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BESSEL COLLOCATION APPROACH FOR SOLVING ONE-DIMENSIONAL WAVE EQUATION WITH DIRICHLET, NEUMANN BOUNDARY AND INTEGRAL CONDITIONS

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Abstract

In this paper, a collocation method based on Bessel functions of first kind is applied to solve the one-dimensional wave equation subject to the Dirichlet, Neumann boundary, and the integral conditions. Firstly, the matrix forms of these functions with two variables are constructed. Secondly, the matrix forms of the solution form and its partial derivatives are organized and thus each terms of wave equation are written in matrix form. Similarly, the matrix forms of the Dirichlet, Neumann boundary, and the integral conditions of the problem are constructed. By using the collocation points, these matrix equations and matrix operations, the wave problem is reduced to a system of linear algebraic equations. Finally, the solutions of this system determine the coefficients of the assume approximate solution in Bessel series form. An error analysis technique

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is presented for the method. To demonstrate the validity and applicability of the technique, some numerical examples are solved. The method is easy to implement and produces accurate results. Also, the results of the method are compared with the results of previous methods in literature.

1. Introduction

The solutions of the hyperbolic non-local initial-boundary value problems are used in the solutions of the model problems in science and engineering. Therefore, the development of numerical methods for the solutions of these problems has been an important research subject in many branches of science and engineering. The hyperbolic partial differential equations with given initial conditions and a standard boundary condition and an integral condition replacing the classic boundary condition are encountered in mathematical modelling of many problems in physics [1-9].

In this study, we deal with the one-dimensional wave equation [10, 11]

$$L[\nu(x,\, t)] = q(x,\, t), \quad L[\nu(x,\, t)] = \frac{\partial^2 \nu}{\partial t^2} - \frac{\partial^2 \nu}{\partial x^2}, \quad x \in [0,\, l],\, t \in [0,\, T], \tag{1}$$

with the Dirichlet boundary condition

$$\nu(x,\, 0) = f_1(x), \quad x \in [0,\, l], \tag{2}$$

$$\nu(0,\, t) = g_1(t), \quad t \in [0,\, T], \tag{3}$$

Neumann boundary condition

$$\nu_l(x,\, 0) = f_2(x),\; x \in [0,\, l], \tag{4}$$

and the nonlocal condition (or the integral condition)

$$\int_0^l \nu(x,\, t)dx = g_2(t), \quad 0 \le t \le T, \tag{5}$$

where $q,\, f_1,\, f_2,\, g_1$ and g_2 are known functions and also $q(x,\, t)$ is defined for $(x,\, t) \in [0, l] \times [0, T],\, f_1(x), f_2(x) \in C[0,\, l],\, g_1(t), g_2(t) \in C[0,\, T]$.

Recently, the one-dimensional wave equations have been solved by using numerical methods, such as the finite difference method [10], the Bernstein Ritz-Galerkin method [11], the method of lines [12], the variational iteration method [13], a numerical method based on an integro-differential formulation [14], the Legendre tau method [15], homotopy perturbation method [16], Lagrange interpolation, and modified cubic B-spline differential quadrature methods [17]. In addition, some partial differential equations considered with the integral condition have been solved with the aid of various numerical methods considered in [2, 5, 9, 18-25]. Also, Yüzbaşı and Şahin [23] have applied the Bessel collocation approach to solve singularly perturbed one-dimensional parabolic convection-diffusion problem.

In this paper, by means of the collocation method in [23], the solutions of one-dimensional wave equations will be computed in the truncated Bessel series form

$$\nu(x, t) = \sum_{r=0}^{N}\sum_{s=0}^{N} a_{r,s} J_{r,s}(x, t); \quad J_{r,s}(x, t) = J_r(x) J_s(t), \tag{6}$$

so that $a_{r,s}$; $r, s = 0, \ldots, N$ are the unknown Bessel coefficients and $J_n(x)$, $n = 0, 1, 2, \ldots, N$ are the Bessel functions of the first kind defined by

$$J_n(x) = \sum_{k=0}^{\left|\frac{N-n}{2}\right|} \frac{(-1)^k}{k!\,(k+n)!}\left(\frac{x}{2}\right)^{2k+n}, \quad n \in \mathbb{N},\ x \in [0, \infty).$$

2. Main Matrix Relations

To obtain the numerical solution of the one-dimensional wave equation with the presented method, we evaluate the Bessel coefficients of the unknown function. For this purpose, let us write the solution function (6) in type [23]

$$\nu(x, t) = \mathbf{J}(x)\mathbf{Q}(t)\mathbf{A}, \tag{7}$$

where

$$\mathbf{J}(x) = [J_0(x)\ J_1(x) \cdots J_N(x)]_{1\times(N+1)},\ \mathbf{Q}(t) = \begin{bmatrix} \mathbf{J}(t) & 0 & \cdots & 0 \\ 0 & \mathbf{J}(t) & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \mathbf{J}(t) \end{bmatrix}_{(N+1)\times(N+1)^2},$$

$$\mathbf{A} = \left[a_{0,0}\ \ a_{0,1}\ \cdots\ a_{0,N}\ \ a_{1,0}\ \ a_{1,1}\ \cdots\ a_{1,N}\ \cdots\ a_{N,0}\ \ a_{N,1}\ \cdots\ a_{N,N}\right]^T,$$

$$\mathbf{J}(x) = \mathbf{X}(x)\mathbf{D}^T,\ \mathbf{X}(x) = \left[1\ \ x\ \ x^2 \cdots x^N\right], \tag{8}$$

