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HANKEL OPERATORS ON COPSON SPACES

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Abstract

We investigate the necessary and sufficient conditions in order that a Hankel operator on the space H_d^2, isomorphic to *Copson space* $\text{cop}(2)$, belongs to some operator ideals such as that of all linear bounded operators, of all compact operators, of all nuclear operators etc.

1. Introduction

In [10], we introduced the scale of spaces

$$H_d^p := Sp(\mathcal{M}_d^+) = \mathcal{M}_d^+ \cap H^p - \mathcal{M}_d^+ \cap H^p, 1 \le p < \infty,$$

where $Sp(\mathcal{M}_d^+)$ is the space generated by $\mathcal{M}_d^+$. These spaces are Banach lattices with respect to the cone $\mathcal{M}_d^+ \cap H^p$, where

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$$\mathcal{M}_d^+ = \{f(z) = \sum_{k=1}^{\infty} a_k z^k;\ |z| < 1,\ f \text{ analytic, and } a_k \downarrow_k 0\},$$

of all analytic functions from the classical Hardy spaces H^p, $1 \leq p < \infty$, with a decreasing sequence of positive Taylor coefficients. In that paper we proved that the Hankel operators with a symbol having also a decreasing sequence of positive Taylor coefficients act boundedly on H_d^2 if and only if the symbol belongs to the $BMOA_d$, the subspace of the classical Banach space $BMOA$ as defined in [12], generated by the cone $\mathcal{M}_d^+ \cap BMOA$.

It is perhaps of interest to remark that H_d^2 is topologically isomorphic to the classical *Copson space* $\text{cop}(2)$, as defined in [2].

This paper is a continuation of [10], and we intend to give necessary and sufficient conditions in order that a Hankel operator as above belongs to different operator ideals like the ideal of compact operators, of nuclear operators, or to the ideal N_2, as defined in [8].

2. Solid Spaces of Sequences

First we recall the definition of the smallest solid superset $S(X)$, of a Banach space of sequences X. (See [1], [4].)

$$S(X) := \{a = (a_n)_{n \geq 0} : \exists b \in X \text{ such that } |b_n| \geq |a_n| \text{ for all } n\}.$$

The norm on $S(X)$ is the following (see [4]):

$$\|a\|_{S(X)} := \inf\{\|b\|_X : \text{for all } b \in X \text{ such that } |b_n| \geq |a_n| \forall n\}.$$

We use also the space $d(a, p)$ introduced by Bennett in [2]:

$$d(a, p) := \{x = (x_n) : \text{such that } \|x\|_{d(a, p)} := \left(\sum_{k=1}^{\infty} a_k \sup_{n \geq k} |x_n|^p\right)^{1/p} < \infty\},$$

where $a = (a_1, a_2, \dots)$, $a_n \geq 0$, and $1 \leq p < \infty$. In the particular case $a_n = 1$, $\forall n$, we denote $d(a, p)$ simply by $d(p)$.

Another space of interest introduced in [2] is $g(a, p) := \{x : \sum_{k=1}^{n} |x_k|^p = \mathcal{O}(A_n)\}$. Here $a = (a_1, a_2, \dots)$, where $A_n = a_1 + a_2 + \cdots + a_n$.

Of course, the norm of $g(a, p)$ is given by

$$\|x\|_{g(a, p)} = \sup_n \left(\frac{1}{A_n} \sum_{k=1}^{n} |x_k|^p\right)^{1/p}.$$

The following result is valid for all $0 < p < \infty$:

Theorem 2.1. *Let* $0 < p < \infty$. *Then* $S(H_d^p) = d(a, p)$, *where* $a_n = n^{p-2}$, $\forall n$, *with equivalent norms.*

Proof. Let $x = (x_n)_n \in d(a, p)$, and $y = (y_n)_n \in H_d^p$. Then $\sum_n n^{p-2} \sup_{k \geq n} |x_k|^p = \|x\|_{d(a, p)}^p < \infty$, and, for $y_n := \sup_{k \geq n} |x_k|$, for all n, we have $y_n \downarrow_n 0, |x_n| \leq |y_n| \,\forall n$, and $y \in H_d^p \cap \mathcal{M}_d^+$, consequently $x \in S(H_d^p)$.

In other words

$$d(a, p) \subset S(H_d^p),$$

and, moreover, $\|x\|_{S(H_d^p)} \leq \|y\|_{H_d^p} = \|x\|_{d(a, p)}$.

Since by using Theorem 6.1 -[4], $S(H_d^p)$ is the smallest Banach lattice containing H_d^p, and $d(a, p)$ is a Banach lattice containing H_d^p, then

$$S(H_d^p) \subset d(a, p).$$

Thus

$$S(H_d^p) = d(a, p),$$

and their corresponding norms are equivalent by closed graph theorem and by inequality $\|x\|_{S(H_d^p)} \leq \|x\|_{d(a, p)}$. □

Also the study of the largest solid subset of a space X, denoted by $s(X)$, where X is a Banach space of sequences, is of interest as showed in [1].

