\varphi}^{(n)}\right) - \Delta^{(n)}\left(\underline{\varphi}^{(n)}\right)$ converges in quadratic mean to 0 for $\underline{\varphi}^{(n)} = \underline{\varphi} + \frac{1}{\sqrt{n}}\underline{\tau}^{(n)}$. For s fixed, we have

$$E\left\|\tilde{\delta}_s^{(n)}\left(\underline{\varphi}_s^{(n)}\right)-\delta_s^{(n)}\left(\underline{\varphi}_s^{(n)}\right)\right\|^2$$

$$=\frac{1}{n}\sum_{r=0}^{m-1}E\left(\left(\widehat{q}_{n,r}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s^{(n)}\right)\right)-\phi_{\sigma_s}\left(Z_{s,r}^{(n)}\left(\underline{\varphi}_s^{(n)}\right)\right)\right)^2\|X(s-1+Sr)\|^2\right)$$

$$=\frac{1}{n}\sum_{r=0}^{m-1}E\left(\|X(s-1+Sr)\|^2\int_{\mathbb{R}}\left(\widehat{q}_{n,r}(x)-\phi_{\sigma_s}(x)\right)^2 f_{\sigma_s}(x)\,dx\right),$$

with $x=Z_{s,r}^{(n)}\left(\underline{\varphi}_s^{(n)}\right)$. As in Bickel ([8], p. 667), we obtain

$$\int_{\mathbb{R}}\left(\frac{\widehat{f}'_{\eta(n),r}\left(x,\underline{\varphi}_s^{(n)}\right)}{\widehat{f}_{\eta(n),r}\left(x,\underline{\varphi}_s^{(n)}\right)}-\frac{f'_{\sigma_s}(x)}{f_{\sigma_s}(x)}\right)^2 f_{\sigma_s}(x)\,dx$$

$$\leq 3\int_{\mathbb{R}}\left(\frac{\widehat{f}'_{\eta(n),r}\left(x,\underline{\varphi}_s^{(n)}\right)}{\widehat{f}_{\eta(n),r}\left(x,\underline{\varphi}_s^{(n)}\right)}-\frac{\widehat{f}'_{\eta(n),r}\left(x,\underline{\varphi}_s^{(n)}\right)}{\widehat{f}_{\eta(n),r}\left(x,\underline{\varphi}_s^{(n)}\right)}\left(\frac{f_{\eta(n)}(x)}{f_{\sigma_s}(x)}\right)^{1/2}\right)^2 f_{\sigma_s}dx$$

$$+\;3\int_{\mathbb{R}}\left(\frac{\widehat{f}'_{\eta(n),r}\left(x,\underline{\varphi}_s^{(n)}\right)}{\widehat{f}_{\eta(n),r}\left(x,\underline{\varphi}_s^{(n)}\right)}\left(\frac{f_{\eta(n)}(x)}{f_{\sigma_s}(x)}\right)^{1/2}-\frac{f'_{\eta(n)}(x)}{f_{\eta(n)}(x)}\left(\frac{f_{\eta(n)}(x)}{f_{\sigma_s}(x)}\right)^{1/2}\right)^2 f_{\sigma_s}dx$$

$$+\;3\int_{\mathbb{R}}\left(\frac{f'_{\eta(n)}(x)}{f_{\eta(n)}(x)}\left(\frac{f_{\eta(n)}(x)}{f_{\sigma_s}(x)}\right)^{1/2}-\frac{f'_{\sigma_s}(x)}{f_{\sigma_s}(x)}\right)^2 f_{\sigma_s}dx.$$

Then

$$E\left\|\tilde{\delta}_s^{(n)}\left(\underline{\varphi}_s^{(n)}\right)-\delta_s^{(n)}\left(\underline{\varphi}_s^{(n)}\right)\right\|^2$$

$$\leq \frac{3}{n}\sum_{r=0}^{m-1} E\left(\|X(s-1+Sr)\|^2 \int_{\mathbb{R}} \widehat{q}_{n,r}^{\,2}\left(\sqrt{f_{\eta(n)}}-\sqrt{f_{\sigma_s}}\right)^2 dx\right)$$

$$+\frac{3}{n}\sum_{r=0}^{m-1} E\left(\|X(s-1+Sr)\|^2 \int_{\mathbb{R}} \left(\widehat{q}_{n,r}+\frac{f'_{\eta(n)}}{f_{\eta(n)}}\right)^2 f_{\eta(n)}dx\right)$$

$$+\frac{3}{n}\sum_{r=0}^{m-1} E\left(\|X(s-1+Sr)\|^2 \int_{\mathbb{R}} \left(\frac{f'_{\eta(n)}}{\sqrt{f_{\eta(n)}}}-\frac{f'_{\sigma_s}}{\sqrt{f_{\sigma_s}}}\right)^2 dx\right).$$

For the remaining proof one can use Lemmas 6.5-6.9 of Kreiss [24] which ensure that each term of the right hand side converges to zero as $n \to \infty$, exactly in the same manner.