and if N is odd,

$$\mathbf{D} = \begin{bmatrix} \frac{1}{0!0!2^0} & 0 & \frac{-1}{1!1!2^2} & \cdots & \frac{(-1)^{\frac{N-1}{2}}}{\left(\frac{N-1}{2}\right)!\left(\frac{N-1}{2}\right)!2^{N-1}} & 0 \\ 0 & \frac{1}{0!1!2^1} & 0 & \cdots & 0 & \frac{(-1)^{\frac{N-1}{2}}}{\left(\frac{N-1}{2}\right)!\left(\frac{N+1}{2}\right)!2^{N}} \\ 0 & 0 & \frac{1}{0!2!2^2} & \cdots & \frac{(-1)^{\frac{N-3}{2}}}{\left(\frac{N-3}{2}\right)!\left(\frac{N+1}{2}\right)!2^{N-1}} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!(N-1)!2^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!\,N!2^N} \end{bmatrix}_{(N+1)\times(N+1)};$$

if N is even,

$$\mathbf{D}=\begin{bmatrix} \frac{1}{0!0!2^0} & 0 & \frac{-1}{1!1!2^2} & \cdots & 0 & \frac{(-1)^{\frac{N}{2}}}{\left(\frac{N}{2}\right)!\left(\frac{N}{2}\right)!2^N} \\ 0 & \frac{1}{0!1!2^1} & 0 & \cdots & \frac{(-1)^{\frac{N-2}{2}}}{\left(\frac{N-2}{2}\right)!\left(\frac{N}{2}\right)!2^{N-1}} & 0 \\ 0 & 0 & \frac{1}{0!2!2^2} & \cdots & 0 & \frac{(-1)^{\frac{N-2}{2}}}{\left(\frac{N-2}{2}\right)!\left(\frac{N+2}{2}\right)!2^N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!\,(N-1)!2^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!\,N!2^N} \end{bmatrix}_{(N+1)\times(N+1)}.$$

The matrix forms of the relations between the matrix $\mathbf{X}(x)$ and the matrices $\mathbf{X}^{(1)}(x)$ and $\mathbf{X}^{(2)}(x)$ becomes as follows [23]:

$$\mathbf{X}^{(1)}(x)=\mathbf{X}(x)\mathbf{B}^T \text{ and } \mathbf{X}^{(2)}(x)=\mathbf{X}(x)\left(\mathbf{B}^T\right)^2, \tag{9}$$

where

$$\mathbf{B}^T=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By using Equations (8) and (9), we have the matrix relation

$$\mathbf{J}^{(1)}(x)=\mathbf{X}(x)\mathbf{B}^T\mathbf{D}^T \text{ and } \mathbf{J}^{(2)}(x)=\mathbf{X}(x)(\mathbf{B}^T)^2\mathbf{D}^T. \tag{10}$$

Since $\mathbf{D}^T$ is an inverse matrix, by using relations (8) and (10) as follows [25]

$$\left.\begin{aligned}&\mathbf{J}^{(k)}(x) = \mathbf{X}^{(k)}(x)\mathbf{D}^T = \mathbf{X}(x)(\mathbf{B}^T)^k\mathbf{D}^T,\ k = 1, 2\\&\mathbf{X}(x) = \mathbf{J}(x)(\mathbf{D}^T)^{-1}\end{aligned}\right\},$$

we gain the relation between the matrix $\mathbf{J}(x)$ and its derivatives $\mathbf{J}^{(1)}(x)$ and $\mathbf{J}^{(2)}(x)$ as

$$\mathbf{J}^{(1)}(x) = \mathbf{J}(x)\mathbf{P} \text{ and } \mathbf{J}^{(2)}(x) = \mathbf{J}(x)\mathbf{P}^2, \tag{11}$$

so that

$$\mathbf{P}^k = (\mathbf{D}^T)^{-1}(\mathbf{B}^T)^k\mathbf{D}^T,\ k = 1, 2.$$

In the same way to Equation (9), the derivatives $\mathbf{Q}^{(1)}(t)$ and $\mathbf{Q}^{(2)}(t)$ can be expressed as follows [23]

$$\mathbf{Q}^{(1)}(t) = \mathbf{Q}(t)\overline{\mathbf{P}} \text{ and } \mathbf{Q}^{(2)}(t) = \mathbf{Q}(t)\overline{\mathbf{P}}^2, \tag{12}$$

where

$$\overline{\mathbf{P}} = \begin{bmatrix}\mathbf{P} & 0 & \cdots & 0\\ 0 & \mathbf{P} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{P}\end{bmatrix}_{(N+1)^2\times(N+1)^2}.$$

3. Method for Solution

With the aid of the relations (7), (11) and (12), we first gain the matrix forms of the terms $\nu_{xx}(x, t)$ and $\nu_{tt}(x, t)$ of Equation (1) and $\nu_t(x, t)$ given in Equation (4) as

$$\nu_t(x, t) = \mathbf{J}(x)\mathbf{Q}(t)\overline{\mathbf{P}}\mathbf{A}, \tag{13}$$

$$u_{xx}(x, t) = \mathbf{J}(x)\mathbf{P}^2\mathbf{Q}(t)\mathbf{A}, \tag{14}$$

and

$$u_{tt}(x, t) = \mathbf{J}(x)\mathbf{Q}(t)\overline{\mathbf{P}}^2\mathbf{A}. \tag{15}$$

We substitute the expressions (14) and (15) into Equation (1) and then find the matrix equation as

$$\left\{\mathbf{J}(x)\mathbf{Q}(t)\overline{\mathbf{P}}^2 - \mathbf{J}(x)\mathbf{P}^2\mathbf{Q}(t)\right\}\mathbf{A} = q(x, t). \tag{16}$$