In what follows we describe the largest solid subspace of the topological dual of H_d^p, $1 \leq p < \infty$.

We denote by A^* the topological dual of the Banach space of sequences A.

First we prove a lemma:

Lemma 2.2. *The space of all coefficient multipliers from* ℓ^∞ *into* $(H_d^p)^*$, *denoted by* $(\ell^\infty, (H_d^p)^*)$, *coïncides with the corresponding space* (H_d^p, ℓ^1).

Proof. Let $m = (m_n)_n$ be a multiplier belonging to $(\ell^\infty, (H_d^p)^*)$, that is $(m_n a_n)_n \in (H_d^p)^* \forall (a_n)_n \in \ell^\infty$.

Let $(b_n)_n \in H_d^p$. The canonical sequence $(e_n)_n$ is a non normalized Schauder basis of H_d^p, $1 \leq p < \infty$, by [10]. Consequently $\sum_n m_n a_n b_n$ is a convergent series $\forall$ $(b_n)_n \in H_d^p$, and $\forall$ $(a_n)_n \in \ell^\infty$.

Thus fix an arbitrary $(b_n)_n \in H_d^p$, and take $a_n = \operatorname{sign}(b_n m_n)$, $\forall n$. Then we have that $\sum_n |m_n||b_n| < \infty$, $\forall (b_n)_n \in H_d^p$, that is $m \in (H_d^p, \ell^1)$.

In other words $(\ell^\infty, (H_d^p)^*) \subset (H_d^p, \ell^1)$.

Conversely, let $m \in (H_d^p, \ell^1)$, and $a = (a_n) \in \ell^\infty$. Then, for $b = (b_n)_n \in H_d^p$, we have:

$$(m_n b_n)_n \in \ell^1 \text{ implies } \sum_n |m_n||b_n| < \infty \text{ thus } \sum_n |m_n||a_n||b_n| < \infty,$$

$\forall a \in \ell^\infty$, $b \in H_d^p$, which in turn implies $(m_n a_n)_n \in (H_d^p)^* \forall (a_n)_n \in \ell^\infty$. It follows

$$m \in (\ell^\infty, (H_d^p)^*).$$

□

Theorem 2.3. *$s((H_d^1)^*) = g(a, 1)$, where $a_n = (\frac{1}{n})_n$, and $g(a, 1) = \{x = (x_n); \|x\|_{g(a,1)} = \sup_{n \geq 0} \frac{\sum_{k=1}^n |x_k|}{\ln(n+1)} < \infty\}$, with the equivalent norms.*

Proof. It is easy to see that $s((H_d^1)^*) = (\ell^\infty, (H_d^1)^*)$. (See [1].)

By using the previous lemma and [4] we have that $s((H_d^1)^*) = (H_d^1, \ell^1)$ $= (S(H_d^1), \ell^1)$.

Now by Theorem 2.1 and by Theorem 3.8 - [2], it follows that $((H_d^1)^*) = (d(a, 1), \ell^1) = g(a, 1)$. □

In the case $1 < p < \infty$, we have:

Theorem 2.4. *Let* $1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$s((H_d^p)^*) = \ell^q \cdot g(a, p),$$

where $\ell^q \cdot g(a, p)$ *means the space of all sequences* $x \cdot y$, *with* $x \in \ell^q$, $y \in g(a, p)$.

Proof. Similarly with the proof of Theorem 2.3 we have

$$s((H_d^p)^*) = (d(a, p), \ell^1) = d(a, p)^\times,$$

where the last term is the Köthe dual of $d(a, p)$.

But, by using Theorem 12.3 -[2], we get the result. □

First recall that, for $0 < p < \infty$,

$$\operatorname{cop}(p) := \left\{ x = (x_k)_k : \sum_{k=0}^{\infty} \left(\sum_{k=n}^{\infty} \frac{|x_k|}{k+1} \right)^p < \infty \right\}, \text{ with the quasi-norm}$$

$$\|x\|_{\operatorname{cop}(p)} = \left(\sum_{n=0}^{\infty} \left(\sum_{k=n}^{\infty} \frac{|x_k|}{k+1} \right)^p \right)^{1/p}.$$

We recall that by the space bv we mean the Banach space of sequences of real numbers $a = (a_n)_{n \geq 0}$ with bounded variation $\|a\|_{bv_0} := |\alpha| + \sum_{n=0}^{\infty} |a_n - a_{n+1}|,$ where $\lim_n a_n = \alpha.$

It is well-known and easy to prove that bv is a vector lattice for the order induced by the cone $C := \{a = (a_n)_{n \geq 0}, a_n \downarrow_n \alpha \geq 0\}.$

For the convenience of the reader we sketch the argument of this: First the cone C is a lattice cone.