Briefly, Equation (16) can be expressed in the matrix form as

$$\mathbf{W}(x, t)\mathbf{A} = q(x, t), \tag{17}$$

where

$$\mathbf{W}(x, t) = \left[w_{1,k}\right]_{1\times(N+1)^2} = J(x)Q(t)\overline{\mathbf{P}}^2 - J(x)\mathbf{P}^2Q(t),\ k = 0, 1, \dots, (N+1)^2.$$

When we substitute the collocation points defined by

$$x_i = \frac{l}{N}i,\ t_j = \frac{T}{N}j, \quad i = 0, 1, \dots, N,\ j = 0, 1, \dots, N, \tag{18}$$

into Equation (17), we obtain a system of the matrix equations

$$\mathbf{J}(x_i)\mathbf{Q}(t_j)\overline{\mathbf{P}}^2 - \mathbf{J}(x_i)\mathbf{P}^2\mathbf{Q}(t_j) = q(x_i, t_j).$$

Briefly, the main matrix equation of this system is written as

$$\mathbf{WA} = \mathbf{Q}. \tag{19}$$

In here,

$$\overline{\mathbf{W}} = [\mathbf{W}(x_0, t_0)\,\mathbf{W}(x_0, t_1)\cdots\mathbf{W}(x_0, t_N)\,\mathbf{W}(x_1, t_0)\,\mathbf{W}(x_1, t_1)\cdots\mathbf{W}(x_1, t_N)\cdots$$
$$\mathbf{W}(x_N, t_0)\,\mathbf{W}(x_N, t_1)\cdots\mathbf{W}(x_N, t_N)]^T_{(N+1)^2\times(N+1)^2},$$

$$\mathbf{A} = \left[a_{0,0}a_{0,1}\cdots a_{0,N}a_{1,0}a_{1,1}\cdots a_{1,N}\cdots a_{N,0}a_{N,1}\cdots a_{N,N}\right]^T_{(N+1)^2\times 1},$$

and

$$\mathbf{F} = [q(x_0, t_0)q(x_0, t_1)\cdots q(x_0, t_N)q(x_1, t_0)q(x_1, t_1)\cdots q(x_1, t_N)\cdots q(x_N, t_0)q(x_N, t_1)\cdots q(x_N, t_N)]^T_{(N+1)^2\times 1}.$$

Since we find the matrix forms of the conditions (2)-(4), we first substitute the relations (7) and (13) into Equations (2)-(4) and thus the corresponding matrix forms of the conditions (2)-(4) are written as

$$\nu(x, 0) = \mathbf{J}(x)\mathbf{Q}(0)\mathbf{A} = f_1(x), \quad 0 \leq x \leq l, \tag{20}$$

$$\nu_t(x, 0) = \mathbf{J}(x)\mathbf{Q}(0)\overline{\mathbf{P}}\mathbf{A} = f_2(x), \quad 0 \leq x \leq l, \tag{21}$$

$$\nu(0, t) = \mathbf{J}(0)\mathbf{Q}(t)\mathbf{A} = g_1(t), \quad 0 \leq t \leq T. \tag{22}$$

To get the matrix form of the condition (5), we put Equation (7) into the condition (5)

$$\int_0^l \mathbf{J}(x)\mathbf{Q}(t)\mathbf{A}dx = \left\{\int_0^l \mathbf{J}(x)dx\right\}\mathbf{Q}(t)\mathbf{A} = \left\{\int_0^l \mathbf{X}(x)dx\right\}\mathbf{D}^T\mathbf{Q}(t)\mathbf{A}, \quad 0 \leq t \leq T,$$

and thus, we have the matrix form of the condition (5) as

$$\mathbf{L}\mathbf{D}^T\mathbf{Q}(t)\mathbf{A} = g_2(t), \; 0 \leq t \leq T, \tag{23}$$

so that

$$\mathbf{L} = \left[l \;\; \frac{l^2}{2} \;\; \frac{l^3}{3} \;\; \cdots \;\; \frac{l^{N+1}}{N+1}\right].$$

When the collocation points (18) is placed into the matrix forms (20)-(23), we have

$$\nu(x_i, 0) = \mathbf{J}(x_i)\mathbf{Q}(0)\mathbf{A} = f_1(x_i), \; \nu_t(x_i, 0) = \mathbf{J}(x_i)\mathbf{Q}(0)\overline{\mathbf{P}}\mathbf{A} = f_2(x_i),$$

$$\nu(0, t_j) = \mathbf{J}(0)\mathbf{Q}(t_j)\mathbf{A} = g_1(t_j), \; \mathbf{L}\mathbf{D}^T\mathbf{Q}(t_j)\mathbf{A} = g_2(t_j).$$

Hence, the fundamental matrix equations of the conditions (2)-(5) are written as follows, respectively,

$$\mathbf{UA} = [\mathbf{F}_1] \quad \text{or} \quad [\mathbf{U};\ \mathbf{F}_1],$$

$$\overline{\mathbf{U}}\mathbf{A} = [\mathbf{F}_2] \quad \text{or} \quad [\overline{\mathbf{U}};\ \mathbf{F}_2],$$

$$\mathbf{VA} = [\mathbf{G}_1] \quad \text{or} \quad [\mathbf{V};\ \mathbf{G}_1],$$

$$\overline{\mathbf{V}}\mathbf{A} = [\mathbf{G}_2] \quad \text{or} \quad [\overline{\mathbf{V}};\ \mathbf{G}_2],$$

so that

$$\mathbf{U} = [\mathbf{U}_0\mathbf{U}_1 \cdots \mathbf{U}_N]^T,\ \overline{\mathbf{U}} = [\overline{\mathbf{U}}_0\overline{\mathbf{U}}_1 \cdots \overline{\mathbf{U}}_N]^T,\ \mathbf{V} = [\mathbf{V}_0\mathbf{V}_1 \cdots \mathbf{V}_N]^T,$$