For instance if $a, b \in C,$ then $\sup(a, b) := a \vee_C b \in C$ is given by

$$a \vee_C b := (c_n)_n, \text{ where } c_n = \sum_{k=n}^{\infty} (a_k - a_{k+1}) \vee (b_k - b_{k+1}), \ \forall k \geq 0.$$

It follows, by using [11] - Chapter V - Proposition 1.2, that bv is a vector lattice with respect to the order induced by the cone C.

We recall that the modulus of $a \in bv,$ denoted by $|a|_{bv},$ is defined by

$$(|a|_{bv})_n := |\alpha| + \sum_{k=n}^{\infty} |a_k - a_{k+1}|, \ \forall n \geq 0.$$

If $\alpha = 0,$ for all sequences from $bv,$ the corresponding space is denoted by $bv_0,$ and the latter is also vector lattice.

We mention that bv is also a Banach lattice, but this is not important for this paper.

Then we have:

Theorem 2.5. *Let* $0 < p < \infty,$ $p \neq 2.$ *Then* H_d^p *is linear-topological isomorphic* (*but not latticially isomorphic*) *to the Copson space of order* p, $\mathrm{cop}(p)$.

The isomorphism $T : H_d^p \to \mathrm{cop}(p)$ *is given, for* $x = (x_n)_{n\geq 0} \in H_d^p$, *by* $T(x) := u = (u_k)_{k\geq 0} \in \mathrm{cop}(p)$, *in the following way*:

$$u_k = (k+1)\Big[(k+1)^{1-2/p} x_k - (k+2)^{1-2/p} x_{k+1}\Big],\ k \geq 0.$$

$T^{-1} : \mathrm{cop}(p) \to H_d^p$ *is given by*

$$x_n = (n+1)^{2/p-1} \sum_{k=n}^{\infty} \frac{u_k}{k+1},\ n \geq 0.$$

Moreover,

$$\|x\|_{H_d^p} = \left(\sum_{n=0}^{\infty} (n+1)^{p-2} (|x_n|_{bv_0})_n^p \right)^{1/p} = \|T(x)\|_{\mathrm{cop}(p)},\ \forall x \in H_d^p.$$

Proof. Obvious by inspection. □

Corollary 2.6. *Let* $1 < p < \infty$, $p \neq 1$. *Then* $(H_d^p)^*$ *is linear-topologically* (*but not latticially*) *isomorphic to* $d(p^*)^{\times}$, *where* $\frac{1}{p^*} + \frac{1}{p} = 1$.

Proof. By using Theorem 6.8 and Corollary of Theorem 12.17 - [2], we get that $\mathrm{cop}(p)^* = d(p^*)$. □

The particular case $p = 2$ is of interest view the fact that Hankel operators act nicely on H_d^2, as we see in the next section.

Corollary 2.7. (1) $H_d^2 \overset{T}{\longleftrightarrow} \mathrm{cop}(2)$ *by the latticial isometry*: $T(x) = u$, *where* $u_k = (k+1)[x_k - x_{k+1}]$, $k \geq 0$, *and*,

$$T^{-1}(u) = x,\ x_n = \sum_{k=n}^{\infty} \frac{u_k}{k+1},\ n \geq 0.$$

Moreover T^* *maps isometrically* $(H_d^2)^*$ *onto* $d(2)$.

(2) H_d^2 *is not isomorphic to* ℓ^2, *but* $(H_d^2)^*$ *is isomorphic to* $S(H_d^2) = d(2)$.

(3) H_d^2 *is a reflexive Banach space*.

Proof. (1) Is a particular case of the previous Theorem and Corollary.

(2) By using a more generally fact - Proposition 15.13 [2].

(3) By (2). □

3. Hankel Operators on H_d^2

We saw in Corollary 2.7 that H_d^2 is not isomorphic to a Hilbert space. However $H_d^2 \subset H^2$, and the restriction of a Hankel operator (with an appropriate symbol) to H_d^2 behaves very likely to a classical Hankel operator on H^2.

More specific we show in Theorem 4.1 [10] that a Hankel operator $H_{\bar{f}}$, with a decreasing sequence of Taylor coefficients, is a linear and bounded operator on H_d^2 if and only if $f \in BMOA_d^+ := \mathcal{M}_d^+ \cap BMOA$.

In order to be self-contained we give the proof of this result.

First we recall the definition of $BMOA$.

Following [12] we consider first a function $f \in L^2(\partial\mathbb{D})$ and I an interval contained in $\partial\mathbb{D}$. We write the mean of f over I as

$$f_I = \frac{1}{|I|}\int_I f(\theta)d\theta,$$

where $|I|$ denotes the length of I. f is said to have bounded mean oscillation on $\partial\mathbb{D}$ if

$$\|f\|_{BMO} = \sup_{I}\left[\frac{1}{|I|}\int_{I}|f(\theta) - f_I|^2 d\theta\right]^{1/2} < \infty.$$

Let BMO denote the space of all functions $f \in L^2(\partial\mathbb{D})$ having bounded mean oscillation. It can be checked that BMO is a Banach space modulo constants. Now let $BMOA$ the intersection of BMO with H^2 and $BMOA(\mathbb{D})$ be the space consisting of harmonic extensions $\widehat{f}$ of functions in $BMOA$.

Since $BMOA$ is only a Banach space modulo constants we consider on the $BMOA$ the norm

$$\|f\| = \|f\|_{BMO} + |\widehat{f}(0)|.$$

$BMOA$ equipped with this norm is then equivalent to the dual of H^1, as it follows by using [12] Theorem 8.3.8 (the celebrated theorem of Fefferman). The bilinear map which realizes this duality is given by

$$< f, g >= \frac{1}{2\pi}\int_{0}^{2\pi} f(\theta)\overline{g(\theta)}d\theta,$$

where $f \in H^1$, $g \in BMOA$.

In the sequel we present some results concerning the behaviour of Hankel matrices on H_d^2.

Let $(a_n)_{n\geq 0}$ be a sequence of positive real numbers with $\sum_n a_n^2 < \infty$. The infinite matrix

$$A := \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & \cdots & \cdots \\ a_3 & a_4 & \cdots & \cdots & \cdots \\ a_4 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

having the constant entries on each skew-diagonal, is called a *Hankel matrix.*

To each Hankel matrix we can associate a Hankel operator mapping H^2 into $\overline{H^2}$, denoted by H_f, where f is an anti-analytic function on the unit disk with Taylor coefficients a_n, $n \geq 0$, in the following way:

$$H_f g = (I - p)(fg),$$

for $g \in H^2$, where P is the Szegö projection, that is the orthogonal projection from $L^2(\partial D)$ onto H^2.

A natural question about this operator is to determine the property of f such that H_f is bounded. It is well-known [12] that the answer is as follows:

Theorem of Nehari *H_f is bounded if and only if f belongs to BMOA.*

Denote by ℓ_d^2 the subspace $C - C$ of ℓ^2, where $C := \{a = (a_n)_{n\geq 0} \in \ell^2;$ $a_n \downarrow_n 0\}$. ℓ_d^2 is equipped with the norm

$$\|a\|: = \inf_{a=a^1-a^2,\, a^1, a^2 \in C} \left((\sum_{n=0}^{\infty} (a_n^1)^2)^{1/2} + (\sum_{n=0}^{\infty} (a_n^2)^2)^{1/2} \right) \sim \left(\sum_{n=0}^{\infty} (|a|_{bv_0})_n^2 \right)^{1/2}.$$

Of course, so equipped, ℓ_d^2 is a Banach lattice isomorphic to H_d^2.

Similarly, in our context we have:

Theorem 3.1. *Let* A *be the Hankel matrix defined as above, where the sequence* $(a_n)_{n\geq 0}$ *is, moreover, monotone decreasing* $a_n \downarrow_n 0$. *Then* A *determine a bounded operator from* ℓ_d^2 *into* ℓ_d^2 *if and only if* $\sup_{n\geq 0}(n+1)a_n < \infty$, *that is the Hankel operator* H_f, *where* $f(z) := \sum_{n=0}^{\infty} a_n \bar{z}^n$, $z \in \mathbb{D}$, *is bounded on* H_d^2 *if and only if* $\bar{f} \in BMOA_d$.

Proof. Assume that $\sup_n(n+1)a_n < \infty$. Then by using [13] we get that $\bar{f} \in BMOA_d$. From [12]-9.2.3, we get that H_f is bounded on H^2, and $\|H_f\|$ is comparable with the norm $\|A\|_{B(\ell^2)}$. Next let us take the sequence $b = (b_n)_{n\geq 0}) \in C := \mathcal{M}_d^+ \cap H^2$, and $g = \sum_{n=0}^{\infty} b_n e^{\text{int}}, t \in [0, 2\pi]$. Then $H_f g = A \cdot b \in C$. Then it is easy to see that H_f is a bounded map on $H_d^2 := C - C$.

Conversely assume that A determines a bounded operator (which is the Hankel operator H_f) on H_d^2. Consider the unit vector function

$$k_z(t) = \frac{\sqrt{1-|z|^2}}{1-\bar{z}e^{it}} \in H^2, \; z \in \mathbb{D}.$$

This function has the Taylor coefficients $b_n = r^n e^{-\text{int}}, n \geq 0$, where $z = re^{it}$. Of course $|b_n| \downarrow_n 0$. It is well-known (see [12]-9.2.2) and easy to see that

$$\| H_f k_z \|_{H^2} = \widehat{|\bar{f}|^2}(z) - |\hat{\bar{f}}(z)|^2,$$

for all $z \in \mathbb{D}$, where $\hat{f}$ is the Poisson extension of f on $\mathbb{D}$.