$$\overline{\mathbf{V}} = [\overline{\mathbf{V}}_0\overline{\mathbf{V}}_1 \cdots \overline{\mathbf{V}}_N]^T,\ \mathbf{F}_1 = [f_1(x_0)\ f_1(x_1) \cdots f_1(x_N)]^T,$$

$$\mathbf{F}_2 = [f_2(x_0)\ f_2(x_1) \cdots f_2(x_N)]^T,\ \mathbf{G}_1 = [g_1(t_0)\ g_1(t_1) \cdots g_1(t_N)]^T,$$

$$\mathbf{G}_2 = [g_2(t_0)\ g_2(t_1) \cdots g_2(t_N)]^T,\ i = 0, 1, \ldots, N,\ j = 0, 1, \ldots, N,$$

$$\mathbf{U}_i = \mathbf{J}(x_i)\mathbf{Q}(0) = \lfloor u_{i1}\ u_{i2}\ u_{i3}\ \cdots\ u_{i(N+1)^2} \rfloor,$$

$$\overline{\mathbf{U}}_i = \mathbf{J}(x_i)\mathbf{Q}(0)\overline{\mathbf{P}} = \lfloor \overline{u}_{i1}\ \overline{u}_{i2}\ \overline{u}_{i3}\ \cdots\ \overline{u}_{i(N+1)^2} \rfloor,$$

$$\mathbf{V}_j = \mathbf{J}(0)\mathbf{Q}(t_j) = \lfloor \nu_{j1}\ \nu_{j2}\ \nu_{j3}\ \cdots\ \nu_{j(N+1)^2} \rfloor$$

and

$$\overline{\mathbf{V}}_j = \mathbf{LD}^T\mathbf{Q}(t_j) = \lfloor \overline{\nu}_{j1}\ \overline{\nu}_{j2}\ \overline{\nu}_{j3}\ \cdots\ \overline{\nu}_{j(N+1)^2} \rfloor.$$

To find the solution of Equation (1) under conditions (2)-(5), we form the augmented matrix [15] as

$$\left[\widetilde{\mathbf{W}};\ \widetilde{\mathbf{Q}}\right] = \begin{bmatrix} \mathbf{U} & ; & \mathbf{F}_1 \\ \overline{\mathbf{U}} & ; & \mathbf{F}_2 \\ \mathbf{V} & ; & \mathbf{G}_1 \\ \overline{\mathbf{V}} & ; & \mathbf{G}_2 \\ \mathbf{W} & ; & \mathbf{Q} \end{bmatrix}. \tag{24}$$

Thus, the Bessel coefficients matrix is

$$\mathbf{A} = \left(\tilde{\tilde{\mathbf{W}}}\right)^{-1} \tilde{\tilde{\mathbf{Q}}}.$$

In here, $\left[\tilde{\tilde{\mathbf{W}}};\ \tilde{\tilde{\mathbf{Q}}}\right]$ is computed by using the Gauss elimination technique and then removing the zero rows of gauss eliminated matrix. The Bessel coefficients matrix is easily calculated by using the command '$\mathbf{A} = \widetilde{\mathbf{W}} \setminus \widetilde{\mathbf{Q}}$' in MATLAB. The determined coefficients is placed in Equation (6) and thus, we obtain the desired approximate solution

$$\nu_N(x, t) = \sum_{r=0}^{N} \sum_{s=0}^{N} a_{r,s} J_{r,s}(x, t). \tag{25}$$

4. Error Estimation for Solution

In this section, by using error computation [28, 29] and the residual correction technique [30, 31], error estimation is made for the suggested method. For our purpose, we deal with the residual function for the present method as

$$R_N(x, t) = L[\nu_N(x, t)] - q(x, t). \tag{26}$$

Here, $\nu_N(x, t)$ is the Bessel series solution (25) of the problem (1)-(5). Hence, $\nu_N(x, t)$ satisfies the equation

$$L[\nu_N(x, t)] = \frac{\partial^2 \nu_N}{\partial t^2} - \frac{\partial^2 \nu_N}{\partial x^2} = q(x, t) + R_N(x, t), \tag{27}$$

with the conditions

$$\begin{cases} \nu_N(x, 0) = f_1(x), \ \nu_{Nt}(x, 0) = f_2(x), \ 0 \le x \le l, \\ \nu_N(0, t) = g_1(t), \ \displaystyle\int_0^l \nu_N(x, t)dx = g_2(t), \ 0 \le t \le T. \end{cases} \tag{28}$$

Now, let us define the error function as

$$e_N(x, t) = \nu(x, t) - \nu_N(x, t). \tag{29}$$

Here, $\nu(x, t)$ is the exact solution of the problem (1)-(5).

By using Equations (1)-(5) and (27)-(28), we obtain the error differential equation

$$L[e_N(x, t)] = L[\nu(x, t)] - L[\nu_N(x, t)] = -R_N(x, t),$$

with the homogeneous conditions

$$\begin{cases} e_N(x, 0) = 0, \ e_{Nt}(x, 0) = 0, \ 0 \le x \le l, \\ e_N(0, t) = 0, \ \displaystyle\int_0^l e_N(x, t)dx = 0, \ 0 \le t \le T, \end{cases}$$

or clearly, the error problem is

$$\begin{cases} \dfrac{\partial^2 e_N}{\partial t^2} - \dfrac{\partial^2 e_N}{\partial x^2} = -R_N(x, t), \\ e_N(x, 0) = 0, \ e_{Nt}(x, 0) = 0, \ 0 \le x \le l, \\ e_N(0, t) = 0, \ \displaystyle\int_0^l e_N(x, t)dx = 0, \ 0 \le t \le T, \end{cases} \tag{30}$$

The error problem (30) in the same way as in Section 3 is solved and thus we gain the approximation, $e_{N,M}(x, t)$ to $e_N(x, t)$.