Let k_z^1 be the analytic function with Taylor coefficients $|b_n|$, for all $n \geq 0$. Then, of course, $k_z^1 \in H_d^2$, with $\|k_z^1\|_{H_d^2} = \|k_z\|_{H^2} = 1$, $z \in \mathbb{D}$.

On the other hand the Taylor coefficients of $H_f k_z$ are dominated by the corresponding Taylor coefficients of $H_f k_z^1$. Moreover, the sequence of Taylor coefficients of $H_f k_z^1$ is a decreasing sequence running to 0.

Consequently $\|H_f k_z\|_{H^2} \leq \|H_f k_z^1\|_{H^2} = \|H_f k_z^1\|_{H_d^2} \leq M < \infty$, where M is the operator norm of H_f on H_d^2.

It follows that $\widehat{|\bar{f}|^2}(z) - |\hat{\bar{f}}(z)|^2$ is a bounded function on $\mathbb{D}$, and, by using Theorem 8.3.4 -[12], $\bar{f} \in BMOA$. Since $a_n \downarrow_n 0$, in view of [13], it follows that $\sup_{n\geq 0}(n+1)a_n < \infty$. □

Now we study Hankel operators on H_d^2, first we recall the definition of the $VMOA$ space. The closure of analytic polynomials in $BMOA$ is called *the* $VMOA$, and denote by $VMOA_d^+$ the cone $\mathcal{M}_d^+ \cap VMOA$. Here, of course, the space $VMOA$ is equipped with the norm induced by that of $BMOA$.

Then we have the analogue of Theorem 9.3.2 [12].

Theorem 3.2. *If* $f \in H_d^2 \cap \mathcal{M}_d^+$, $H_{\bar{f}}$ *is a compact operator from* H_d^2 *to* $(H_d^2)^\perp := (H^2)^\perp \cap L^2(\partial\mathbb{D})_d$ *if and only if* $VMOA_d^+$.

Proof. Since $H_{\bar{f}}$ is given by the matrix:

$$A := \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & \cdots & \cdots \\ a_3 & a_4 & \cdots & \cdots & \cdots \\ a_4 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

where $f(\theta) = \sum_{k=0}^{\infty} a_k e^{ik\theta}$, $\theta \in [0, 2\pi)$, $a_k \downarrow_k 0$, it is easy to see that $H_{\bar{f}}$ maps H_d^2 into $L^2(\partial\mathbb{D})_d^- := \{g \in L^2(\partial\mathbb{D}),\ g(\theta) = \sum_{k=1}^{\infty} a_k e^{-ik\theta};\ (a_k)_k \in \ell_d^2\}$.

Assume that $H_{\bar{f}}$ is a compact operator defined on H_d^2 into $L^2(\partial\mathbb{D})_d^-$, denoted simply by $\overline{H_d^2}$. We know that $H_{\bar{f}}$ maps any sequence weakly convergent to zero in a sequence norm-convergent to zero, with respect to the norm of ℓ_d^2.

Now, let k_z^1 be the analytic function $k_{|z|}(t) = \dfrac{\sqrt{1-|z|^2}}{1-\bar{z}e^{it}}$, $z \in \mathbb{D}$, and $t \in \mathbb{R}$. Clearly $k_z^1 \in H_d^2 \cap \mathbb{M}_d^+$. Since $\|k_z^1\|_{H_d^2} = \|k_{|z|}\|_{H^2} = 1$, $\forall z \in \mathbb{D}$, the set $\{k_z^1;\ z \in \mathbb{D}\}$ is bounded in H_d^2.

Because H_d^2 is a reflexive Banach space, $\{k_z^1;\ z \in \mathbb{D}\}$ is a relatively weakly compact set in H_d^2. By using Eberlein's theorem (see Theorem 11.1 - Chapter IV [11]), each subsequence of $\{k_{z_n}^1\}_n$ contains a weakly convergent subsequence in H_d^2.

Since $k_{z_n}^1 \to 0$ whenever $|z_n| \to 1^-$, it follows that each subsequence of $(H_{\bar{f}}(k_{z_n}^1))_n$ contains a norm-convergent subsequence in $\overline{H_d^2}$, hence also in $\overline{H^2}$. By using the proof of Theorem 9.3.2 - [12] it follows that this limit is equal to 0.

Consequently $\|H_{\bar{f}}(k^1_{z_n})\|_{H^2_d} \to 0,$ and also

$$\widehat{|\bar{f}|^2}(|z_n|) - |\hat{\hat{\bar{f}}}(z)|^2 = \|H_{\bar{f}}(k^1_{z_n})\|^2_{H^2} \to 0.$$

By using Theorem 8.4.2 [12], it follows that $f \in VMOA$. By the hypothesis concerning f, we have $f \in VMOA^+_d$.