Consequently, if the exact solution of Equation (1) is not known, then the error function can be guessed by $e_{N,M}(x, t)$.

5. Numerical Examples

In this section, some examples will be investigated to show the reliability and the efficiency of the proposed scheme in this paper. The errors have been computed by using

$$L_2 = \|\nu(x, t) - \nu_N(x, t)\|_2 = \left(\int_0^T \int_0^l (\nu(x, t) - \nu_N(x, t))^2 dxdt\right)^{1/2},$$

and

$$L_\infty = \|\nu(x, t) - \nu_N(x, t)\|_\infty = \max\{|\nu(x, t) - \nu_N(x, t)|,\ 0 \le x \le l,\ 0 \le t \le T\}.$$

Application of the error estimation introduced in Section 4 is made in Example 1. The computations associated with the examples have been done on an Intel PC using MATLAB.

Example 1 ([12]). We first consider Equations (1)-(5) with $l = T = 1$, $f_1(x) = 0$, $f_2(x) = xe^{-x}$, $g_1(t) = 0$, $g_2(t) = -2te^{-t-1} + te^{-t}$ and $q(x, t) = -2(x - t)e^{-x-t}$. The exact solution of the problem is [2, 10] $u(x, t) = xte^{-x-t}$.

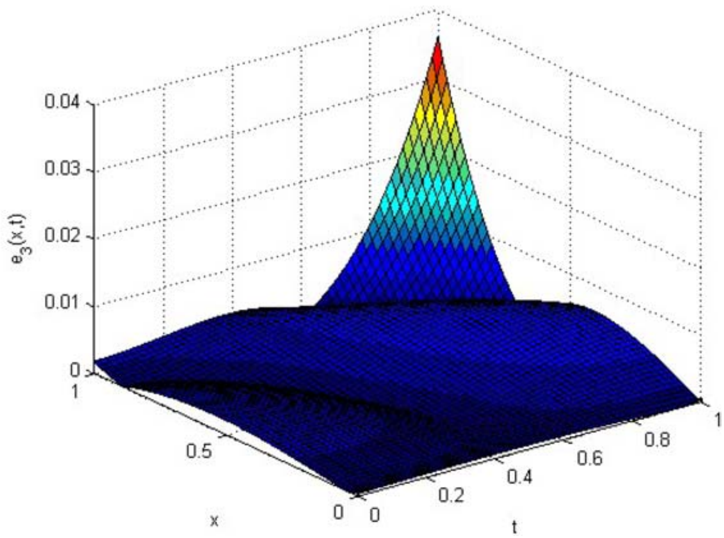
By applying the scheme described in Section 3, we find the approximate solutions of the problem for $N = 3, 5, 7, 10$. In Table 1, we show the values of L_2 and L_∞ for $N = 3, 5, 7, 10$. The actual and the estimated maximum absolute errors are tabulated for some values (N, M). In addition, Figure 1(a)-(d) show graphs of the absolute error functions $e_N(x, t) = |\nu(x, t) - \nu_N(x, t)|$ for $N = 3, 5, 7, 10$. The estimated absolute error functions, $e_{N,M}(x, t)$ for $(N, M) = (3, 4), (7, 8)$ are given in Figure 1(e)-(f). It is observed from Figure 1 and Table 2 that the error estimation defined in Section 4 is very accurate.

Table 1. The errors L_2 and L_∞ for Example 1

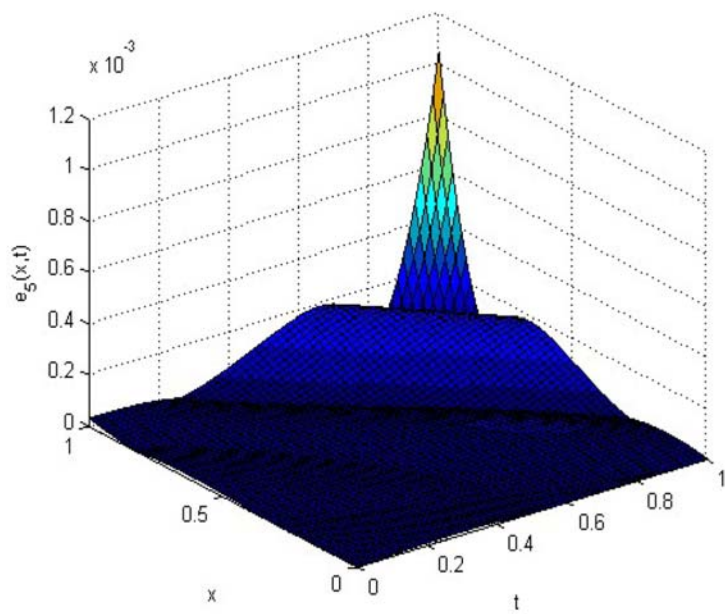
N	3	5	7	10
L_2	3.89×10^{-3}	6.89×10^{-5}	4.19×10^{-7}	1.06×10^{-9}
L_∞	3.58×10^{-2}	1.05×10^{-3}	7.9×10^{-6}	4.2×10^{-9}

Table 2. Comparison of maximum absolute errors (actual and estimation) for some values (N, M) for Example 1