Conversely, let $f \in VMOA^+_d$, $f(\theta) = \sum_{k=0}^{\infty} a_k e^{ik\theta}$, with $a_k \downarrow_k 0$. By using Theorem 8.4.7 [12], it follows that there is a $g \in C(\partial\mathbb{D})$ such that $f = Pg$, P being the Szego projection. By 9.2.3 -(4) -[12] we have $H_{\bar{f}} = H_{\bar{g}}$.

It remains, consequently, to prove that $H_g : H^2_d \to \overline{H^2_d}$, where $g \in C(\partial\mathbb{D})$, is a compact operator. By using [12], we know that $f \to H_f$ is a bounded map from $L^\infty(\partial\mathbb{D})$ into the space $B(H^2, (H^2)^\perp)$, and the norm $\|H_f\|_{B(H^2_d, \overline{H^2_d})}$ is, clearly, dominated by $\|H_f\|_{B(H^2, (H^2)^\perp)}$. Thus it is enough to show that $H_f : H^2_d \to \overline{H^2_d}$, where $f(t) = e^{\text{int}}$, $n \in \mathbb{Z}$, is a compact operator.

But it is easy to see that, for $n \in \mathbb{N}$, $H_f = 0$, and, for $n < 0$, H_f is a finite rank operator. □

As we have seen above a Hankel operator $H_{\bar{f}}$, with $f(z) = \sum_{k=0}^{\infty} a_k e^{ikz}$, $z \in \mathbb{D}$, $a_k \downarrow_k 0$, acting on H^2_d, is bounded if and only if $\sup_n (n+1)a_n < \infty$.

It is natural to ask ourselves if there is something similar in the compact case.

The following result gives us the answer:

Theorem 3.3. *Let* $f(z) = \sum_{k=0}^{\infty} a_k e^{ikz}$, $z \in \mathbb{D}$, $a_k \downarrow_k 0$, *an analytic function.*

Then $H_{\bar{f}}$ *is a compact operator acting on* H_d^2 *if and only if* $\lim_{n\to\infty} (n+1)a_n = 0$.

Proof. Necessity. Let $H_{\bar{f}}$ be a compact operator. Then by Theorem 3.2, $f \in VMOA \subset \mathcal{B}_0$, where $\mathcal{B}_0$ is the closure of all analytic polynomials in the Bloch space $\mathcal{B}$. By using Theorem 1.10(a) and Remark 1.19 - [3], it follows that

$$\sum_{j=0}^{n} j a_j = o(n), \ \forall n.$$

Since $a_j \downarrow_j 0$, we have

$$a_n \frac{n(n+1)}{2} = o(n), \text{ thus } \lim_{n\to\infty} (n+1)a_n = 0.$$

Sufficiency. Assume that $a_n \downarrow_n 0$, and $\lim_n (n+1)a_n = 0$. By Theorem 7.2.2-(3), vol. I-[6], it follows that $g(t) := \sum_{n=1}^{\infty} a_n \sin nt$, $t \in [0, 2\pi)$, is a continuous function.

If $P : L^2 \to H^2$ is the Szegö projection, $P(h)(t) := \sum_{n=0}^{\infty} b_n e^{\text{int}}$, for $h(t) = \sum_{n=-\infty}^{\infty} b_n e^{\text{int}}$, $\forall (b_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, then, by using Theorem 8.4.7 [12], $P(C(\partial\mathbb{D})) = VMOA$.

Consequently,

$$P(g)(t) = \sum_{n=1}^{\infty} \frac{1}{2i} a_n e^{\text{int}} = \frac{1}{2i} [f(t) - a_0] \in VMOA,$$

thus $f \in VMOA$, and, by using Theorem 3.2, it follows that $H_{\bar{f}}$ is a compact operator. Here, of course, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$. □

Remark 3.4. (1) By using the proof of Theorem 3.1 we get easily that, for $f \in BMOA_d^+$, the subspace, $Hank_{BMOA_d}$, of all Hankel operators $H_{\bar{f}} \in B(H_d^2, \overline{H_d^2})$ is a Banach subspace of the similar space $Hank_{BMOA}$.

(2) Similarly, for $f \in VMOA_d^+$, the space, Hank $comp_{VMOA_d}$, of all compact Hankel operators from H_d^2 into $\overline{H_d^2}$, is a Banach subspace of the similar space Hank $comp_{VMOA}$.

We intend now to characterize the nuclear Hankel operators $H_{\bar{f}}$, $f \in \mathcal{M}_d^+$ acting on H_d^2.

Let $S_1(H_d^2, \overline{H_d^2})$ the space of all nuclear operators acting from H_d^2 into $\overline{H_d^2}$, equipped with the usual nuclear norm.