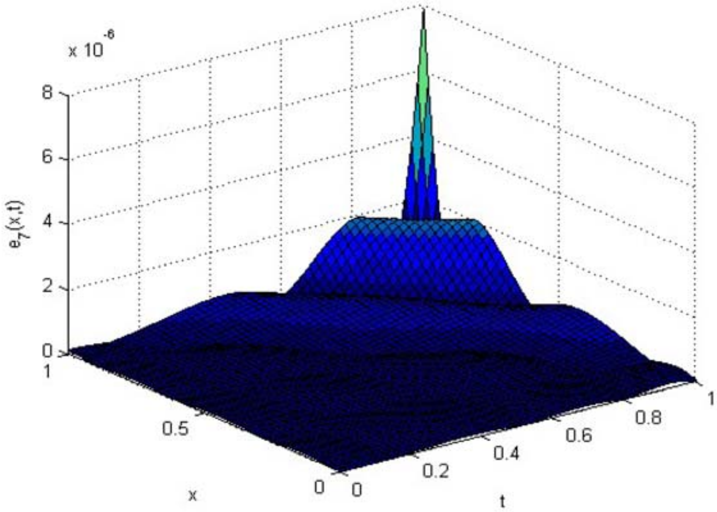
(N, M)	Actual maximum absolute error L_∞	Estimated maximum absolute error
(3, 4)	3.5809×10^{-2}	1.9944×10^{-2}
(4, 5)	8.1408×10^{-3}	6.2206×10^{-4}
(4, 6)	8.1408×10^{-3}	3.6640×10^{-4}
(5, 7)	1.0435e-003	2.5119e-004
(7, 8)	7.8252×10^{-6}	1.3249×10^{-6}
(8, 9)	5.8052×10^{-7}	2.7452×10^{-7}
(9, 10)	8.6434×10^{-8}	6.9753×10^{-8}
(10, 11)	4.2730×10^{-9}	3.3826×10^{-8}



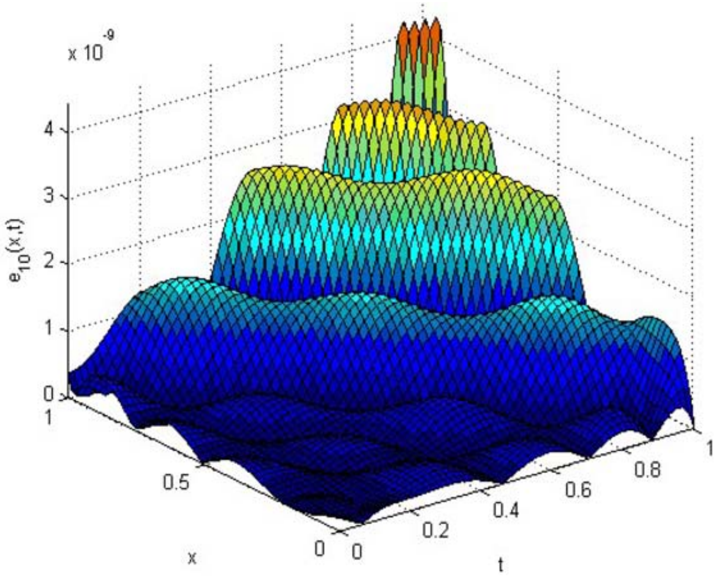
(a) Plot of the absolute error function $e_3(x, t)$ for $N = 3$.



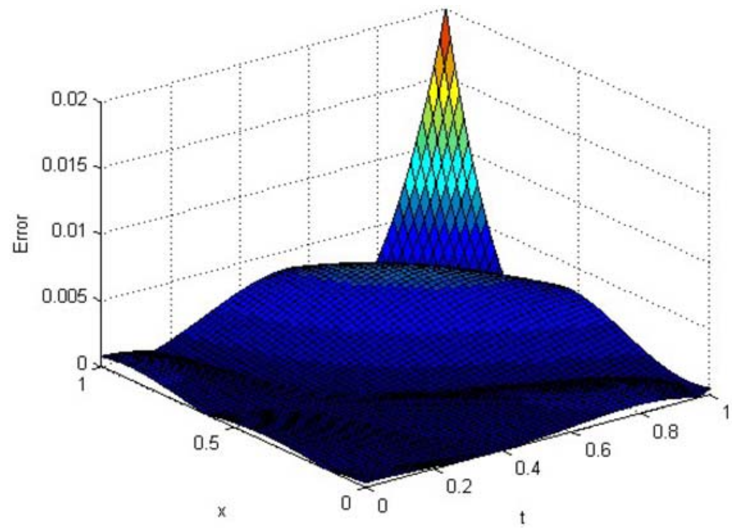
(b) Plot of the absolute error function $e_5(x, t)$ for $N = 5$.



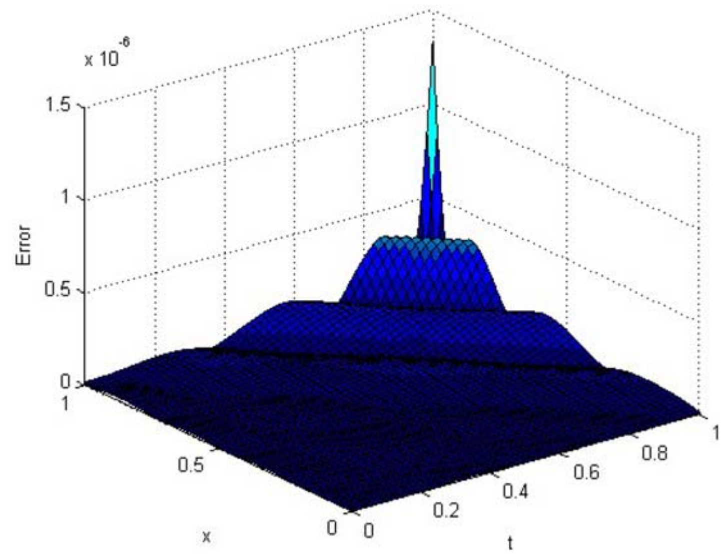
(c) Plot of the absolute error function $e_7(x,\ t)$ for $N = 7$.