Denote by $(\mathrm{Bes}_1)_d^+ = \{f(z) = \sum_{n=0}^{\infty} a_n z^n,\ a_n \downarrow_n 0;\ \|f\|_{\mathrm{Bes}_1}$

$$:= \int_D |f''(z)| dA(z) < \infty\}.$$

Then we have the analogue of Theorem 9.4.4 - [12]:

Theorem 3.5. *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n$, *with* $a_n \downarrow_n 0$, *then* $f \in (\mathrm{Bes}_1)_d^+$ *if and only if* $H_{\bar{f}} \in S_1(H^2, \overline{H_d^2})$. *Moreover the corresponding norms are equivalent.*

Proof. Let $f \in (\mathrm{Bes}_1)_d^+$. Then, by using the proof of Theorem 9.4.4 - [12], we get

$$H_{\bar{f}} = \int_{\mathbb{D}} (1 - |w|^2) \overline{g}(w) H_{\overline{K_w}} dA(w),$$

where $g(w) = f''(z) / \overline{w}^2 \in L^1(\mathbb{D}, dA)$, and $K_w(t) = \dfrac{1}{1 - \overline{w} e^{it}}$, with $|w| < 1$.

Since $H_{\overline{K}_w} h = \widehat{h}(w)(\overline{K}_w - 1)$, $\forall h \in H_d^2$, we have that $H_{\overline{K}_w}$ is a rank 1 operator on H_d^2, into $\overline{H_d^2}$. Now, since, for $w = |w|e^{i\theta}$,

$$(\overline{K}_w - 1)(t) = (\frac{1}{1 - we^{-it}} - 1) = |w|e^{i\theta}e^{-it}(\sum_{n=0}^{\infty} w^n e^{-\text{int}}),$$

we have

$$\sum_{n=k}^{\infty} \left| |w|^{n+1} e^{i(n+1)\theta} - |w|^{n+2} e^{i(n+2)\theta} \right| = \sum_{n=k}^{\infty} |w|^{n+1} |1 - w|, \ \forall k \in \mathbb{N},$$

consequently $\|\overline{K}_w - 1\|_{\overline{H_d^2}} \leq \frac{1}{(1 - |w|^2)^{1/2}} < \infty$, and $H_{\overline{K}_w} : H_d^2 \to \overline{H_d^2}$ is a rank 1 linear operator.

Thus, by Remark 3.4 - (1), and by the proof of Theorem 9.4.4 - [12], we have that

$$\|H_{\overline{K}_w}\|_{B(H_d^2, \overline{H_d^2})} \approx \frac{|w|}{1 - |w|^2}.$$

Consequently, reasoning as in [12]-pages 200-201, we have that:

$$\|H_{\bar{f}}\|_{S_1(H_d^2, \overline{H_d^2})} \text{ is dominated by } \|f\|_{(\text{Bes}_1)_d^+} < \infty.$$

Moreover, there is a constant $C > 0$, independent of f, such that

$$\|H_{\bar{f}}\|_{S_1(H_d^2, \overline{H_d^2})} \leq C\|f\|_{(\text{Bes}_1)_d},$$

for $f \in (\text{Bes}_1)_d^+$.

Conversely, let $H_{\bar{f}} \in S_1(H_d^2, \overline{H_d^2})$, with $a_n \downarrow_n 0$. Since H_d^2 has a Schauder basis [10], it follows that $H_{\bar{f}} \in (H_d^2)^* \widehat{\otimes}_\pi \overline{H_d^2}$, that is $H_{\bar{f}} = \sum_{n=1}^{\infty} \lambda_n f_n' \otimes g_n$, with $f_n' \in (H_d^2)^*$, $g_n \in \overline{H_d^2}$, and $\sum_n \lambda_n \|f_n'\|_{(H_d^2)^*} \|g_n\|_{H_d^2} < \infty$.

By using Theorem 2.1 we can assume $f_n' \in d(2) \subset H^2$. Then we have, uniformly with respect to n, $\|f_n'\|_{(H_d^2)^*} \approx \|f_n'\|_{d(2)}$, and $\|f_n'\|_{(H_d^2)^*} \geq C\|f_n'\|_{H^2}$, where $C > 0$ is a constant not depending on n.

Consequently we have $\|H_{\bar{f}}\|_{S_1(H^2,\overline{H^2})} = \sum_n \lambda_n \|f_n'\|_{H^2}\|g_n\|_{H^2} \leq C^{-1}\sum_n \lambda_n \|f_n'\|_{(H_d^2)^*}\|g_n\|_{\overline{H_d^2}} = \|H_{\bar{f}}\|_{S_1(H_d^2,\overline{H_d^2})} < \infty$, that is $H_{\bar{f}} \in S_1(H^2,\overline{H^2})$, and, by using the proof of Theorem 9.4.4 -p. 201 -[12], it follows that

$$\int_{\mathbb{D}} |f''| dA(w) < \infty.$$

Thus $f \in (\mathrm{Bes}_1)_d^+$. □

What about the characterization of Hankel operators from other ideals of operators, similar to Schatten classes S_p, $1 < p < \infty$?