(d) Plot of the absolute error function $e_{10}(x,\ t)$ for $N = 10$.



(e) Plot of the absolute error function $e_{3,4}(x, t)$.

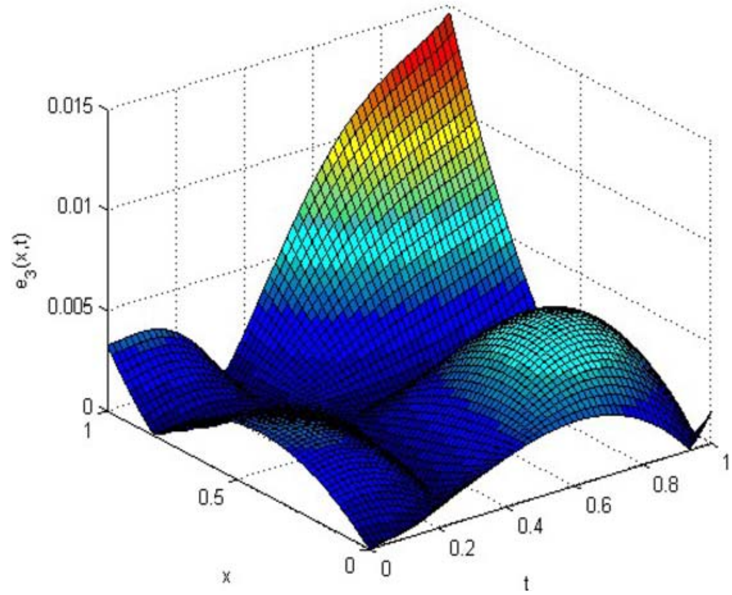


(f) Plot of the absolute error function $e_{7,8}(x, t)$.

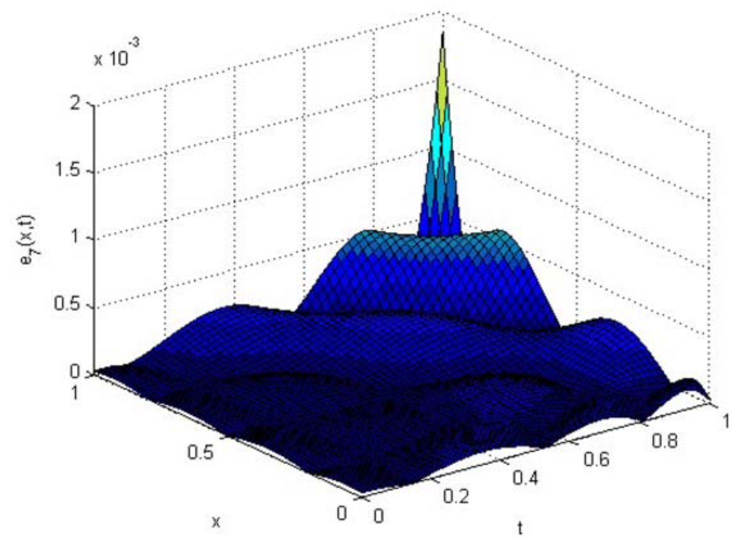
Figure 1. For Example 1 (a)-(d), graphs of the absolute error functions $e_N(x, t) = |\nu(x, t) - \nu_N(x, t)|$ for $N = 3, 5, 7, 10$ and (e)-(f) the estimated absolute error functions, $e_{N,M}(x, t)$ for $(N, M) = (3, 4), (7, 8)$.

Example 2 ([11])**.** As a second example, let us consider Equations (1)-(5) with $l = T = 1$, $f_1(x) = 0$, $f_2(x) = 0$, $g_1(t) = 0$, $g_2(t) = 0$, $q(x, t) = (2x - 3x^2)\left(\frac{-4t^2}{(1+t^2)^2} + \frac{2}{1+t^2}\right) + 6\ln(1+t^2)$ and the exact solution $\nu(x, t) = \ln(1+t^2)(2x - 3x^2)$.

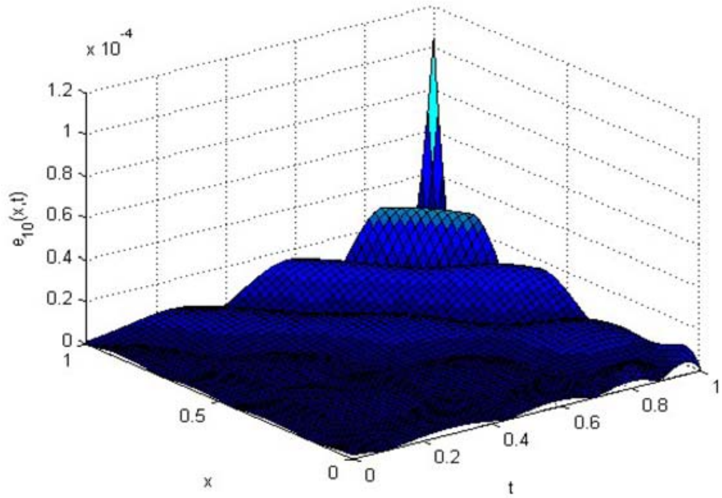
By using the presented technique with $N = 3, 7, 10$, the approximate solutions are computed for $N = 3, 7, 10$ of the Example 2. Table 3 presents some values of absolute error functions $e_N(x, t) = |\nu(x, t) - \nu_N(x, t)|$ for $N = 3, 7, 10$. In Figure 2, the absolute error functions $e_N(x, t) = |\nu(x, t) - \nu_N(x, t)|$ for $N = 3, 7, 10$ are plotted. Table 3 and Figure 2 show that the accuracy increases when N is increased.