We were only been able to give a characterization of Hankel operators mapping H_d^2 into $\overline{H^2}$, which are 2-*nuclear operators*. (See [8], [9].)

We recall the definition of *the analytic Besov space* Bes_2, [12] as follows:

$$\mathrm{Bes}_2 := \{f : \mathbb{D} \to \mathbb{C},\ f \text{ analytic function such that}$$

$$\|f\|_{\mathrm{Bes}_2} := \left(\int_{\mathbb{D}} |f'(z)|^2 dA(z)\right)^{1/2} < \infty\}.$$

An operator $T : X \to Y$, X, Y Banach spaces is called a 2-nuclear operator [9], and is writting $T \in N_2(X, Y)$, if

$$T = \sum_{i=1}^{\infty} a_i \otimes y_i, \text{ such that } l_{2,X'}(a) := (\sum_{i=1}^{\infty} \|a_i\|^2)^{1/2} < \infty,$$

$$\text{and } w_{2,Y}(y) := \sup_{\|y'\|_{F'} \leq 1} (\sum_{i=1}^{\infty} |y'(y_i)|^2)^{1/2} < \infty.$$

$T \in N_2(X, Y)$ is a Banach space when equipped with the norm

$$\|T\|_{N_2(X, Y)}; = \inf\{l_{2, X'}(a) w_{2, Y}(y)\}.$$

Now let us denote by $(\mathrm{Bes}_2)_d^+ := \mathrm{Bes}_2 \cap \mathcal{M}_d^+$. Then we have:

Theorem 3.6. *Let* $f \in \mathcal{M}_d^+$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $z \in \mathbb{D}$. *Then*

$$H_{\bar{f}} \in N_2(H_d^2, \overline{H^2}) \Leftrightarrow f \in (\mathrm{Bes}_2)_d^+ \Leftrightarrow \sum_{k=0}^{\infty} a_k^2 (k+1) < \infty.$$

Moreover

$$\|H_{\bar{f}}\|_{N_2(H_d^2, \overline{H^2})} \approx \Big(\sum_{k=0}^{\infty} a_k^2 (k+1)\Big)^{1/2}.$$

Proof. The second equivalence in the statement of the theorem is nothing else than a particular case of Corollary 1.3 - [5].

So, it remains only to prove

$$H_{\bar{f}} \in N_2(H_d^2, \overline{H^2}) \Leftrightarrow f \in (\mathrm{Bes}_2)_d^+.$$

Let $f \in (\mathrm{Bes}_2)_d^+$. By using Theorem 9.4.13 - [12], and Proposition 2.11.27-[9], we get that $H_{\bar{f}} \in N_2(H_d^2, \overline{H^2})$.

Conversely, let $H_{\bar{f}} \in N_2(H_d^2, \overline{H^2})$. Then, $H_{\bar{f}} T^{-1} \in N_2(\mathrm{cop}(2), H^2)$, where $T^{-1} : \mathrm{cop}(2) \to H_d^2$ is the isomorphism given by Corollary 1.7,

$$T^{-1}(u) = x,\ x_n = \sum_{k=n}^{\infty} \frac{|u_k|}{k+1},\ n \geq 0.$$

Equivalently, it follows that the operator given by the product of matrices

$$H := \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & \cdots & \cdots \\ a_3 & a_4 & \cdots & \cdots & \cdots \\ a_4 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{1}{4} & \cdots \\ 0 & 0 & 0 & \frac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

belongs to $N_2(\mathrm{cop}(2), H^2)$.

Hence, by using relation (6.5) - [2] and Proposition 2.11.27 - [9], we get that

$$H = \begin{pmatrix} a_1 & \frac{a_1 + a_2}{2} & \frac{a_1 + a_2 + a_3}{3} & \frac{a_1 + a_2 + a_3 + a_4}{4} & \cdots \\ a_2 & \frac{a_2 + a_3}{2} & \frac{a_2 + a_3 + a_4}{3} & \cdots & \cdots \\ a_3 & \frac{a_3 + a_4}{2} & \cdots & \cdots & \cdots \\ a_4 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$\in S_2(\ell_2, \ell_2).$$

Thus, it follows that

$$\sum_{i=1}^{\infty} a_i^2 + \frac{1}{2^2} \sum_{i=1}^{\infty} (a_i + a_{i+1})^2 + \frac{1}{3^2} \sum_{i=1}^{\infty} (a_i + a_{i+1} + a_{i+2})^2 + \cdots < \infty.$$

Since the sequence $(a_i)_i$ is monotone decreasing, we get

$$\sum_{i=1}^{\infty} i a_i^2 = \sum_{i=1}^{\infty} a_i^2 + \sum_{i=2}^{\infty} a_i^2 + \sum_{i=3}^{\infty} a_i^2 + \cdots < \infty,$$

that is $f \in \mathrm{Bes}_2^+$.

□

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