(a) Plot of the absolute error function $e_3(x,\ t)$ for $N = 3$.



(b) Plot of the absolute error function $e_7(x,\ t)$ for $N = 7$.



(c) Plot of the absolute error function $e_{10}(x, t)$ for $N = 10$.

Figure 2. Graphs of the absolute error functions $e_N(x, t) = |\nu(x, t) - \nu_N(x, t)|$ for $N = 3, 7, 10$.

Table 3. Comparison of the absolute errors of $\nu(x, t)$ for $N = 3, 7, 10$ of Example 2

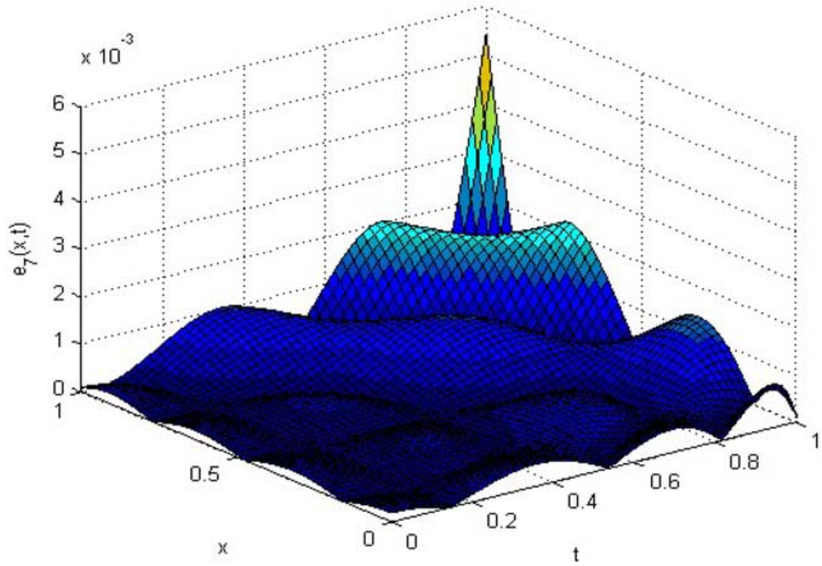
(x_i, t_j)	$N = 3$, $e_3(x_i, t_j)$	$N = 7$, $e_7(x_i, t_j)$	$N = 10$, $e_{10}(x_i, t_j)$
(0, 0)	5.1378e-005	3.4366e-005	1.8486e-006
(0.1, 0.1)	1.2218e-003	2.3335e-005	4.2913e-007
(0.2, 0.2)	1.7303e-003	2.2185e-006	5.6527e-007
(0.3, 0.3)	9.2713e-004	3.2261e-005	1.9086e-006
(0.4, 0.4)	5.9512e-004	1.7129e-005	6.2070e-007
(0.5, 0.5)	1.6496e-003	1.7738e-007	1.7617e-006
(0.6, 0.6)	1.2108e-003	1.1823e-004	7.7937e-006
(0.7, 0.7)	1.0464e-003	2.2970e-004	8.7023e-007
(0.8, 0.8)	4.6978e-003	8.0386e-005	1.3817e-005
(0.9, 0.9)	9.1120e-003	4.8088e-004	2.3497e-005
(1,1)	1.4561e-002	1.8319e-003	1.0376e-004

Example 3 ([10]). Finally, we consider Equations (1)-(5) with $l = T = 1$, $f_1(x) = 0$, $f_2(x) = \pi\cos(\pi x)$, $g_1(t) = \sin(\pi t)$, $g_2(t) = 0$ and $q(x, t) = 0$. The exact solution of this problem is $\nu(x, t) = \cos(\pi x)\sin(\pi t)$.

Table 4 denotes a comparison of the present method and the finite difference method [10] for $N = 7$. This comparison shows that our method is very effective. Also, we give the absolute error function $e_7(x, t) = |\nu(x, t) - \nu_7(x, t)|$ for $N = 7$ in Figure 3.

Table 4. Comparison of the absolute errors for $\nu(x_i, 0.5)$ of the Example 3

x_i	Finite difference method [10]	Present method
		$N = 7$, $e_7(x_i, 0.5)$
0.1	1.5e-003	2.8149e-004
0.2	1.4e-003	2.5674e-004
0.3	1.7e-003	1.6368e-004
0.4	1.6e-003	7.3575e-005
0.5	1.5e-003	5.5188e-005
0.6	1.5e-003	2.5172e-004
0.7	1.9e-003	5.3196e-004
0.8	1.8e-003	7.8042e-004
0.9	1.7e-003	7.6521e-004
1	1.6e-003	2.6242e-004



Plot of the absolute error function $e_7(x, t)$ for $N = 7$.

Figure 3. Graphs of the absolute error functions $e_N(x, t) = |\nu(x, t) - \nu_N(x, t)|$ for $N = 7$.

6. Conclusion

In this article, a collocation approach is presented for the approximate solutions of onedimensional wave equation subject to Dirichlet, Neumann boundary and nonlocal integral conditions. We demonstrate the accuracy and efficiency of our technique with examples. It seems from Tables and Figures that the errors decrease as N is increased. By using the error estimation introduced in Section 4, the absolute error functions are estimated and they are shown in Figure 1(e)-(f). It is seen from Figure 1 and Table 2 that the error estimation is very effective. When the exact solution of the problem is not known, then the errors can be guessed with the error function $e_{N,M}(x, t)$. In addition, we compare our method with the finite difference method [1] and this comparison indicates that our method is very effective and accurate and easy to apply as well. The approximate solutions of Equation (1) by the suggested method are calculated easily in shorter time with the computer programs such as MATLAB, Maple, and Mathematica.

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